__Lecture 17__: Orthogonality!

__Today__: Review 11:30 - 1:30
736 Evans

__Thursday__ Midterm 2 through 6.3

__Friday__: No quiz.

Warmup! Is the following list orthogonal? orthonormal?

$$\underline{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix} \qquad \underline{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \underline{v}_3 = \begin{pmatrix} -1 \\ 1 \\ 3 \\ 1 \end{pmatrix}$$

$$\underline{v}_1 \cdot \underline{v}_3 = (1)(-1) + (2)(1) + (0)(3) + (-1)(1)$$
$$= 0$$

$\underline{v}_1$ is orthog to $\underline{v}_3$

$\underline{v}_2$ orthog to everything.

Orthogonal ✓ Not orthonormal $\|\underline{v}_2\| = 0$.

Warmup^2 a) Show following list is orthog. basis

$$\underline{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \quad \underline{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \underline{v}_3 = \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$$

Check: $\underline{v}_1 \cdot \underline{v}_2 = 0$, $\underline{v}_1 \cdot \underline{v}_3 = 0$

$\underline{v}_2 \cdot \underline{v}_3 = 0$

So orthog.

Thus lin indep. So span

Basis as well.

b) Find coeffs of vector

$$\underline{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

w.r.t. previous basis.

$$\underline{v} = x_1 \underline{v}_1 + x_2 \underline{v}_2 + x_3 \underline{v}_3$$

Traditional method:

$$A = \begin{bmatrix} | & | & | \\ \underline{v}_1 & \underline{v}_2 & \underline{v}_3 \\ | & | & | \end{bmatrix} \quad \text{Solve } A\underline{x} = \underline{v}$$

Easier method since basis is orthog.

$$\underline{v} \cdot \underline{v}_1 = x_1 \underline{v}_1 \cdot \underline{v}_1 + x_2 \underbrace{\underline{v}_2 \cdot \underline{v}_1}_{=0} + x_3 \underbrace{\underline{v}_3 \cdot \underline{v}_1}_{=0}$$

since basis is orthog

So $x_1 = \dfrac{\underline{v} \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1} = \dfrac{2}{5}$

Similarly we find

$$\underline{v} = \frac{\underline{v} \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1} \underline{v}_1 + \frac{\underline{v} \cdot \underline{v}_2}{\underline{v}_2 \cdot \underline{v}_2} \underline{v}_2 + \frac{\underline{v} \cdot \underline{v}_3}{\underline{v}_3 \cdot \underline{v}_3} \underline{v}_3$$

$$\underline{v} = \underset{\underset{x_1}{\uparrow}}{\frac{2}{5}}\,\underline{v}_1 + \underset{\underset{x_2}{\uparrow}}{\frac{2}{6}}\,\underline{v}_2 + \underset{\underset{x_3}{\uparrow}}{\frac{4}{30}}\,\underline{v}_3$$

<u>Warmup 3</u> Suppose $\mathcal{U}$ is an $n \times n$ matrix

When does $\mathcal{U}$ preserve inner prod?

Show $(\mathcal{U}\underline{x}) \cdot (\mathcal{U}\underline{y}) = \underline{x} \cdot \underline{y}$ all $\underline{x}, \underline{y}$

if and only if cols of $\mathcal{U}$ are orthonormal basis

<u>Proof</u>. Suppose $(\mathcal{U}\underline{x}) \cdot (\mathcal{U}\underline{y}) = \underline{x} \cdot \underline{y}$ all $\underline{x}, \underline{y}$.

In particular, let's take $\underline{x} = \underline{e}_i$
$\underline{y} = \underline{e}_j$

$\mathcal{U}\underline{e}_i = \underline{v}_i$ ith col

$\mathcal{U}\underline{e}_j = \underline{v}_j$ jth col

$$\underline{v}_i \cdot \underline{v}_j = (\mathcal{U}\underline{e}_i)\cdot(\mathcal{U}\underline{e}_j) = \underline{e}_i \cdot \underline{e}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Conclude cols of $\mathcal{U}$ are orthonormal so form orthonormal basis

<u>Exer</u>. prove converse part of assertion.

