Loops and Traces

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Overview

Broad question:

What is harmonic analysis of higher categorical symmetries?
(... group actions on linear categories...)

More specific question:

What can it tell us about classical symmetries?
(... group actions on vector spaces...)

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Outline

1. Motivation: Geometric Langlands via topological field theory
2. Warmup: traces of multiplication operators
3. Traces of Hecke operators, character sheaves

Automorphic moduli

G complex reductive group
C smooth projective complex curve

$\mathrm{Bun}_G(C)$ moduli of principal G-bundles over C

Adelic/locally symmetric description

$$\mathrm{Bun}_G(C) \quad \simeq \quad G(C_{rat}) \backslash \prod\nolimits'_c G(\mathcal{K}_c) / \prod\nolimits_c G(\mathcal{O}_c)$$

Linearization

Geometric representation theory:

Focus on equations of harmonic analysis
rather than their solutions in specific function spaces

Local operators

multiplication and differentiation $\rightsquigarrow$ $\mathcal{D}$-modules

Global operators

pullback and integration $\rightsquigarrow$ complexes of $\mathcal{D}$-modules

Automorphic derived category:

$$\mathcal{D}(\mathrm{Bun}_G(C)) \quad \simeq \quad \mathcal{D}(G(C_{rat})\backslash \textstyle\prod_c' G(\mathcal{K}_c)/\prod_c G(\mathcal{O}_c))$$

Quantum Field Theory interpretation

Where in nature do we find categories?
Hint: linearity results from superposition...

1d quantum field theory (quantum mechanics, particles)

Evolution occurs along 1-manifolds (line segments) and possible states at the end points form a vector space.

Symmetries: traditional harmonic analysis of linear operators.

2d quantum field theory (strings)

Evolution occurs along 2-manifolds (surfaces) and possible boundary conditions form a linear category.

Symmetries: categorical harmonic analysis of linear endofunctors.

Mirror symmetry for 2d field theories

When do we understand a category?

Spectral description: quasicoherent sheaves (or variants) on a scheme or stack.

Rough Conjecture (Geometric Langlands)

$$\mathcal{D}(\mathrm{Bun}_G(C)) \overset{?}{\longleftrightarrow} \mathcal{Q}(\mathrm{Conn}_{G^\vee}(C))$$

Quantum version of duality of Hitchin systems

$$\begin{tikzcd}
T^* \mathrm{Bun}_G(C) \arrow[d] & T^* \mathrm{Bun}_{G^\vee}(C) \arrow[d] \\
\mathcal{A}_{\mathfrak{g}}(C) \arrow[r, equal] & \mathcal{A}_{\mathfrak{g}^\vee}(C)
\end{tikzcd}$$

Symplectic deformation   Complex deformation

Geometric Fourier Transform (T-duality) with singularities.

Toy example: mirror symmetry with symmetries

Easiest example of torus fibration

$$T^*S^1 \simeq S^1 \times \mathbb{R} \longrightarrow \mathbb{R}$$

Automorphic side

Constructible sheaves on S^1 with one marked point $s \in S^1$.
Quantum symplectic objects on T^*S^1 via microlocalization.
Hecke operators: Dehn twists of $T^*S^1 \to \mathbb{R}$.

Spectral side

Coherent sheaves on $\mathbb{P}^1 = \{T^*S^1$ with ends collapsed$\}$.
Quantum complex objects, linearization of subvarieties.
Multiplication operators: tensor with line bundles.

Matching: both sides are modules over quiver $\bullet \rightrightarrows \bullet$

$$\text{skyscraper } \mathbb{C}_s \text{ at marked point} \longleftrightarrow \text{structure sheaf } \mathcal{O}_{\mathbb{P}^1}$$
$$\text{constant sheaf on complement } \mathbb{C}_{S^1 \setminus s} \longleftrightarrow \text{line bundle } \mathcal{O}_{\mathbb{P}^1}(-1)$$

Symmetries from 4d field theory

Symmetries compatible with Hitchin systems

labelled 3d cobordisms $\rightsquigarrow$ linear endofunctors

Monoidal category of endofunctors with extra commutativity: multiplication of operators indexed by points of $\mathbb{R}^3$.

