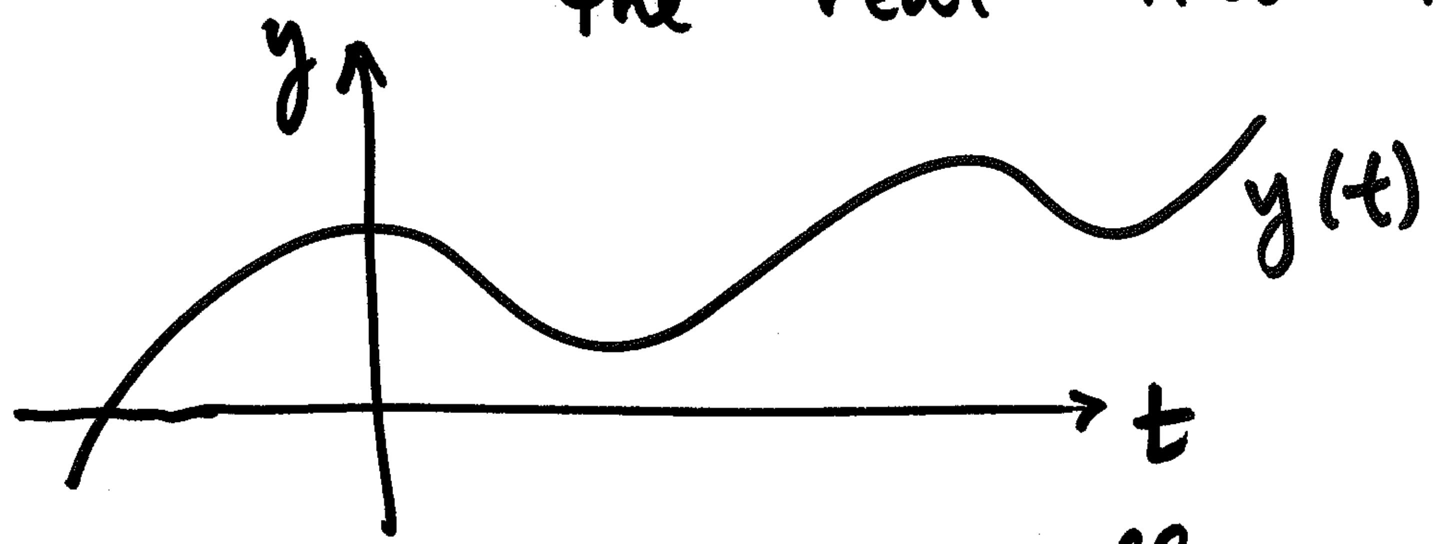
Lecture 19 On to Differential Equations!

(Don't Believe the Hype...)

<u>Today</u> Off. Hours 12-2pm 736 Evans

<u>Friday</u> Quiz through 6.5

Warmup! Let $y = y(t)$ be a fn on the real line $\mathbb{R}$



Find all solns to the diff eqn

$$y'' = 3t$$

1) <u>Guess some</u> soln: $y = \frac{t^3}{2}$

2) <u>Consider homogeneous version</u>

$$y'' = 0$$

General soln: $y = at + b$

a, b numbers

Altogether, general solution of inhomog. eqn $y = \frac{t^3}{2} + at + b$

__Lin Alg Interpretation__ V vect sp of fns on $\mathbb{R}$

$T : V \to V$ lin transf.

$T(y) = y''$

We calculated:

