Lecture 25 Fourier Series

Or "How I Became a Mathematician"

Today Off Hrs 12-2pm 736 Evans

Fri Quiz through 9.6

Next Week: Reviews here during lecture times

Off Hrs Thurs 12-2pm 736 Evans

Warmup Find heat $u(x,t)$ in rod of length $L = \pi$ with $\beta = 7$

given boundary valves

$$u(0,t) = 0 = u(\pi,t)$$

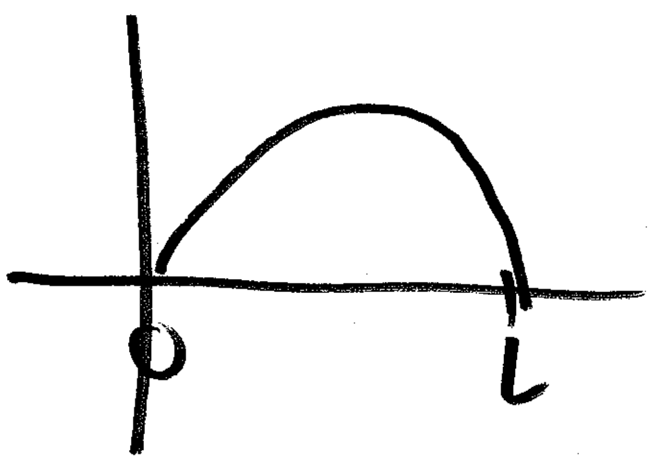
and initial valves

$$u(x,0) = 3\sin(2x) - 6\sin(5x)$$

Soln Last lecture we found list of solns to heat eqn with given boundary values

$$u_n(x,t) = e^{-\beta\left(\frac{n\pi}{L}\right)^2 \cdot t} \sin\left(\frac{n\pi x}{L}\right)$$

$n = 1, 2, 3, \ldots$



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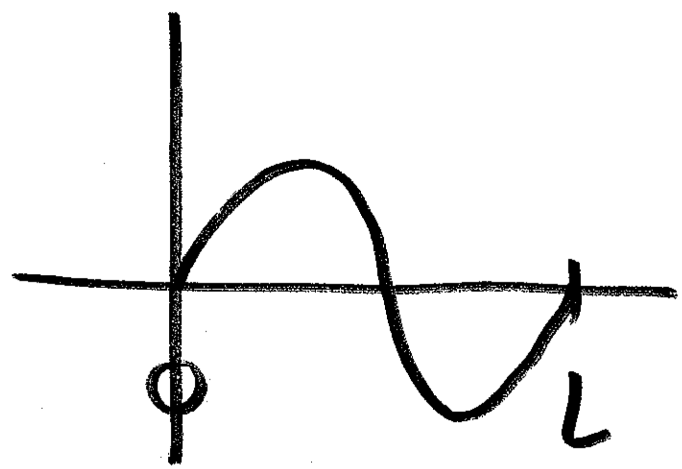


, ...

Here $L = \pi$, $\beta = 7$ so

$$u_n(x,t) = e^{-7n^2t} \sin(nx), \quad n = 1, 2, 3, \ldots$$

We hope we can write

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t)$$

Find c_n so that initial value is correct!

Set $t=0$:

$$u(x,0) = \sum_{n=1}^{\infty} c_n u_n(x,0)$$

$$= \sum_{n=1}^{\infty} c_n \sin(nx)$$

Want:

$$u(x,0) = 3\sin(2x) - 6\sin(5x)$$

So take $c_2 = 3$, $c_5 = -6$

and all others $= 0$

We have found the soln

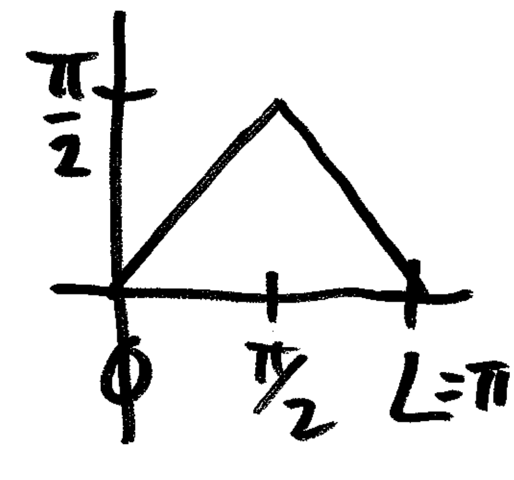
$$\boxed{u(x,t) = 3e^{-7(2)^2 t}\sin(2x) - 6e^{-7(5)^2 t}\sin(5x)}$$