__Def__ A $n \times n$ matrix $\mathcal{U}$ with orthonormal cols is called an __orthogonal matrix__

__Ex__ 1) $\mathcal{U} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is orthogonal.

(rotation by $\frac{\pi}{2}$)

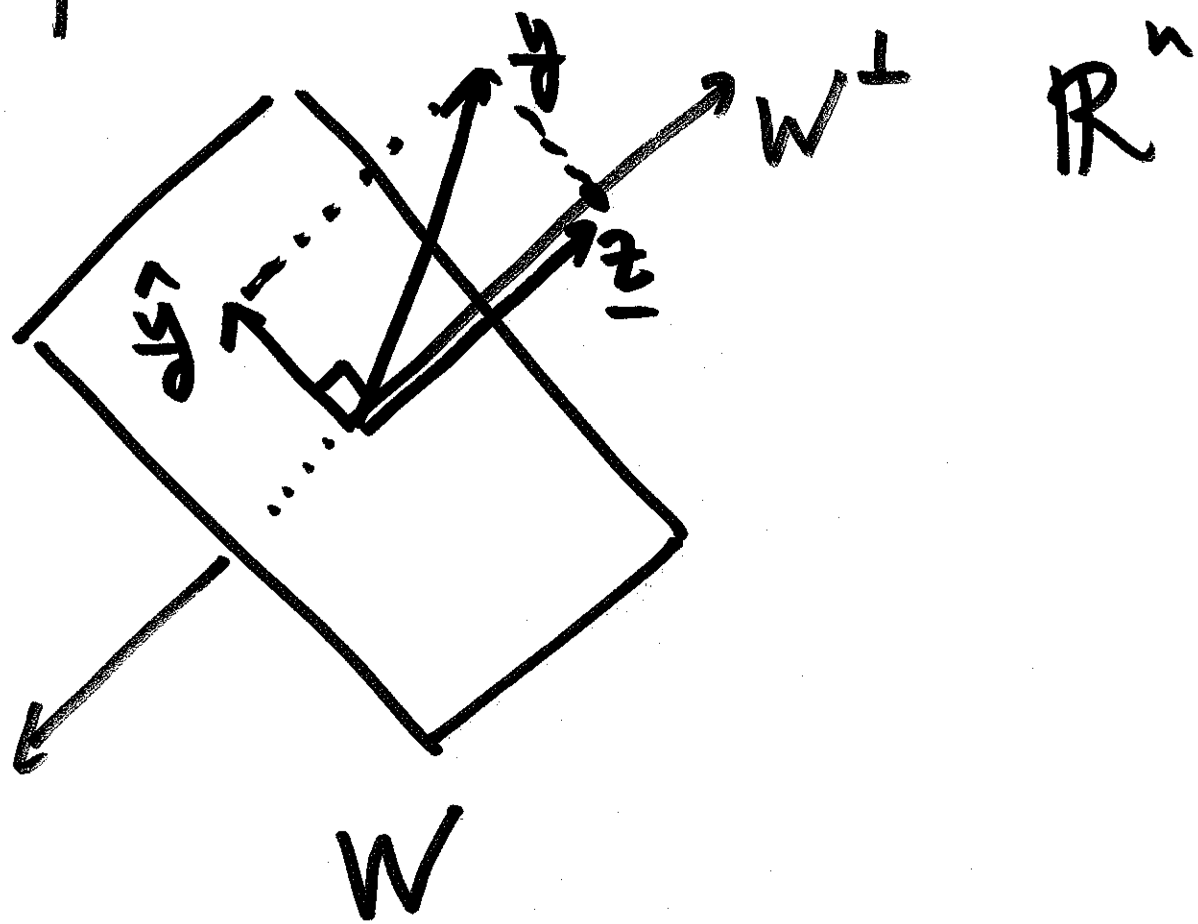
2) $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is __not__ orthogonal

__Remark__ U is orthogonal

if and only if

U invertible
and
$U^{-1} = U^T$

<u>Orthog projections</u> recall given subspace W in $\mathbb{R}^n$, we have subspace $W^\perp$ in $\mathbb{R}^n$



