Good Morning! Welcome to
Lecture 11: abstraction continued!

Today: Off Hrs/Review: 12-2pm
736 Evans

Website: GSI Off Hrs/Reviews
Practice Midterm

Fri Quiz through 4.4

Clarification: polynomial $1 = \underset{a_0}{\underset{\uparrow}{1}} + \underset{a_1}{\underset{\uparrow}{0}} \cdot x + \cdots + \underset{a_n}{\underset{\uparrow}{0}} \cdot x^n$

Warmup! $V = \mathbb{R}^3$ Find vector $\underline{v}$ in $\mathbb{R}^3$ with coords $[\underline{v}]_B = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ w.r.t. basis

$$B = \left\{ \underline{b}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \underline{b}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \underline{b}_3 = \begin{bmatrix} 7 \\ 1 \\ 3 \end{bmatrix} \right\}$$

Soln: $$\underline{v} = 1 \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} 7 \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -17 \\ -2 \\ -10 \end{bmatrix}$$

Warmup 2 $V = \mathbb{P}_2$ Find coords of $x - 2x^2 = \underline{v}$ w.r.t. basis $\mathcal{B} = \{ \underline{b}_1 = 1 + x^2, \underline{b}_2 = 1 + x, \underline{b}_3 = 1 + x + x^2 \}$

Soln Find $[\underline{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ so that

$$\underline{v} = c_1 \underline{b}_1 + c_2 \underline{b}_2 + c_3 \underline{b}_3$$

$$x - 2x^2 = c_1(1 + x^2) + c_2(1 + x) + c_3(1 + x + x^2)$$

Lin syst:

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_2 + c_3 &= 1 \\ c_1 \qquad + c_3 &= -2 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -2 \end{array}\right]$$ aug. matrix

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

<u>Soln</u>

$$[\underline{v}]_B = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

__Thm__ V vect sp, $B = \{\underline{b}_1, \ldots, \underline{b}_n\}$ basis

Then we have two inverse lin transfs

$$T_B : V \longrightarrow \mathbb{R}^n$$

$$T_B(\underline{v}) = [\underline{v}]_B$$

$$T_B^{-1} : \mathbb{R}^n \longrightarrow V$$

$$T_B^{-1}\left(\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}\right) = c_1\underline{b}_1 + \cdots + c_n\underline{b}_n$$

Def. We say a lin transf $T: V \to W$ is an isomorphism if there is a lin transf $S: W \to V$ so that

$$TS = I_W \qquad ST = I_V$$

If so, we say $T^{-1} = S$ is the inverse of T.

Remark: Same criteria for invt-ble matrix hold here: $\operatorname{Nul} T = \{0\}$ + $\operatorname{Range} T = W$
(injective) (surjective)

Exer Find an isomorphism

$$T : \mathbb{P}_n \longrightarrow \mathbb{R}^{n+1}$$

Soln: Choose std basis $B = \{1, x, x^2, \dots, x^n\}$ for $\mathbb{P}_n$

Take $T = T_B$ coord. map

$$T_B(p) = [p]_B = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$p(x) = a_0 + a_1 x + \cdots + a_n x^n$

__Thm__ Suppose V vect sp with $\mathcal{B} = \{\underline{b}_1, \ldots, \underline{b}_n\}$ basis

Then V is isomorphic to $\mathbb{R}^n$

__Proof__. Take $T = T_{\mathcal{B}} : V \longrightarrow \mathbb{R}^n$

$$T(\underline{v}) = [\underline{v}]_{\mathcal{B}}$$

Check that this has an inverse! ∎

<u>Thm</u> V vect sp, $\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$ basis

1) $\underline{v}_1, \dots, \underline{v}_k$ lin indep in V $\Rightarrow$ $k \leq n$

2) $\underline{v}_1, \dots, \underline{v}_k$ spans V $\Rightarrow$ $k \geq n$

Proof of 1) (Leave 2) as exercise)

Suppose $\underline{v}_1, \dots, \underline{v}_k$ lin indep but $k > n$

Apply $T_B : V \to \mathbb{R}^n$ (isomorphism)

We find $T_B(\underline{v}_1), \dots, T_B(\underline{v}_k)$ in $\mathbb{R}^n$

must be lin dep since $k > n$.

