(1) Why take Math 54?

It's fun!

It's easy! <u>Linear</u>!

It's powerful! Solve equations by algorithms!

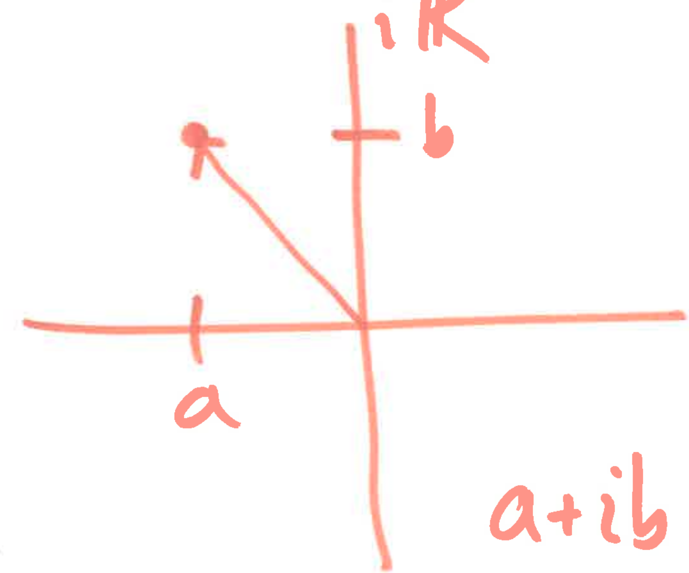
It's sexy!

Don't take Math 54 if you do not want to work <u>consistently</u>!

② <u>Notation</u>

$\mathbb{R}$ = real numbers

$\mathbb{C}$ = complex numbers



In this course all <u>numbers</u> will be real or complex.

$a, b, c, \ldots$ <u>numbers</u>

$x, y, z, \ldots$ <u>variables</u>

Definition A linear equation in n vars. is an eqn that can be put in form:

variables

$$a_1 x_1 + \cdots + a_n x_n = b$$

numbers

Exer. Is the following a lin. eq.?

$$3(x_1 - 2x_2) = 5(4 - x_3)$$

Yes!

$$3(x_1^2 - 2x_2) = 7$$

No!

4 <u>Def</u>. A <u>system of lin eqns</u> / <u>linear system</u> in n variables is a finite collection of lin. eqns.

Can be put in form:

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Ex

$$2x_1 - 3x_2 + 4x_3 = 8$$
$$x_1 + 0 \cdot x_2 + 2x_3 = 7$$

$m = 2 \qquad n = 3$

Ex

$$x_1 + x_2 - 3x_3 + 4x_4 - x_5 = 2$$

$x_1 = 7$	More solns expected when
$2x_1 = 3$	
$4x_2 = 0$	$n > m$.

Def. <u>Solution set</u> of a lin. syst. is the set of all $(s_1, \ldots, s_n)$ that solve all the equations.

n-tuples of numbers

Exer Find soln sets:

1) $x_1 - x_2 = 1$
   $2x_1 \qquad = 0$

   Soln set $= \{(0, -1)\}$

2) $x_1 - 2x_2 = 0$
   $2x_1 - 4x_2 = 1$

   Soln set $= \emptyset$

3) $x_1 - x_2 + x_3 = 0$
   $x_2 + x_3 = 0$

   Soln set $= \{(2s, s, -s)\}$

   s any number

Three possibilities:

1) No soln – inconsistent

2) Unique soln

3) Infinitely many soln

$\left.\begin{array}{l}\text{2) Unique soln}\\\text{3) Infinitely many soln}\end{array}\right\}$ – consistent

Ex For what c does following lin sys. satisfy each of above:

$$\left.\begin{aligned} x_1 + cx_2 &= 1 \\ 2x_1 + 2x_2 &= 0 \end{aligned}\right\} \Rightarrow (2-2c)x_2 = -2$$

$$\Rightarrow x_2 = \frac{-2}{2-2c}$$

$\Rightarrow$ $\underline{c=1}$ inconsistent, $c \neq 1$ unique soln $x_1 = \frac{2}{2-2c}$

Def Two lin sys are <u>equivalent</u> if they have the same soln set.