Automorphic side: global integral operators

Spherical Hecke category $\mathcal{H}^{sph} = \mathcal{D}(G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O}))$

$$\mathcal{H}^{sph} \curvearrowright \mathcal{D}(\mathrm{Bun}_G(C)) \text{ via Hecke correspondences}$$

Satake Equivalence (underived version)

$$\mathcal{H}_G^{sph} \simeq \mathrm{Rep}(G^\vee)$$

Spectral side: local multiplication operators

$$\mathrm{Rep}(G^\vee) \curvearrowright \mathcal{Q}(\mathrm{Conn}_{G^\vee}(C)) \text{ via pullbacks}$$

Tame ramification

Automorphic side: modifications of bundles with flags
Affine Hecke category: $\mathcal{H}^{aff} \simeq \mathcal{D}(I \backslash G(K)/I)$

Monoidal category with central functor

$$\mathcal{H}^{sph} \to \mathcal{H}^{aff}$$

Spectral description: due to Kazhdan-Lusztig, Bezrukavnikov
Modifications of connections with poles and compatible flags

Geometric characters

Goal: Develop homotopical theory of traces of Hecke operators.

Guide: Compactification in quantum field theory

correlations on product with circle $M \rightsquigarrow M \times S^1$

Inspiration: Trace Formulas from Frobenius to Weyl to Lefschetz to Harish Chandra to Arthur-Selberg

Characters of representations $\leftrightsquigarrow$ Geometry of fixed points

Payoffs:

1. Rich invariants/structures of Hecke operators.
2. Langlands duality for traces and characters.
3. Character sheaves for loop groups.
4. Elliptic representation theory.

Abstract starting point

Categorical harmonic analysis

monoidal category $\mathcal{A}$ and $\mathcal{A}$-module category $\mathcal{M}$

Working context

1. homotopical algebra: derived operations, capture universal constructions not only their classical results.
2. stable ∞-infinity categories: categorical vector spaces, modern notion of triangulated category.

Challenge

1. Calculate spectrum of $\mathcal{A}$ in the form of a symmetric monoidal category of sheaves on an algebraic stack.
2. Decompose $\mathcal{M}$ over the spectrum in the form of a module category with known fibers over the spectrum.

What is the dimension of a category?

Recall: the dimension of a dualizable $\mathbb{C}$-vector space V is given by the composition

$$\dim V : \mathbb{C} \to \mathrm{End}(V) \simeq V^* \otimes V \to \mathbb{C}$$

Suppose $\mathcal{C}$ is a dualizable $\mathbb{C}$-linear category.
Dualizability is context-dependent but quite general.
For the moment, let us assume $\mathcal{C}$ is small, so that its dual is $\mathcal{C}^{op}$.

Then the dimension of $\mathcal{C}$ is given by the composition

$$\dim \mathcal{C} : \mathit{Vect}_{fd} \xrightarrow{\mathrm{id}_{\mathcal{C}}} \mathrm{End}(\mathcal{C}) \simeq \mathcal{C}^{op} \otimes \mathcal{C} \xrightarrow{\mathrm{Hom}_{\mathcal{C}}} \mathit{Vect}_{fd}$$

We can package $\dim \mathcal{C}$ into the data of a single vector space by taking its value on the one-dimensional vector space $\mathbb{C}$.

Examples of dimensions of categories

Another name for $\dim \mathcal{C}$ is Hochschild homology $HH(\mathcal{C})$. In particular, if $\mathcal{C} = A$-mod, then

$$\dim \mathcal{C} = A \otimes_{A \otimes A^{op}} A$$

Exercises:

1. $\mathcal{C} = Vect$, then $\dim \mathcal{C} \simeq \mathbb{C}$.
2. $\mathcal{C} = \mathbb{C}[G]$-mod, for G finite, then $\dim \mathcal{C} \simeq \mathbb{C}[G/G]$.

Algebraic geometry: X scheme

1. $\mathcal{C} = \mathcal{Q}(X)$, then $\dim \mathcal{C} \simeq H^*_{Dol}(X)$.
2. $\mathcal{C} = \mathcal{D}(X)$, then $\dim \mathcal{C} \simeq H^*_{dR}(X)$.

Representation theory: G affine

1. $\mathcal{C} = \mathcal{Q}(BG)$, then $\dim \mathcal{C} \simeq H^*_{Dol}(G/G)$.
2. $\mathcal{C} = \mathcal{D}(BG)$, then $\dim \mathcal{C} \simeq H^*_{dR}(G/G)$.