Goal: Solve similar problem for arbitrary fn $f(x)$ with

$$f(0) = 0 = f(L)$$

Representative Exer. Find heat $u(x,t)$ in rod of length $L = \pi$ with $\beta = 1$ given initial values

$$u(x,0) = \begin{cases} x & 0 \le x \le \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \le x \le \pi \end{cases}$$



Informal but Miraculously Effective Idea:

$u_n(x,0) = \sin\left(\frac{n\pi x}{L}\right)$ form

$n = 1, 2, 3, \ldots$

an "orthogonal basis" for

$$V = \{ f : [0,L] \to \mathbb{R} \mid f(0) = 0 = f(L) \}$$

Recall V inner product vect sp

$\underline{u}_1, \dots, \underline{u}_k$ list of vectors

1) is orthogonal if $\langle \underline{u}_i, \underline{u}_j \rangle = 0$ $i \neq j$

2) is orthogonal basis if

a) orthogonal and a

b) basis

Given $\underline{u}_1, \dots, \underline{u}_k$ orthogonal basis any vector $\underline{v}$ in V can be written

$$\underline{v} = \sum_{i=1}^{k} \underbrace{\frac{\langle \underline{v}, \underline{u}_i \rangle}{\langle \underline{u}_i, \underline{u}_i \rangle}}_{\text{numbers } c_i} \underline{u}_i$$

What is the inner product on

$$V = \{ f : [0,L] \to \mathbb{R} \mid f(0) = 0 = f(L) \}$$

?

$$\langle f, g \rangle = \int_0^L f(x) g(x) \, dx$$

We can calculate:

$$\langle u_m, u_n \rangle = \int_0^L \sin\left(\frac{m\pi x}{L}\right) \cdot \sin\left(\frac{n\pi x}{L}\right) dx$$

$m, n = 1, 2, 3, \ldots$

$$= \int_0^L \frac{1}{2}\left[\cos\left(\frac{(m-n)\pi x}{L}\right) - \cos\left(\frac{(m+n)\pi x}{L}\right)\right] dx$$

$$= \begin{cases} 0 & m \neq n \\ \frac{L}{2} & m = n \end{cases}$$

list is orthogonal!

In fact the list is an "orthogonal basis"

(Not really a basis since we will need ∞ sums.)

<u>Thm</u> Any diff. fn $f : [0, L] \longrightarrow \mathbb{R}$ with $f(0) = 0 = f(L)$ is equal to its <u>Fourier Series</u>!

Fourier Series:

