Good Morning!

<u>Announcements</u>:

Office Hours: <u>today</u> 12:30 - 2:30
736 Evans

<u>Friday</u>: Quiz up to and
including Section 1.4

Warmup Exer 1) T/F Is it possible for two vectors $\underline{v}_1, \underline{v}_2$ in $\mathbb{R}^3$ to span all of $\mathbb{R}^3$?

$$\text{Span}\{\underline{v}_1, \underline{v}_2\} = \{ a_1\underline{v}_1 + a_2\underline{v}_2 \}$$

any numbers

If $\text{Span}\{\underline{v}_1, \underline{v}_2\} = \mathbb{R}^3$, this means any $\underline{u}$

$$\underline{u} = a_1\underline{v}_1 + a_2\underline{v}_2 \quad \text{some } a_1, a_2$$

Reinterpret as lin sys:

$$3\left\{\left[\begin{array}{cc|c} | & | & \\ \underline{v}_1 & \underline{v}_2 & \underline{u} \\ | & | & \end{array}\right]\right.$$
$$\underbrace{\qquad\qquad\qquad\qquad}_{3}$$

in form of aug. matrix.

Want a_1, a_2 to solve sys.

RREF $\rightsquigarrow$ $\left[\begin{array}{cc|c} | & | & | \\ \underline{w}_1 & \underline{w}_2 & \underline{r} \\ | & | & | \end{array}\right]$

What are the possible RREF for $3\left\{\underbrace{\begin{bmatrix} | & | \\ \underline{w}_1 & \underline{w}_2 \\ | & | \end{bmatrix}}_{2}\right.$?

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Last row is always $[0 \;\; 0]$!

Choose $\underline{r} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ Last row becomes

$$\left[\begin{array}{cc:c} 0 & 0 & 1 \end{array}\right]$$

No soln!

<u>Warmup</u> 2) For what $\underline{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ can we solve: $\begin{aligned} 2x_1 - x_2 &= b_1 \\ -4x_1 + 2x_2 &= b_2 \end{aligned}$?

Aug matrix: $\left[\begin{array}{cc|c} 2 & -1 & b_1 \\ -4 & 2 & b_2 \end{array}\right]$

REF $\rightsquigarrow$ $\left[\begin{array}{cc|c} 2 & -1 & b_1 \\ 0 & 0 & b_2 + 2b_1 \end{array}\right]$

$\boxed{\text{Soln} \iff b_2 + 2b_1 = 0}$.