<u>Thm</u> Every vector $\underline{y}$ in $\mathbb{R}^n$ can be uniquely expressed as

$$\underline{y} = \hat{\underline{y}} + \underline{z}$$

where $\hat{\underline{y}}$ is in W, $\underline{z}$ is in $W^{\perp}$

Moreover, if $\underline{u}_1, \ldots, \underline{u}_k$ is an orthog basis for W, then

$$\hat{\underline{y}} = \frac{\underline{y} \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1} \cdot \underline{u}_1 + \cdots + \frac{\underline{y} \cdot \underline{u}_k}{\underline{u}_k \cdot \underline{u}_k} \underline{u}_k$$

Proof Choose orthog basis of W

$\underline{u}_1, \dots, \underline{u}_k$

Set $\hat{y} = \dfrac{\underline{y} \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1} \underline{u}_1 + \cdots + \dfrac{\underline{y} \cdot \underline{u}_k}{\underline{u}_k \cdot \underline{u}_k} \underline{u}_k$

Set $z = y - \hat{\underline{y}}$

Note $\hat{\underline{y}}$ in $\mathrm{Span}\{\underline{u}_1, \dots, \underline{u}_k\} = W$

Check: $\underline{z}$ in $W^\perp$

$\underline{z} \cdot \underline{u}_i = (\underline{y} - \hat{\underline{y}}) \cdot \underline{u}_i = \underline{y} \cdot \underline{u}_i - \underline{y} \cdot \underline{u}_i = 0.$

So far we have expressed

$$\underline{y} = \underbrace{\hat{\underline{y}}}_{\substack{\text{in } W \\ \text{given by} \\ \text{desired formula}}} + \underbrace{\underline{z}}_{\text{in } W^{\perp}}$$

<u>Exer</u> complete proof by showing $\hat{\underline{y}}$, $\underline{z}$ are unique so that $\underline{y} = \hat{\underline{y}} + \underline{z}$ with $\hat{\underline{y}}$ in W, $\underline{z}$ in $W^{\perp}$

__Def.__ Given subspace W in $\mathbb{R}^n$ and vector $\underline{y}$ in $\mathbb{R}^n$

$$\text{proj}_W(\underline{y}) = \hat{y}$$

orthog. projection of $\underline{y}$ onto W

We've seen if $\underline{u}_1, \dots, \underline{u}_k$ is orthog basis of W then

$$\text{proj}_W(\underline{y}) = \hat{y} = \frac{\underline{y} \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1} \underline{u}_1 + \cdots + \frac{\underline{y} \cdot \underline{u}_k}{\underline{u}_k \cdot \underline{u}_k} \cdot \underline{u}_k$$

Exer. Let $W = \text{span}\left\{ \underline{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ in $\mathbb{R}^3$

Find orthog projs. of $\underline{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ to $W, W^\perp$.

Soln: Find orthog basis of W

Apply formula for $\hat{y}$

Take $\underline{u}_1 = \underline{v}_1$

$$\left.\begin{aligned} \underline{u}_1 &= \underline{v}_1 \\ \underline{u}_2 &= \underline{v}_2 - \frac{\underline{v}_2 \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1} \underline{v}_1 \end{aligned}\right\} \text{orthog basis}$$

$$\underline{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\underline{u}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} \\ -\frac{7}{6} \\ \frac{4}{6} \end{bmatrix}$$

$$\hat{y} = \frac{y \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1}\underline{u}_1 + \frac{y \cdot \underline{u}_2}{\underline{u}_2 \cdot \underline{u}_2} \cdot \underline{u}_2$$

$$= \frac{-2}{6}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \frac{-\frac{10}{6}}{\frac{66}{36}}\begin{bmatrix} -\frac{1}{6} \\ -\frac{7}{6} \\ \frac{4}{6} \end{bmatrix} = \ldots$$

Take $\text{proj}_{W^\perp}(\underline{y}) = \underline{y} - \text{proj}_W(\underline{y})$

$$= y - \hat{y} = \ldots$$