Unifying perspective: loop spaces

The loop space of a scheme or stack X is the derived intersection

$$LX = \Delta_X \times_{X \times X} \Delta_X$$

Locally functions on LX are calculated by a tor-complex.

LX is the "universal nonlinear dimension" of X.

Examples:

1. X reasonable scheme, $LX \simeq \mathrm{Spec}_{\mathcal{O}_X} \Omega_X^{-\bullet}$.
2. $X = BG$, for G affine, $LX \simeq G/G$.
3. $X = Y/G$, as above, loop space is derived inertia stack

$$LX \simeq \{(x, g) \in X \times G \,|\, gx = x\}/G$$

Proposition

If X is a reasonable stack, then

$$\dim \mathcal{Q}(X) \simeq H^*_{Dol}(LX) \qquad \dim \mathcal{D}(X) \simeq H^*_{dR}(LX)$$

Functoriality of dimension

A continuous functor $\ell : \mathcal{C} \to \mathcal{D}$ induces a morphism

$$\dim \ell : \dim \mathcal{C} \to \dim \mathcal{D}$$

In particular, a compact object $c \in \mathcal{C}$ gives a continuous functor $\ell_c : \mathit{Vect} \to \mathcal{C}$ and thus a morphism

$$\dim \ell_c : \mathbb{C} \to \dim \mathcal{C}$$

We can package $\dim \ell_c$ into the data of a single vector

$$\dim c \in \dim \mathcal{C}$$

by taking its value on $1 \in \mathbb{C}$.

Proposition (Abstract Grothendieck-Riemann-Roch)

$$\mathcal{B} \xrightarrow{k} \mathcal{C} \xrightarrow{\ell} \mathcal{D} \implies \dim(\ell \circ k) \simeq \dim \ell \circ \dim k$$

Functoriality via loops

A morphism $f : X \to Y$ of schemes or stacks induces a morphism of loop spaces

$$Lf : LX \to LY$$

Theorem

In any linearization (with proper adjunctions and base change), functoriality of dim *is given by integration along* Lf.

Example (Nonlinear character formula)

Given $f : X/G \to BG$, fibers of induced loop map

$$L(X/G) \to L(BG) \simeq G/G$$

are derived fixed points $X^g = \{x \in X \,|\, gx = x\}$.
Induction: If $X = \Gamma\backslash G$, then $X/G = \Gamma\backslash G/G \simeq B\Gamma$, and we obtain

$$\Gamma/\Gamma \to G/G.$$

Applications: fixed point formulas

The following are formal consequences of the above observations. They hold in great generality given sufficient dualizability.

1. Frobenius character formula for induced and permutation representations.
2. Lefschetz trace formula for equivariant $\mathcal{D}$-modules or constructible complexes.
3. Atiyah-Bott fixed point formula for equivariant quasicoherent sheaves (or variants).

Peculiar traditional formulas reflect derived nature of loops.

Example: Weyl character formula

Start with basic Eisenstein induction diagram

$$BT \xleftarrow{p} BB \xrightarrow{q} BG$$

Character $\lambda : T \to \mathbb{C}^\times \rightsquigarrow$ Line bundle $L_\lambda \in \mathcal{Q}(BT) \rightsquigarrow$
Representation $V_\lambda = q_! p^* L_\lambda \in \mathcal{Q}(BG)$.

Taking loops gives diagram with Grothendieck-Springer resolution

$$T/T \xleftarrow{Lp} B/B \simeq \tilde{G}/G \xrightarrow{Lq} G/G$$

Weyl Character Formula

Character of V_λ is distribution
$Lq_! Lp^* \lambda =$ (weighted) measure of Springer fibers

We will soon see Harish Chandra characters as well.

