Lecture 24: Solving the Heat Equation!

Thurs. Off. Hrs 12-2pm 736 Evans

Fri. Quiz through 9.6

Next week: Reviews during lecture times

Warmup 1) Find general soln of heat eqn for $u = u(x,t)$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

that are <u>independent of t</u>.

2) Solve <u>BVP</u> for u with

$$u(0,t) = u_0$$
$$u(L,t) = u_L$$

with u as above.

<u>Soln</u> 1) u indep of t means

$$\frac{\partial u}{\partial t} = 0.$$

So $0 = \dfrac{\partial^2 u}{\partial x^2}$

Thus general soln is

$$u = ax + b$$

2) $u(0) = a \cdot 0 + b = u_0$

So $b = u_0$

$u(L) = a \cdot L + u_0 = u_L$

So $a = \dfrac{u_L - u_0}{L}$

Thus $u = \left(\dfrac{u_L - u_0}{L}\right) x + u_0$

solves BVP

Let's solve heat eqn in general!

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}$$

IVP: $u(x,0) = f(x)$

BVP: $u(0,t) = 0$

$u(L,t) = 0$

Method Separation of Variables

Hope for soln $u(x,t) = X(x)\,T(t)$

$$\frac{\partial u}{\partial t} = X(x)\,T'(t)$$

$$\frac{\partial^2 u}{\partial x^2} = X''(x)\,T(t)$$

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}$$ implies

$$X(x)T'(t) = \beta X''(x)T(t)$$

Thus

$$\underbrace{\frac{1}{\beta}\frac{T'(t)}{T(t)}}_{\text{indep of } x} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{indep of } t}$$

So we conclude

$$\frac{1}{\beta}\frac{T'(t)}{T(t)} = \underbrace{-\lambda}_{\text{some constant}} = \frac{X''(x)}{X(x)}$$

We arrive at decoupled ODEs

$$T'(t) + \lambda\beta T(t) = 0 \qquad X''(x) + \lambda X(x) = 0$$

Return to BVP and see what is implied: $u(x,t) = X(x)T(t)$

$$\begin{array}{ccccc} u(0,t) & = & 0 & = & u(L,t) \\ \parallel & & & & \parallel \\ X(0)T(t) & & & & X(L)T(t) \end{array}$$

To find a nontriv soln, let's impose:

$$X(0) = 0 = X(L)$$

Now solve ODE (homogeneous)

$$X''(x) + \lambda X(x) = 0$$

given Boundary Values

$$X(0) = 0 = X(L)$$

Three cases: $\lambda > 0$, $\lambda = 0$, $\lambda < 0$ ...

Only $\lambda > 0$ leads to nontriv soln.

$$X(x) = c_1 \cos(\sqrt{\lambda} \cdot x) + c_2 \sin(\sqrt{\lambda} \cdot x)$$

This is general soln
before imposing boundary values

Boundary values imply

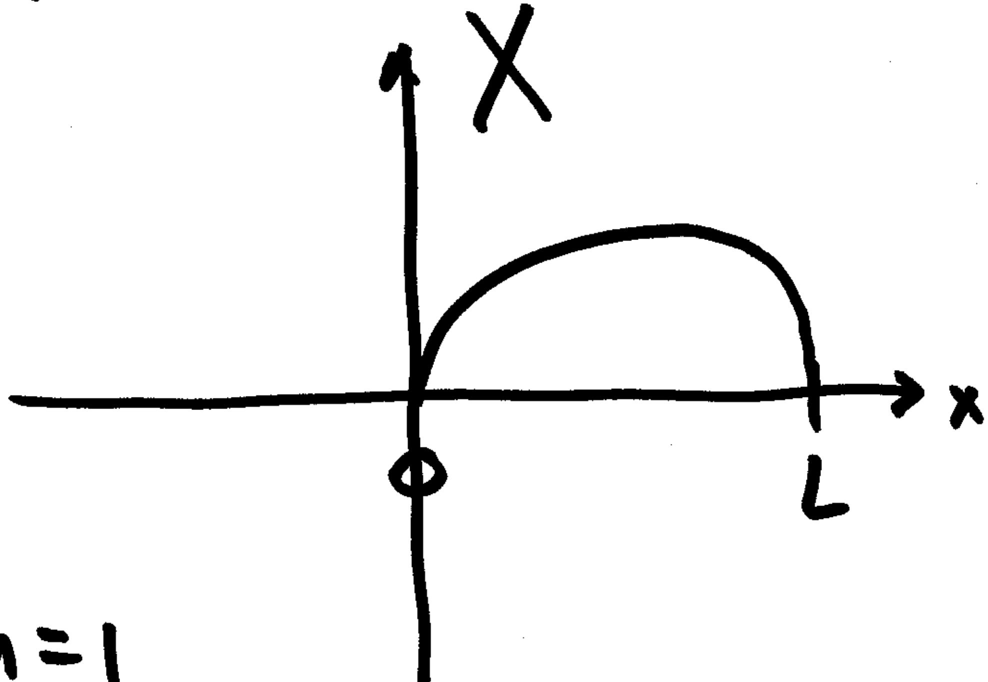
$$X(x) = C_2 \sin\left(\frac{n\pi}{L} \cdot x\right) \qquad n = 1, 2, 3, \ldots$$

