Lecture 12: Changes of Basis

"There is no absolute point of view from which real and ideal can be finally separated and labelled." T.S. Eliot

Today: Off. Hrs 12-2pm, 736 Evans
(New time!)

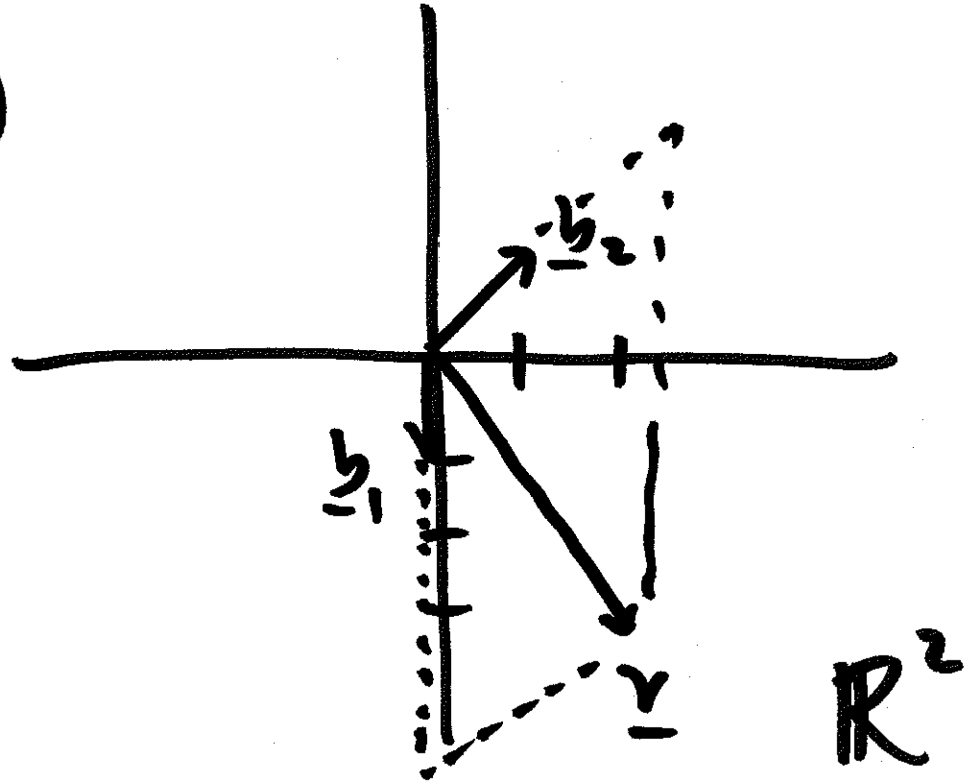
Fri Quiz thru 4.6

<u>Warmup</u> 1) Find coord vector $[\underline{v}]_B$ of

$\underline{v} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ w.r.t. $B = \left\{ \underline{b}_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \underline{b}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

Solve in 2 ways...

1)



<u>Want</u> Find a_1, a_2 s.t.

$\underline{v} = a_1 \underline{b}_1 + a_2 \underline{b}_2$

So $\begin{bmatrix} 2 \\ -3 \end{bmatrix} = a_1 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Aug matrix $\left[\begin{array}{cc|c} 0 & 1 & 2 \\ -1 & 1 & -3 \end{array}\right]$

Solve with $a_2 = 2$, $a_1 = 5$

$$[\underline{v}]_B = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

<u>Solution 2)</u> P_B = matrix that transforms B corrds to usual coords

This means $\underline{v} = P_B [\underline{v}]_B$

We know $P_B = \begin{bmatrix} | & | \\ \underline{b}_1 & \underline{b}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$

Now invert $P_B^{-1} \underline{v} = [\underline{v}]_B$

$$P_B^{-1} \underline{v} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

<u>More</u> equitable notation:

$$\mathcal{E} = \left\{ \underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \underline{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$$

Std basis

If $\underline{v} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ is a vector,

then $[\underline{v}]_{\mathcal{E}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

Also, we'll write $\underset{\mathcal{E} \leftarrow \mathcal{B}}{P} = P_{\mathcal{B}}$

So $[\underline{v}]_{\mathcal{E}} = \underset{\mathcal{E} \leftarrow \mathcal{B}}{P} [\underline{v}]_{\mathcal{B}}$