Example: Spectral side of Trace Formula

Recall: for $c \in C$, local multiplication operators

$$\mathrm{Rep}(G^\vee) \curvearrowright \mathcal{Q}(\mathrm{Conn}_{G^\vee}(C))$$

arise via pullback along restriction map

$$f_c : \mathrm{Conn}_{G^\vee}(C) \longrightarrow BG^\vee$$

Passing to loops induces map

$$Lf_c : L\,\mathrm{Conn}_{G^\vee}(C) \longrightarrow L(BG^\vee) \simeq G^\vee/G^\vee$$

We obtain decomposition of dimension

$$\dim \mathcal{Q}(\mathrm{Conn}_{G^\vee}(C)) \simeq H^*_{Dol}(L\,\mathrm{Conn}_{G^\vee}(C)) \text{ over } G^\vee/G^\vee$$

Above $\kappa \in G^\vee$, fiber admits endoscopic interpretation

$$\dim \mathcal{Q}(\mathrm{Conn}_{G^\vee}(C))|_\kappa \simeq H^*_{Dol}(\mathrm{Conn}_{G^\vee_\kappa}(C))$$

Partial example: Geometric side of Trace Formula

For $c \in C$, Hecke operators arise via global bundle modifications

$$\mathcal{H}^{sph} \curvearrowright \mathcal{D}(\mathrm{Bun}_G(C))$$

But identity Hecke operator (and its symmetries) arise via pullback along restriction map

$$f_c : \mathrm{Bun}_G(C) \simeq G(C_{rat}) \backslash \prod_c' G(\mathcal{K}_c) / \prod_c G(\mathcal{O}_c) \longrightarrow B(G(\mathcal{O}_c))$$

Passing to loops gives local Hitchin map

$$Lf_c : L\,\mathrm{Bun}_G(C) \longrightarrow L(B(G(\mathcal{O}_c))) \simeq G(\mathcal{O}_c)/G(\mathcal{O}_c)$$

Varying $c \in C$, we obtain decomposition of dimension

$$\dim \mathcal{D}(\mathrm{Bun}_G(C)) \simeq H^*_{dR}(L\,\mathrm{Bun}_G(C))$$

over global Hitchin base $\mathcal{A}_G(C)$.
Nothing yet about non-identity Hecke operators...

Traces of Hecke operators

To organize traces of Hecke operators (integral operators as opposed to multiplication operators), we will return to the general setting of categorical harmonic analysis:

monoidal category $\mathcal{A}$ and $\mathcal{A}$-module category $\mathcal{M}$

Each object $a \in \mathcal{A}$ provides an endofunctor $F_a : \mathcal{M} \to \mathcal{M}$.

Let us assume $\mathcal{M}$ is dualizable. Then the trace of an endofunctor $F_a : \mathcal{M} \to \mathcal{M}$ is given by the composition

$$\operatorname{tr} F_a : \mathit{Vect} \xrightarrow{coev_{\mathcal{M}}} \mathcal{M}^* \otimes \mathcal{M} \xrightarrow{\mathrm{id}_{\mathcal{M}^*} \otimes F_a} \mathcal{M}^* \otimes \mathcal{M} \xrightarrow{ev_{\mathcal{M}}} \mathit{Vect}$$

We can package $\operatorname{tr} F_a$ into the data of a single vector space by taking its value on $\mathbb{C}$.

What is the structure of the traces $\operatorname{tr} F_a$ as we vary $a \in \mathcal{A}$? What kind of object is the character of $\mathcal{M}$ and where should it live?

Characters of module categories

The characters (collection of traces) of modules $\mathcal{M}$ for a monoidal category $\mathcal{A}$ are objects of its Hochschild homology category

$$\operatorname{Tr}\mathcal{A} = \mathcal{A} \otimes_{\mathcal{A}\otimes\mathcal{A}^{op}} \mathcal{A}$$

A dualizable $\mathcal{A}$-module category $\mathcal{M}$ defines a character object

$$\chi(\mathcal{M}) \in \operatorname{Tr}\mathcal{A}$$

Formal analogy with preceding:

- $\operatorname{Tr}\mathcal{A}$ is the dimension of the dualizable 2-category $\mathcal{A}$-mod.
- $\chi(\mathcal{M}) \in \operatorname{Tr}\mathcal{A}$ results from functoriality of dimension.

Characters for matrix categories

Now let us consider Hecke categories or matrix categories:

linearizations of fiber products $X \times_Y X$

Example

$Y = BG$, $X = BH$, for $H \subset G$, then

$$X \times_Y X \simeq H \backslash G / H$$

For quasicoherent sheaves, Morita theory holds.

Theorem

$X \to Y$ *reasonable, then*

$$\mathrm{Tr}(\mathcal{Q}(X \times_Y X)) \simeq \mathcal{Q}(LY)$$

Characters for Hecke categories

For $\mathcal{D}$-modules, Tannakian theory fails hence Morita theory fails.