$$\left(\sqrt{\lambda} = \frac{n\pi}{L}\right)$$

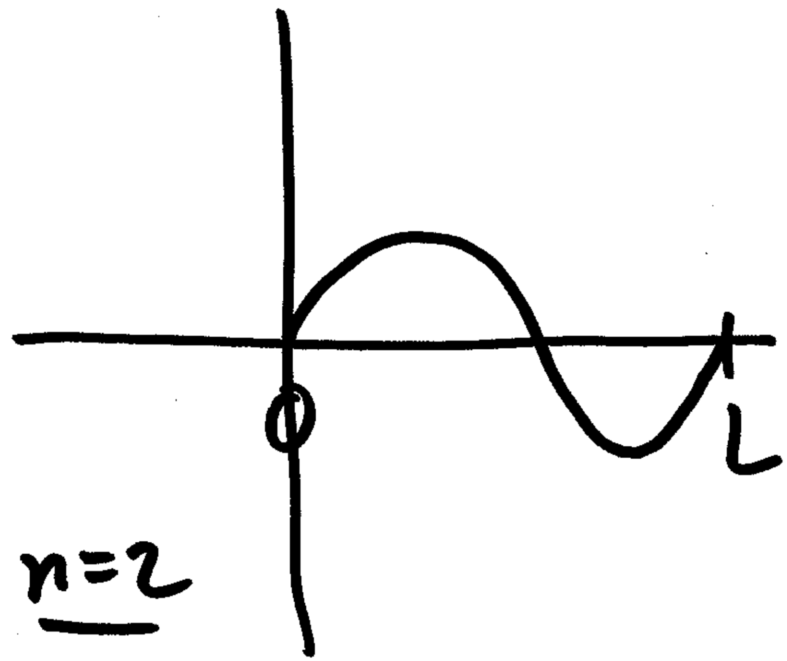
<u>Define</u> $X_n(x) = \sin\left(\frac{n\pi}{L} \cdot x\right)$

for $n = 1, 2, 3, \ldots$

What do these solns look like?

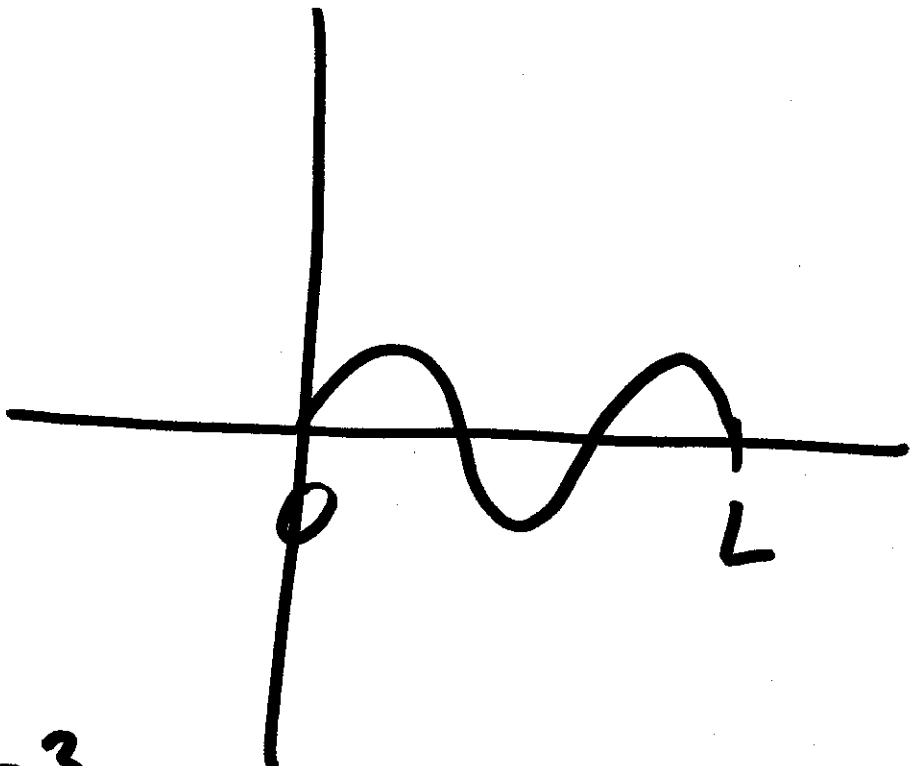


n=1



n=2

$$X_1(x) = \sin\left(\frac{\pi}{L} x\right) \qquad X_2(x) = \sin\left(\frac{2\pi}{L} \cdot x\right)$$



<u>$n=3$</u>

$$X_3(x) = \sin\left(\frac{3\pi}{L}\cdot x\right)$$

Now back to other ODE

$$T'(t) + \beta \lambda T(t) = 0$$

We've found already $\lambda = \left(\frac{n\pi}{L}\right)^2$

<u>Define</u> $T_n(t) = e^{-\beta\left(\frac{n\pi}{L}\right)^2 \cdot t}$

for $n = 1, 2, 3, \ldots$

These are solns of above ODE

Collecting our work, we find a <u>list of solns</u>

$$u_n(x,t) = X_n(x)\,T_n(t)$$

$$= \sin\left(\frac{n\pi}{L}\cdot x\right) e^{-\beta\left(\frac{n\pi}{L}\right)^2 \cdot t}$$

for $n = 1, 2, 3, \ldots$

Remaining missing ingredient:

given initial valve $u(x,0)=f(x)$

Suppose our soln is a "<u>lin comb</u>"

$$u(x,t)=\sum_{n=1}^{\infty} c_n u_n(x,t)$$

This implies

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n u_n(x,0)$$

$$= \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L} \cdot x\right)$$

<u>Next</u> time: make precise

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L} \cdot x\right)$$

and find coeffs c_n, $n=1,2,\dots$