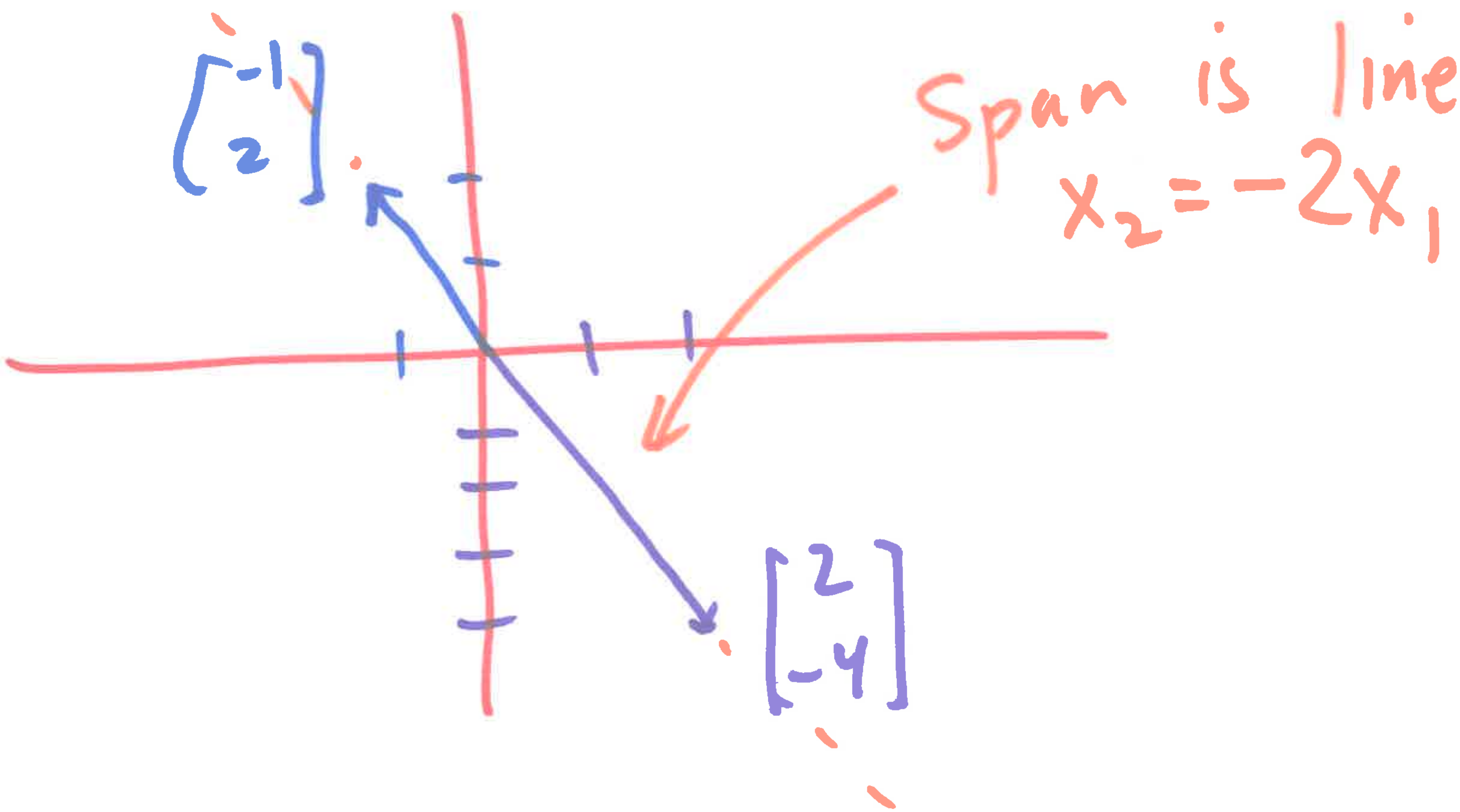
If $b_2 + 2b_1 = 0$ then $\left[\begin{array}{cc|c} 2 & -1 & b_1 \\ 0 & 0 & 0 \end{array}\right]$

x_2 any number

$x_1 = \dfrac{b_1 + x_2}{2}$

Problem was same as asking when is $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ in

$$\text{Span}\left\{ \begin{bmatrix} 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\} ?$$



__Last time__ 3 equivalent notations

1) Lin Sys:

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ -2x_1 &= 4 \\ 3x_1 - 6x_2 &= -1 \end{aligned}$$

2) Aug matrix

$$\left[\begin{array}{cc:c} 1 & 2 & 3 \\ -2 & 0 & 4 \\ 3 & -6 & -1 \end{array}\right]$$

3) Matrix eqn: $A\underline{x} = \underline{b}$

$$\begin{bmatrix} 1 & 2 \\ -2 & 0 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$$

Product of a matrix and a vector:

Input: A $m \times n$ matrix

$\underline{x}$ n-vector

Output: $\underline{b}$ m-vector

$$m\left\{\underbrace{\begin{bmatrix} & A & \end{bmatrix}}_{n}\right. \quad n\left\{\begin{bmatrix} \underline{x} \end{bmatrix}\right. \quad \begin{bmatrix} \underline{b} \end{bmatrix}\Big\}m$$

$$\underline{b} = \begin{bmatrix} | & & | \\ \underline{a}_1 & \cdots & \underline{a}_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \cdot \underline{a}_1 + x_2 \underline{a}_2 + \cdots + x_n \cdot \underline{a}_n$$

m-vectors

Exer ~~For what~~ what is $\underline{a}_3$ if $A = \begin{bmatrix} 2 & -1 & \underline{a}_3 \\ 0 & 3 & \end{bmatrix}$

$\underline{x} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$ and $A\underline{x} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$?

$$A\underline{x} = \begin{bmatrix} 2 & -1 & \underline{a}_3 \\ 0 & 3 & \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 2 \\ 0 \end{bmatrix} + (-2)\begin{bmatrix} -1 \\ 3 \end{bmatrix} + (-1)\underline{a}_3 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