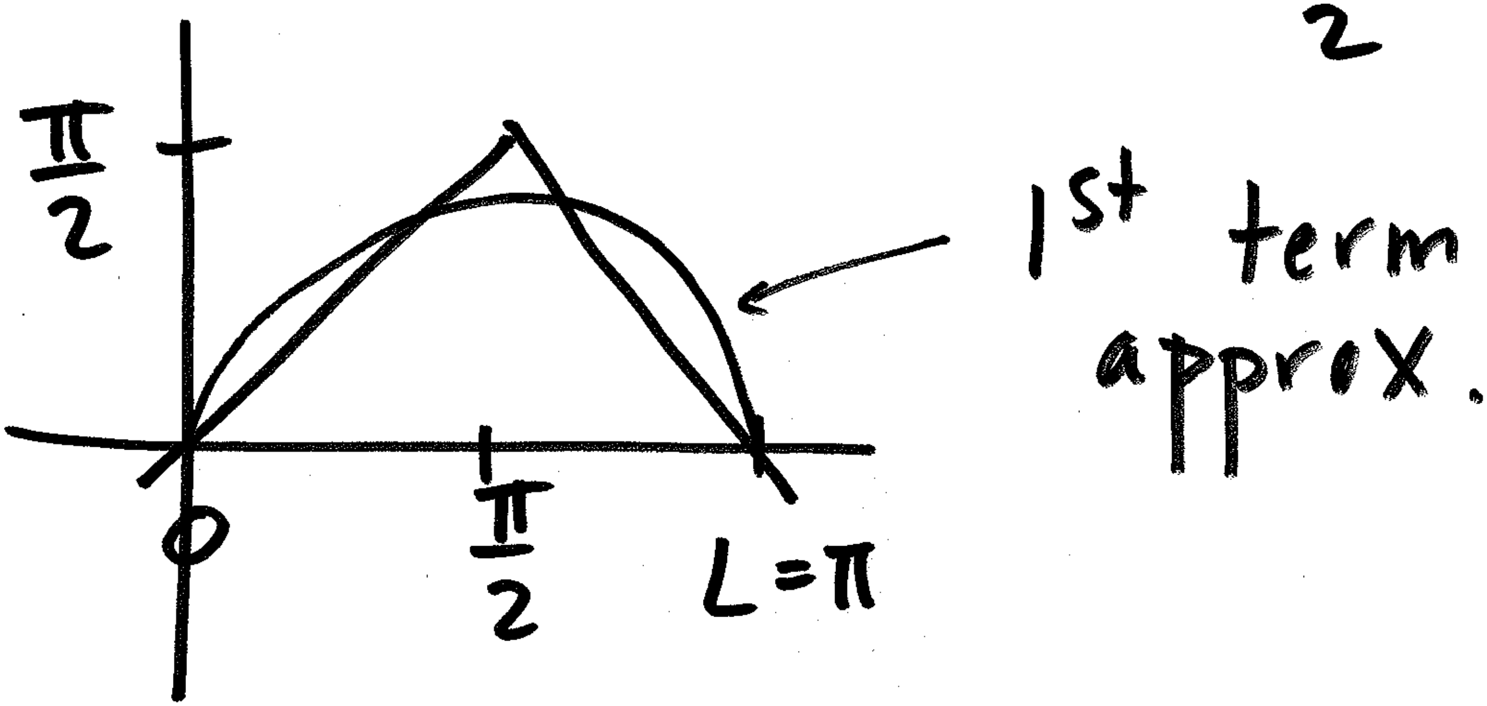
$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$$

Fourier coeffs:

$$c_n = \frac{\langle f, u_n \rangle}{\langle u_n, u_n \rangle} = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Back to representative exer:

$$L=\pi, \quad f(x)=\begin{cases} x & 0 \le x \le \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \le x \le \pi \end{cases}$$



Integrating by parts . . . we find

$$c_n = \begin{cases} 0 & n \text{ even} \\ \dfrac{4(-1)^{(n-1)/2}}{\pi n^2} & n \text{ odd} \end{cases}$$

Thus

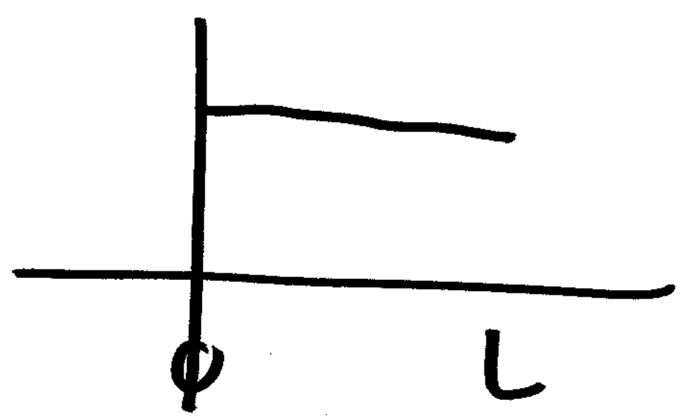
$$f(x) = \frac{4}{\pi}\left(\sin(x) - \frac{1}{9}\sin(3x) + \frac{1}{25}\sin(5x) - \cdots\right)$$