Note $\underset{\mathcal{B} \leftarrow \mathcal{E}}{P} = \left(\underset{\mathcal{E} \leftarrow \mathcal{B}}{P}\right)^{-1} = P_{\mathcal{B}}^{-1}$

__Warmup 2__ Find basis $\mathcal{B} = \{\underline{b}_1, \underline{b}_2\}$ of $\mathbb{R}^2$

so that $\left[\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]_B = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\left[\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

this means $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = (-1)\cdot\underline{b}_1 + (2)\underline{b}_2$

We know the matrix equation:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underset{\mathcal{E} \leftarrow B}{P} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

Solve now

$$\underset{\mathcal{E}\leftarrow B}{P} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}^{-1} = \frac{1}{-1}\begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$$

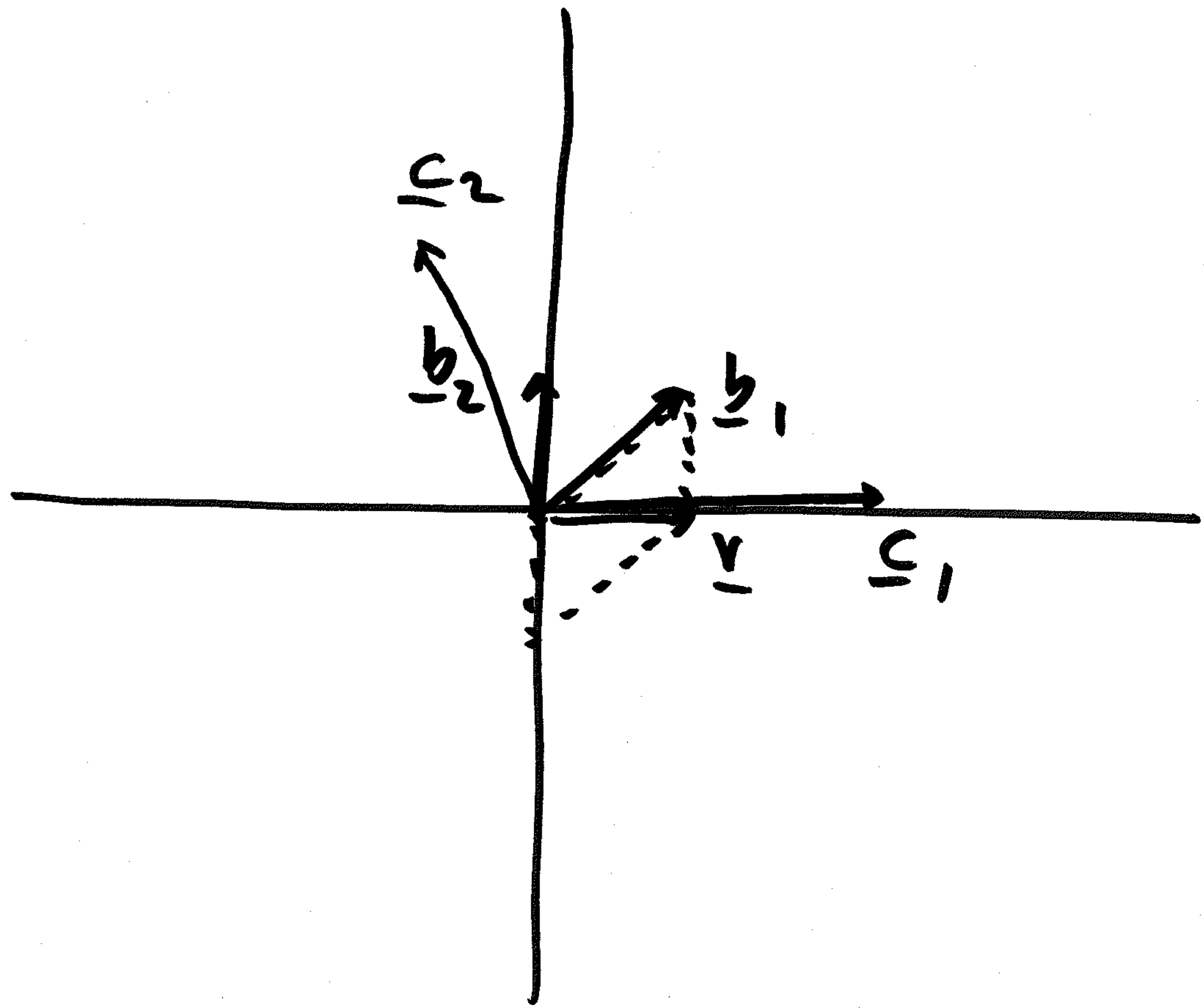
$$= \begin{array}{cc} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \\ \uparrow \quad\;\; \uparrow \\ \underline{b}_1 \quad \underline{b}_2 \end{array}$$

Soln: $B = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

What if we have two bases and neither is $\mathcal{E}$? Relation between coords wrt each?