Warmup for affine Hecke category
Finite Hecke category

$$\mathcal{H}^{fin} = \mathcal{D}(B\backslash G/B)$$

or twisted generalizations over Harish Chandra center.

Theorem

$$\mathrm{Tr}(\mathcal{H}^{fin}) \simeq \textit{Lusztig's character sheaves } Ch_G \subset \mathcal{D}(G/G)$$

Paradox

Cuspidal characters via categorical principal series.

Corollary (Noncommutative Fourier Transform)

Koszul equivalence $\mathcal{H}^{fin}_G \simeq \mathcal{H}^{fin}_{G^\vee}$ *implies character equivalence*

$$Ch_G \simeq Ch_{G^\vee}$$

Character of regular Hecke module

Warmup for affine Springer theory
Character of regular $\mathcal{H}^{fin}$-module is Grothendieck-Springer sheaf

$$\chi(\mathcal{H}^{fin}) \simeq R\mu_*\mathcal{O}_{B/B} \in Ch_G \subset \mathcal{D}(G/G)$$

$$\mu : B/B \simeq \widetilde{G}/G \to G/G$$

Universal equations describing characters of modules with fixed infinitesimal character.

Application: Harish Chandra characters

Warmup for automorphic module category
Symmetric $\mathcal{H}^{fin}$-module

$$\mathcal{M}^{fin} = \mathcal{D}(K\backslash G/B)$$

Two interpretations:

1. Beilinson-Bernstein localization: $\mathcal{M}^{fin} \longleftrightarrow (\mathfrak{g}, K)$-modules.
2. Intertwiners between categorical modules.

Application of preceding theory to Harish Chandra characters:

1. Refined Harish Chandra characters as algebraic solutions

$$M \in \mathcal{M}^{fin} \rightsquigarrow \chi(M) : \chi(\mathcal{H}^{fin}) \longrightarrow \chi(\mathcal{M}^{fin})$$

   Traditional characters are "solutions" in distributions.

2. Character formulas follow from fixed point geometry and microlocalization.

Application: character varieties

Recall spectral moduli space $\mathrm{Conn}_{G^\vee}(C)$.
If we puncture curve $C \setminus c$, then have local monodromy map

$$\mathrm{Conn}_{G^\vee}(C \setminus c) \longrightarrow G^\vee / G^\vee$$

Decompose punctured cohomology

$$H^*_{dR}(\mathrm{Conn}_{G^\vee}(C \setminus c))$$

as $\mathcal{D}$-module on $G^\vee / G^\vee$.
Fiber over $1 \in G^\vee$ is unpunctured cohomology

$$H^*_{dR}(\mathrm{Conn}_{G^\vee}(C))$$

Now can analyze in terms of character sheaves, take Fourier transform and obtain Langlands dual description.

Elliptic Character Sheaves

Characters for affine Hecke category $\mathcal{H}^{aff}$ should be character sheaves on loop group LG. Technically difficult...

Algebraic answer Hochschild homology category $\mathrm{Tr}(\mathcal{H}^{aff})$.

Topological field theory answer

$\mathcal{H}^{aff}$ correlations on cylinder $\rightsquigarrow$ $\mathrm{Tr}(\mathcal{H}^{aff})$ correlations on torus

Conjecture

E elliptic curve, $\mathcal{N} \subset T^* \mathrm{Bun}_G(E)$ nilpotent Higgs fields

$$\mathrm{Tr}(\mathcal{H}^{aff}) \simeq \mathcal{D}_{\mathcal{N}}(\mathrm{Bun}_G(E))$$

Idea of proof: categorical Verlinde formula for (monadic study of) automorphic category under degenerations of curves.

Endoscopic parameter

Langlands duality for affine Hecke category implies topological geometric Langlands for elliptic curves

$$\mathcal{D}_{\mathcal{N}}(\mathrm{Bun}_G(E)) \longleftrightarrow \mathcal{Q}_{\mathcal{T}}(\mathrm{Conn}_{G^\vee}(E))$$

Topological invariance results from restriction:

$$\text{Nilpotent Higgs fields} \longleftrightarrow \text{Torsion sheaves}$$

Endoscopic parameter $\kappa \in G^\vee$ is eigenvalue for circle action on E.

Many other symmetries...