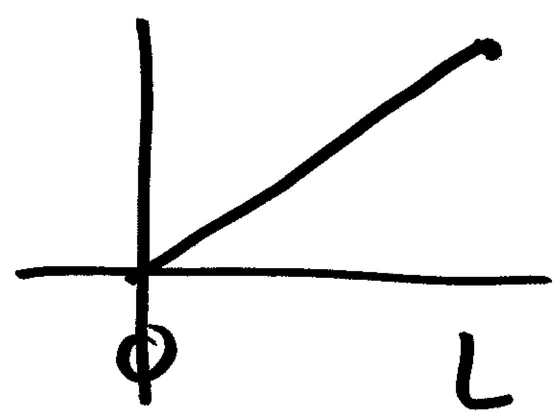
So the soln to heat eqn with this initial value is:

$$u(x,t) = \frac{4}{\pi}\left(e^{-t}\sin(x) - \frac{1}{9}e^{-9t}\sin(3x) + \frac{1}{25}e^{-25t}\sin(5x) - \cdots\right)$$

More general Fourier Series

What happens if we don't impose $f(0) = 0 = f(L)$?

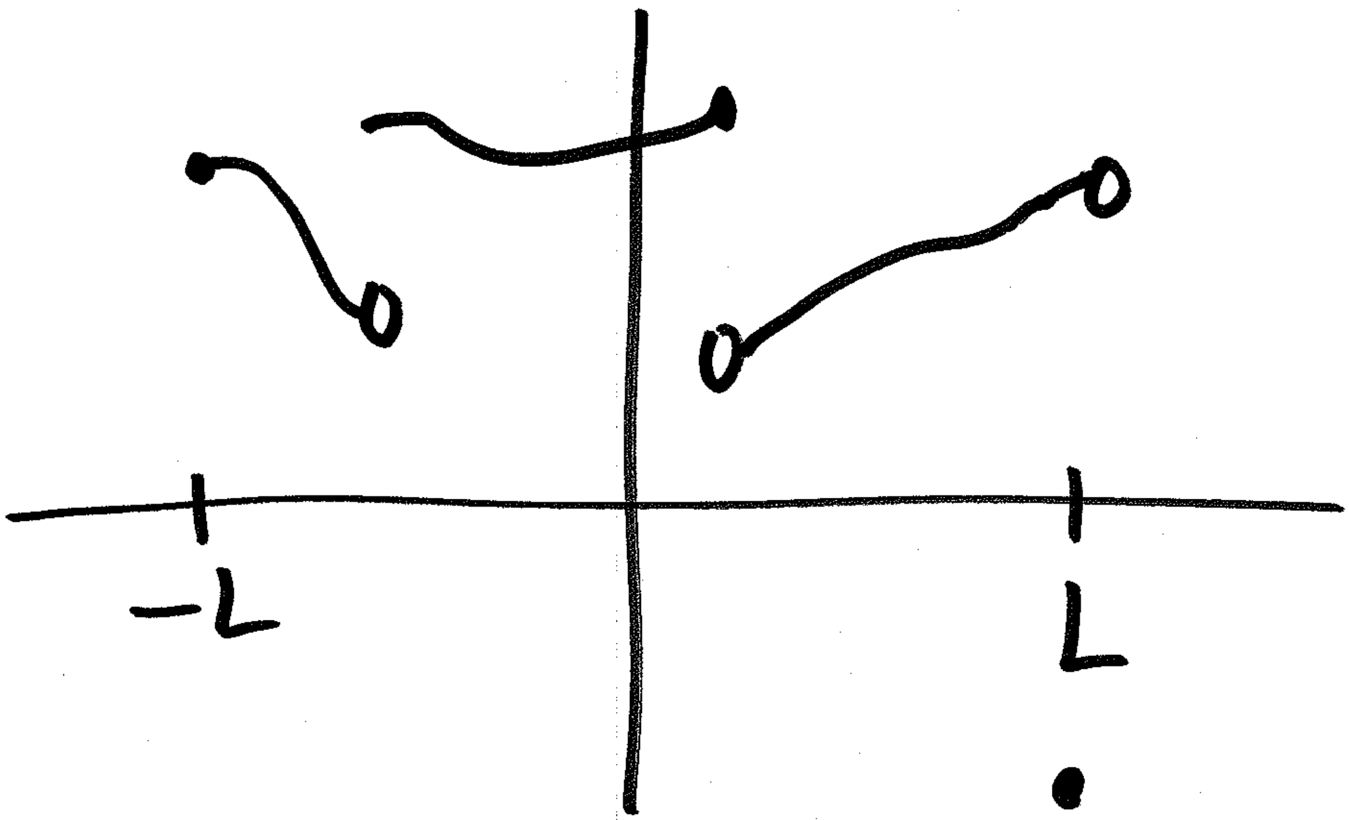




Use cos as well!