Exer For what c are the following equiv?

(1) $x_1 - cx_2 = 0$
$x_1 + x_3 = 0$

(2) $2x_1 - x_2 + x_3 = 0$
$x_2 + x_3 = 0$

Better (1)
$$x_1 - cx_2 = 0$$
$$x_1 + x_3 = 0$$

$c = -1$?
(1) $(-1, 1, 1) \neq (-s, -s, s)$ (2)
$s = 1$ any s.

(1) $x_1 = cx_2$
$x_1 = -x_3$
Soln set $= \{(cs, s, -cs)\}$
~~[illegible]~~ s number

(2) $x_2 = -x_3$
$x_1 = -x_3$
Soln set $= \{(-s, -s, s)\}$
s number

$\boxed{c = 1}$

Matrix Notation

Ex $\begin{aligned} x_1 - 2x_2 + x_3 &= 7 \\ 4x_1 \qquad - x_3 &= 0 \end{aligned} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 7 \\ 4 & 0 & -1 & 0 \end{array}\right]$

m eq^{ns} in n vars $\rightsquigarrow$ $m\left\{\underbrace{\left[\begin{array}{ccc|c} & & & \\ & & & \end{array}\right]}_{n+1}\right.$

Matrix: m rows
$n+1$ columns

Called: Augmented matrix of lin sys.

<u>Now</u>: Solve Lin Systs!

<u>Strategy</u>: Change to <u>easier</u> sys using:

R1) Add multiple of any row to any other

R2) Interchange rows.

R3) Scale a row. by nonzero number.

<u>Observation</u> R1, R2. R3 take you to equiv systs.

Goal: Row Echelon Form (REF)

$$\begin{bmatrix} 0 \cdots 0 & \blacksquare & * & \cdots & \cdots & \cdots & * \\ 0 \cdots\cdots 0 & \cdots & 0 & \blacksquare & * & \cdots & * \\ 0 & \cdots & \cdots & 0 & \blacksquare & * \cdots & * \\ \vdots & & & & & \ddots & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

$\blacksquare \neq 0 \longleftarrow$ pivot entries / leading terms

$*$ = any number

Ex: $\begin{bmatrix} 0 & 2 & 3 & 9 & 1 \\ 0 & 0 & 0 & -1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Reduced Row Echelon Form

$$\begin{bmatrix} 0 \cdots 0 & 1 & * & * & 0 & * & \cdots & 0 & * & \cdots & \vdots \\ 0 & 0 & \cdots & & 1 & * & \cdots & 0 & * & \cdots & \vdots \\ \vdots & \vdots & & & 0 & 0 \cdots 0 & & 1 & * & \cdots & \vdots \\ \vdots & \vdots & & & \vdots & & & & & & \vdots \\ 0 & \cdots & & & & & & & & \cdots & 0 \end{bmatrix}$$

$\boxtimes$ became $=$ 1.

Above $\boxtimes$ became $= 0$.

Ex $\begin{pmatrix} 0 & \underline{1} & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

<u>Easy</u> to solve lin systs in RREF.

<u>Exer</u> Solve
$$\begin{array}{c} \begin{array}{cccccc} 1 & & 3 & & & \end{array} \\ \left[\begin{array}{cccc|c} \boxed{1} & 2 & 0 & 2 & 3 \\ 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

<u>Pivot vars</u> x_1, x_3 correspond to pivot cols.

<u>Free vars</u> x_2, x_4 others.

<u>Soln set</u>: x_2, x_4 any numbers.

$x_1 = -2x_2 - 2x_4 + 3$

$x_3 = -x_4$

Conclusion 1) Lin sys is inconsistent if and only if in REF there is a row of form $[0 \; \cdots \; 0 \,\vdots\, b \neq 0]$

2) Else: We can specify free vars arbitrarily and uniquely solve for pivot vars.

__Exer__ For what c, d is below sys inconsistent?

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & c & d \end{array}\right]$$

$$\overset{R1}{\rightsquigarrow} \left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & c+6 & d-2 \end{array}\right]$$

Inconsistent $\iff c = -6,\ d \neq 2$.