~~Nul T = Span{1, t}~~

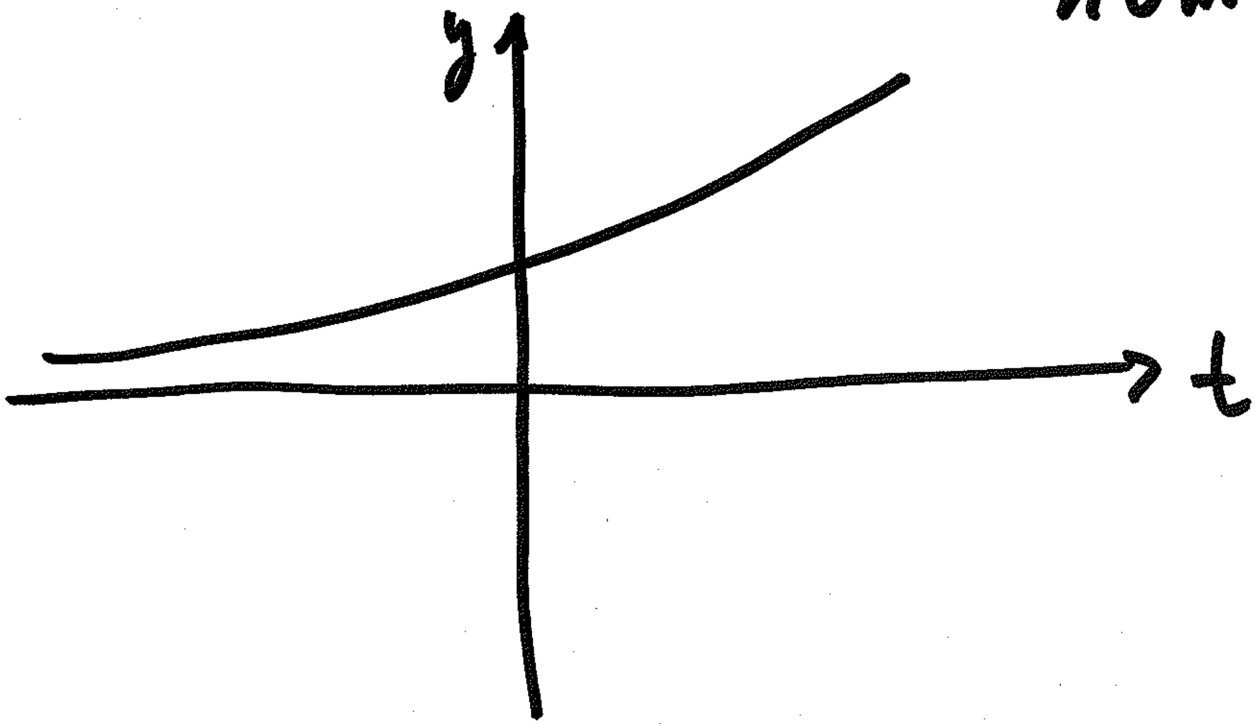
__Step__ 2) $1, t$ Basis of Nul T

__Step__ 1) $3t$ is in image of T

Warmup 2 Find all solns to diff eqn

$$y' = ay, \quad a \text{ any number}$$

Soln: $y = ce^{at}$, c any number



<u>Claim</u> These are all solns

<u>Pf</u>. Suppose y is some soln.

$$\frac{d}{dt}\left(\frac{y}{e^{at}}\right) = \frac{d}{dt}\left(ye^{-at}\right)$$

$$= y'e^{-at} + y(-a)e^{-at}$$

$$= aye^{-at} + y(-a)e^{-at} = 0$$

So $\frac{y}{e^{at}} = c$ constant!

We conclude $y = ce^{at}$, some c.

<u>Lin alg interpretation</u>: V vect sp of fns on $\mathbb{R}$

$T : V \longrightarrow V$ lin transf.

$T(y) = y'$

We found eigenvectors ce^{at} with eigenvalue a

A word about: $y = e^{at}$

1) Solves $y' = ay$ and any other soln is a scale

2) Satisfies $e^{(a+b)t} = e^{at} e^{bt}$

Power series expansion:

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \dots$$

We now know (remember) how to solve $y' + by = 0$

$\underline{\text{Solns}}$: $y = ce^{-bt}$

Let's solve

$$y'' + by' + cy = 0 \quad !$$

__Lin alg interpretation:__ V vect sp of fns on $\mathbb{R}$

$$T : V \longrightarrow V$$

$$T(y) = y'' + by' + cy$$

__Goal:__ find basis for $\mathrm{Nul}\, T$

Change notation

$$T = \left(\frac{d}{dt}\right)^2 + b\frac{d}{dt} + cI$$

<u>Def</u> <u>Auxiliary eqn</u> is

$$r^2 + br + c = 0$$

<u>Ex</u> $\boxed{y'' - 3y' - 4y = 0}$

Aux. eqn. $r^2 - 3r - 4 = 0$

Factor $(r-4)(r+1) = 0$

(order of factorization does not matter)

Return to diff eqn

$$\boxed{\left(\frac{d}{dt} - 4I\right)\left(\frac{d}{dt} + 1I\right) y = 0}$$

Now easy to solve!