<u>Exer</u> Suppose $\underline{v}$ in $\mathbb{R}^2$ has coords $[\underline{v}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ w.r.t. $B = \left\{ \underline{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \underline{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

Find $[\underline{v}]_C$ w.r.t. $C = \left\{ \underline{c}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \underline{c}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$



$\underline{c}_2$
$\underline{b}_2$
$\underline{b}_1$
$\underline{v}$
$\underline{c}_1$

Solve in 2 ways: 1) Directly
2) Via std basis $\mathcal{E}$

1) We know $\underline{v} = 1 \cdot \underline{b}_1 + (-1)\, \underline{b}_2$

$$= 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(So $\underline{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ but this is in std coords and we want $[\underline{v}]_C$!)

Go back to $\underline{v} = 1 \cdot \underline{b}_1 + (-1)\,\underline{b}_2$

Take coord wrt C everywhere in eqn

$$[\underline{v}]_C = 1\,[\underline{b}_1]_C + (-1)\,[\underline{b}_2]_C$$

So
$$[\underline{v}]_C = \underbrace{\begin{bmatrix} | & | \\ [\underline{b}_1]_C & [\underline{b}_2]_C \\ | & | \end{bmatrix}}_{\text{Call this } \underset{C \leftarrow B}{P}} [\underline{v}]_B$$

$$\boxed{[\underline{v}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

To solve problem, we know $[\underline{v}_B] = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

We need to find $\underset{C\leftarrow B}{P}$.

To find $\underset{C\leftarrow B}{P} = \begin{bmatrix} [\underline{b}_1]_C & [\underline{b}_2]_C \end{bmatrix}$ we must

$$= \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$$

solve

$\underline{b}_1 = x_1 \underline{c}_1 + x_2 \underline{c}_2 \qquad \underline{b}_2 = y_1 \underline{c}_1 + y_2 \underline{c}_2$

So $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = y_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + y_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Aug matrix

$$\begin{array}{c} \left[\begin{array}{cc:cc} 2 & -1 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{array}\right] \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \underline{c}_1 & \underline{c}_2 & \underline{b}_1 & \underline{b}_2 \end{array} \end{array}$$

RREF
$$\rightsquigarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

So $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix}$ $\quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix}$

Thus $\underset{C \leftarrow B}{P} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

So $[\underline{v}]_C = \underset{C \leftarrow B}{P} [\underline{v}]_B = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$

Thm $\underset{C \leftarrow B}{P} = \begin{bmatrix} | & & | \\ [\underline{b}_1]_C & \cdots & [\underline{b}_n]_C \\ | & & | \end{bmatrix}$

$$\underset{B \leftarrow C}{P} = \begin{bmatrix} | & & | \\ [\underline{c}_1]_B & \cdots & [\underline{c}_n]_B \\ | & & | \end{bmatrix}$$

$$= \left(\underset{C \leftarrow B}{P} \right)^{-1}$$

Special case $C = \mathcal{E}$ std basis

$$P_B = \underset{\mathcal{E} \leftarrow B}{P} = [\, \underline{b}_1 \; \cdots \; \underline{b}_n \,]$$

<u>Method 2</u>) We want to find $\underset{C \leftarrow B}{P}$

We can pass through std basis!

$$\underset{C \leftarrow B}{P} = \left(\underset{C \leftarrow \mathcal{E}}{P}\right)\left(\underset{\mathcal{E} \leftarrow B}{P}\right)$$ matrix mult.

$$= P_C^{-1} \, P_B$$

$$= \begin{bmatrix} | & & | \\ \underline{c}_1 & \cdots & \underline{c}_n \\ | & & | \end{bmatrix}^{-1} \begin{bmatrix} | & & | \\ \underline{b}_1 & \cdots & \underline{b}_n \\ | & & | \end{bmatrix}$$

Back to problem

$$\underset{C \leftarrow B}{P} = P_C^{-1} P_B = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$