(**Why?** $n\left\{\underbrace{\begin{bmatrix} | & & | \\ T_B(\underline{v}_1) & \cdots & T_B(\underline{v}_k) \\ | & & | \end{bmatrix}}_{k}\right.$ $k > n$ $\Rightarrow$ there is a non-pivot col)

So there exist $c_1, \ldots, c_k$ <u>not</u> all 0

so that

$$c_1 T_B(\underline{v}_1) + \cdots + c_k T_B(\underline{v}_k) = \underline{0}$$

So since T_B is linear, we have

$$T_B(\underbrace{c_1 \underline{v}_1 + \cdots + c_k \underline{v}_k}_{\text{is in } \mathrm{Nul}\, T_B}) = \underline{0}$$

But T_B is an isom so $\mathrm{Nul}\, T_B = \{\underline{0}\}$

So $c_1\underline{v}_1 + \cdots + c_k\underline{v}_k = \underline{0}$

But we know $c_1, \ldots, c_k$ are <u>not</u> all 0

So $\underline{v}_1, \ldots, \underline{v}_k$ lin dep $\lightning$ Contradiction!

$\blacksquare$

Exer Find basis for Nul T where

$$T : \mathbb{P}_2 \longrightarrow \mathbb{R}$$

$$T(p) = \frac{dp}{dx}(1) + p(0)$$

Remark: By Theorem, since Nul T is a subspace of $\mathbb{P}_2$, any basis of Nul T is lin indep in $\mathbb{P}_2$ so must have ≤ 3 elements.

<u>Soln</u> $p(x) = a_0 + a_1 x + a_2 x^2$

$$T(p) = \left(a_1 + 2a_2 \cdot (1)\right) + a_0$$

$$= a_0 + a_1 + 2a_2$$

<u>Solve lin syst</u>: $\begin{array}{cccc} \left[\,1\right. & 1 & 2 & \left.\vdots\ 0\,\right] \\ \uparrow & \uparrow & \uparrow & \\ \text{pivot col} & \text{free} & \text{cols} & \end{array}$

<u>Basis</u>: $-1 + x,\ -2 + x^2$

__Def.__ V vect sp. We define __dimension__ $\dim V$ to be the size of a basis.

(If there is not a fin. basis,
then we say $\dim V = \infty$)

__Ex.__ 1) $\dim \mathbb{R}^n = n$

2) $\dim \mathbb{P}_n = n+1$

(3) $\dim \mathbb{S} = \infty$)

__Thm__ If V vect sp has two bases $\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$, $\mathcal{C} = \{\underline{c}_1, \dots, \underline{c}_k\}$

then $n = k$.

So $\dim V$ is well-defined
(not ambiguous)

__Proof of Theorem__ Suppose say $k > n$

But $\underline{c}_1, \dots, \underline{c}_k$ are lin indep since $\mathcal{C}$ is basis. So by prev. Thm $k \le n$ $\lightning$

__Recall__ If V vect sp and $\underline{v}_1, \dots, \underline{v}_k$ is a spanning collection, then we can extract a subcollection that is a basis.

__Thm__ Suppose V is a fin. dim. vect sp.
If $\underline{v}_1, \dots, \underline{v}_k$ is lin indep collection then we can extend it to a basis

$$B = \{\underline{v}_1, \dots, \underline{v}_k, \underline{v}_{k+1}, \dots, \underline{v}_n\}$$

where $n = \dim V$.

Thm Suppose V is a fin diml. vect sp.
Suppose H is a subspace.

Then $\dim H \le \dim V$