Suffices to solve $\left(\frac{d}{dt} + 1 \cdot I\right)y = 0$

$$\left(\frac{d}{dt} - 4I\right)y = 0$$

We find solns $y = e^{-t}$

$$y = e^{4t}$$

General soln $y = c_1 e^{-t} + c_2 e^{4t}$

__Thm__ If aux. eqn. has two distinct real roots r_1, r_2 then general soln to $y'' + by' + cy = 0$ is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Lin __alg interpretation__ $e^{r_1 t}, e^{r_2 t}$ are basis for $\text{Nul}\, T$ where $T = \left(\frac{d}{dt}\right)^2 + b\frac{d}{dt} + cI$

__Exer__ Solve __Initial Value Problem (IVP)__

$$y'' + 2y' - y = 0$$

$$y(0) = 0, \quad y'(0) = -1$$

__Aux eqn__ $r^2 + 2r - 1 = 0$

__Roots__ $r_1 = -1 + \sqrt{2}$, $r_2 = -1 - \sqrt{2}$

__Gen. soln:__ $y = c_1 e^{(-1+\sqrt{2})t} + c_2 e^{(-1-\sqrt{2})t}$

$$y(0) = c_1 + c_2 = 0$$

$$y'(0) = (-1+\sqrt{2})c_1 + (-1-\sqrt{2})c_2 = -1$$

$$\begin{bmatrix} 1 & 1 \\ -1+\sqrt{2} & -1-\sqrt{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

<u>Soln</u>: $c_1 = \frac{-\sqrt{2}}{4}$ $\quad c_2 = \frac{\sqrt{2}}{4}$

<u>IVP soln</u>: $y = \frac{-\sqrt{2}}{4} e^{(-1+\sqrt{2})t} + \frac{\sqrt{2}}{4} e^{(-1-\sqrt{2})t}$

Thm If aux eqn has two distinct real roots, then can uniquely solve the IVP

$$y'' + by' + cy = 0$$
$$y(0) = Y_0, \quad y'(0) = Y_1$$

Rmk: Given gen soln $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ we're left to solve lin syst:

invertible since $r_1 \neq r_2$ $\rightarrow$ $$\begin{bmatrix} 1 & 1 \\ r_1 & r_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} Y_0 \\ Y_1 \end{bmatrix}$$

General story $y'' + by' + cy = 0$

$\rightsquigarrow$ $r^2 + br + c = 0$

Can always factor (though not always with distinct real roots)

$$(r - r_1)(r - r_2) = 0$$

Thm Can always uniquely solve

IVP $\quad y'' + by' + cy = 0$

$\qquad y(0) = Y_0, \quad y'(0) = Y_1$

Cases: 1) aux eqn has repeated real root r

Gen soln: $y = c_1 e^{rt} + c_2 t e^{rt}$

2) aux eqn has complex roots

$r_1 = \alpha + i\beta, \quad r_2 = \alpha - i\beta$

Gen soln: $y = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t)$

How to go from gen soln

$$y = c_1 y_1 + c_2 y_2$$

to soln of IVP where $y(0) = Y_0$

$y'(0) = Y_1$

<u>Note</u>: y_1, y_2 are lin indep in all cases considered

<u>Wronskian Lemma</u>: If y_1, y_2 solve

eqn $y'' + by' + cy = 0$ and

are lin indep, then the

vectors $\begin{bmatrix} y_1(0) \\ y_1'(0) \end{bmatrix}, \begin{bmatrix} y_2(0) \\ y_2'(0) \end{bmatrix}$

are also lin indep!

Thus can always solve uniquely lin syst

$$\begin{bmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} Y_0 \\ Y_1 \end{bmatrix}$$

lin indep cols means matrix is invertible