So $\underline{a}_3 = -\begin{bmatrix} 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} + (-2)\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} 4 \\ -9 \end{bmatrix}$

Focus on matrix eqns: $A\underline{x} = \underline{0}$

Col perspective:

$$A = \begin{bmatrix} | & & \\ \underline{v}_1 & \cdots\cdots & \underline{v}_k \\ | & & \end{bmatrix} \qquad \underline{x} = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}$$

We are writing $\underline{0}$ as a lin comb of cols:

$$A\underline{x} = a_1\underline{v}_1 + a_2\underline{v}_2 + \cdots + a_k\underline{v}_k = \underline{0}$$

<u>Def</u> List of vectors $\underline{v}_1, \dots, \underline{v}_k$ is

1) <u>linearly dependent</u> if there are coeffs $a_1, \dots, a_k$ not all zero (at least one is non-zero) such that

$$a_1 \underline{v}_1 + \cdots + a_k \underline{v}_k = \underline{0}.$$

We say $A\underline{x} = \underline{0}$ has a <u>nontrivial</u> soln.

2) linearly independent if whenever

$$a_1 \underline{v}_1 + \cdots + a_k \underline{v}_k = \underline{0}$$

we always have $a_1 = \cdots = a_k = 0$

We say $A\underline{x} = \underline{0}$ has only the trivial soln $\underline{x} = \left.\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}\right\} k$

Exer Which subsets of the following list are lin indep?

$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $\underline{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\underline{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\underline{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\underline{v}_5 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

1 elt subsets: ~~We need~~ A single vector

$\underline{v}$ is lin indep $\iff$ $\underline{v} \neq \underline{0}$.

Lin indep means $a \cdot \underline{v} = \underline{0} \Rightarrow a = 0$.

Then we must have had $\underline{v} \neq \underline{0}$.

$\underline{v} \neq \underline{0}$ means $a \cdot \underline{v} = \underline{0} \Rightarrow a = 0$

This is exactly the def of Lin indep

Lin indep 1 <u>elt subsets</u> are:

$\underline{v}_1$ or $\underline{v}_3$ or $\underline{v}_4$ or $\underline{v}_5$

<u>2 elt subsets</u> Lin indep means

$a\underline{u} + b\underline{v} = \underline{0}$ implies $a = b = 0$.

<u>Check</u>: $\underline{v}_1, \underline{v}_2$? Lin dep (not lin indep!)

$$0 \cdot \underline{v}_1 + 1 \cdot \underline{v}_2 = \underline{0}$$

this coeff is not 0.

Check $\underline{v}_1, \underline{v}_3$? Suppose $a_1\underline{v}_1 + a_3\underline{v}_3 = \underline{0}$

This means: $\begin{bmatrix} a_1 + a_3 \\ 0 - a_3 \\ -a_1 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We find: $a_1 = a_3 = 0$

So Lin indep!

<u>Check</u> $\underline{v}_1, \underline{v}_3, \underline{v}_4$? Suppose $a_1\underline{v}_1 + a_3\underline{v}_3 + a_4\underline{v}_4 = \underline{0}$

$$\begin{bmatrix} a_1 + a_3 + a_4 \\ 0 - a_3 + a_4 \\ -a_1 + 0 + a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We find:

$a_1 = a_4$

$a_3 = a_4$

$3a_1 = 0$

Conclude: $a_1 = a_3 = a_4 = 0$.

So Lin Indep!

Span	vs Lin Indep
Want many vectors in small space	Want few vectors in big space.
Adding vectors to list only helps	Deleting vectors from list only helps
$A = \begin{bmatrix} \vert & & \vert \\ \underline{v}_1 & \cdots & \underline{v}_k \\ \vert & & \vert \end{bmatrix}$ $A\underline{x} = \underline{b}$ has soln $\iff \underline{b} \in \text{Span}\{\underline{v}_1, \dots, \underline{v}_k\}$	$A\underline{x} = \underline{0}$ has only triv soln $\underline{x} = \underline{0}$ $\iff \underline{v}_1, \dots, \underline{v}_k$ lin indep