Natural Setup: $f: [-L, L] \to \mathbb{R}$

so that f, f' are piece-wise cont.



"Orthogonal basis"

$$\cos\left(\frac{n\pi x}{L}\right) \qquad n = 0, 1, 2, \ldots$$

$$\sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots$$

Def Fourier series of $f(x)$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

Fourier coeffs:

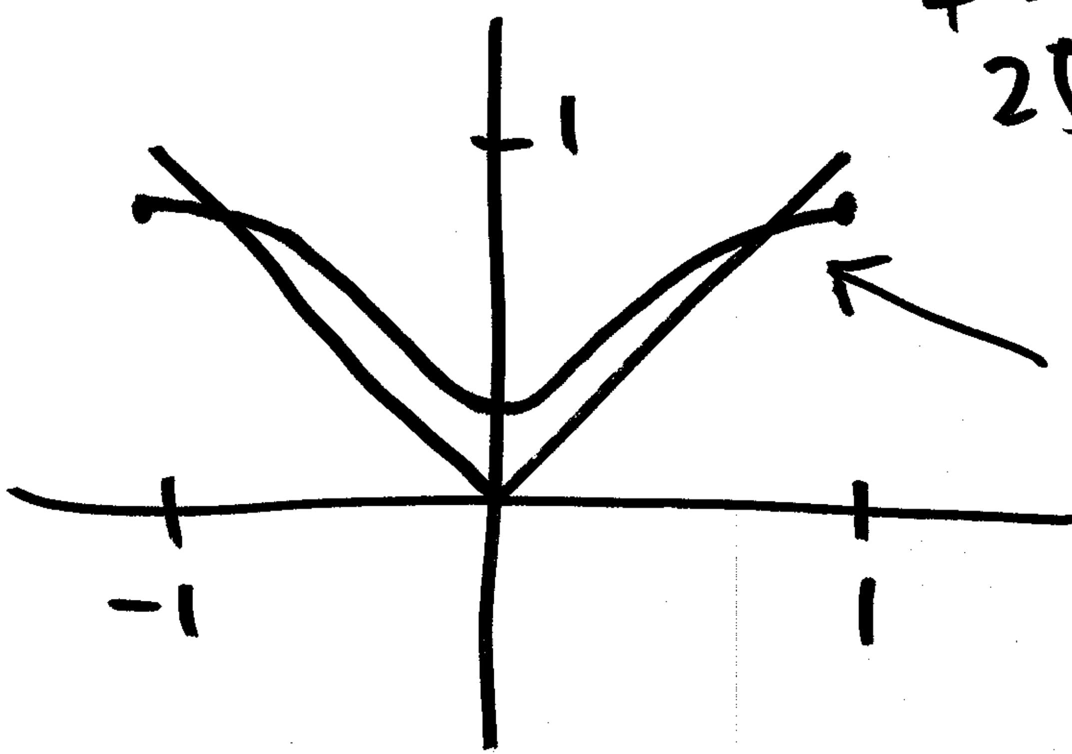
$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

<u>Thm</u> If f, f' are p.w. cont then for any x in $(-L, L)$ we have

$$\text{Fourier series of } f \text{ at } x = \begin{cases} f(x) & \text{if } f \text{ cont at } x \\ \frac{1}{2}\left[f(x^+) + f(x^-)\right] & \text{if } f \text{ is not nec cont at } x \end{cases}$$

__Example__ $L = 1$, $f(x) = |x|$

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\left(\cos(\pi x) + \frac{1}{9}\cos(3\pi x) + \frac{1}{25}\cos(5\pi x) + \dots\right)$$



approx after a few steps