Lecture 21 Higher Order Linear Diff Eqns

No Off. Hrs. this week

Find your GSI and get to know him!

Friday: Quiz through 4.5

Warmup Solve IVP

$$y''' - y' = t + \sin(t)$$

$$y(0) = 2,\ y'(0) = 0,\ y''(0) = 0$$

Soln Analyze homog. version

$$y''' - y' = 0$$

$$\underbrace{\left(\left(\frac{d}{dt}\right)^3 - \frac{d}{dt}\right)}_{\text{lin transf}} y = 0$$

vector

Aux eqn $r = \frac{d}{dt}$

$$r^3 - r = 0$$

Factor: $r(r^2-1) = r(r-1)(r+1)$

Roots: $r = 0, 1, -1$

$$\frac{d}{dt}\left(\frac{d}{dt} - 1\right)\left(\frac{d}{dt} - (-1)\right) y = 0$$

(order doesn't matter)

Basis for soln set

$$e^{0\cdot t} = 1, \quad e^{t}, \quad e^{-t}$$

Now to solve nonhomog. version

want to show $t + \sin(t)$ is in range of $\left(\left(\frac{d}{dt}\right)^3 - \frac{d}{dt}\right)$

1) Find soln to $y''' - y' = t$

<u>Guess</u> $t^2 \rightsquigarrow 0 - 2t = -2t$

<u>Take</u> $-\frac{t^2}{2}$

2) Find soln to $y''' - y' = \sin(t)$

<u>Guess</u> $A\cos(t) + B\sin(t)$

$\rightsquigarrow A\sin(t) + B(-1)\cos(t)$

$-(-1)A\sin(t) - B\cos(t)$

$$= 2A\sin(t) - 2B\cos(t)$$

Take: $A = \frac{1}{2}$ $\quad B = 0$

$$\frac{1}{2}\cos(t)$$

Gen soln to nonhomog eqn

$$y = -\frac{t^2}{2} + \frac{1}{2}\cos(t) + c_1 \cdot 1 + c_2 e^{t} + c_3 e^{-t}$$

Now solve IVP:

$$y(0) = \frac{1}{2} + c_1 + c_2 + c_3 = 2$$

$$y'(t) = -t - \frac{1}{2}\sin(t) + c_2 e^{t} - c_3 e^{-t}$$

$$y'(0) = c_2 - c_3 = 0$$

$$y''(t) = -1 - \frac{1}{2}\cos(t) + c_2 + c_3$$

$$y''(0) = -1 - \frac{1}{2} + c_2 + c_3 = 0$$

Organize into lin syst:

$$\begin{bmatrix} \frac{3}{2} \\ 0 \\ +\frac{3}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Now solve for $c_1, c_2, c_3 \ldots$

Higher Order IVP Theorem

$g, p_1, \dots, p_n$ fns on (a,b)

$\gamma_0, \dots, \gamma_{n-1}$ numbers

x_0 a point of (a,b)

There exists a unique soln to IVP

$$y^{(n)} + p_1 y^{(n-1)} + \dots + p_n y = g$$

$$y(x_0) = \gamma_0, \; y'(x_0) = \gamma_1, \; \dots, \; y^{(n-1)}(x_0) = \gamma_{n-1}$$

Same analysis as we've seen:

Diff op $L = \left(\frac{d}{dt}\right)^n + p_1 \left(\frac{d}{dt}\right)^{n-1} + \cdots + p_n$

(lin transf)

$L y = g$ nonhomog eqn

$L y = 0$ homog eqn

There exists basis for soln set

$y_1, \ldots, y_n$

Find particular soln y_p to nonhomog eqn

$$Ly_p = g$$

Gen soln is of form

$$y = y_p + c_1 y_1 + \cdots + c_n y_n$$

To solve IVP, need to solve lin syst

$$\begin{bmatrix} \gamma_0 - y_p(x_0) \\ \gamma_1 - y_p'(x_0) \\ \vdots \\ \gamma_{n-1} - y_p^{(n-1)}(x_0) \end{bmatrix} = \begin{bmatrix} y_1(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & & y_n'(x_0) \\ \vdots & & \vdots \\ y_1^{(n-1)}(x_0) & & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Why does this lin syst have a unique soln?

<u>Def</u> <u>Wronskian</u> $W[y_1, \dots, y_n](x)$ is det. of

$$\begin{bmatrix} y_1(x) & \cdots & y_n(x) \\ \vdots & & \vdots \\ y_1^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{bmatrix}$$

So we can uniquely solve IVP at x_0
iff $W[y_1, \ldots, y_n](x_0) \neq 0$ } with any given values

Lemma The following are equivalent

1) $W[y_1, \ldots, y_n](x_0) \neq 0$
2) $W[y_1, \ldots, y_n](x) \neq 0$ for all x
3) $y_1, \ldots, y_n$ lin. indep.

__Exer__ Calculate Wronskian from Warmup problem

$$y_1 = 1 \,, \quad y_2 = e^{t} \,, \quad y_3 = e^{-t}$$

$$W[y_1, y_2, y_3] = \det \begin{bmatrix} 1 & e^{t} & e^{-t} \\ 0 & e^{t} & -e^{-t} \\ 0 & e^{t} & e^{-t} \end{bmatrix}$$

$$= 2$$

Now let's specialize to case of constant coeffs

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g$$

numbers

(Gen theory was for fn coefs

$$y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_0 y = g)$$

fns

To solve such an eqn, we proceed exactly as for second order case.

Exer Solve $y^{(4)} - y = 0$

Aux eqn: $r^4 - 1 = 0$

Factor: $(r^2-1)(r^2+1) = 0$

$(r-1)(r+1)(r+i)(r-i) = 0$

Basis of solns

$$e^{t}, e^{-t}, e^{-it}, e^{it}$$

Alternative basis

$$e^{t}, e^{-t}, \cos t, \sin t$$

Exer $y^{(4)} - y^{(2)} = 0$ ← Solve

Aux eqn: $r^4 - r^2 = 0$

Factor $r^2(r^2-1) = 0$

$r^2(r-1)(r+1) = 0$

Basis of solns

$$e^t, \; e^{-t}, \; 1, \; t$$

Question: Can I aways solve in this way?

Fund Thm of Alg: Can always factor aux eqn with complex roots

Basis for solns if aux eqn has root r

distinct real root : e^{rt}

complex root $e^{(\alpha+i\beta)t}$, $e^{(\alpha-i\beta)t}$

alt basis $e^{\alpha t}\cos\beta t$, $e^{\alpha t}\sin\beta t$

repeated root of mult k $e^{rt}, te^{rt}, \ldots, t^{k-1}e^{rt}$