RESEARCH STATEMENT, MELODY CHAN

1. Introduction

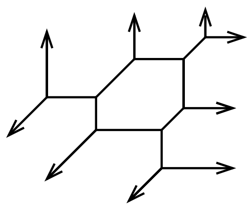
I study tropical geometry, in particular tropical curves, moduli, and determinantal varieties. I have written five papers in tropical geometry: three on tropical curves and their moduli [Cha11a, Cha11b, CMV11], and two on tropical determinantal varieties [CC09, CJR09], joint with D. Cartwright, A. N. Jensen, and E. Rubei. After a brief introduction to tropical geometry, I will propose four main directions for future research and mention some of my main results along the way.

1.1. **What is tropical geometry?** In just ten years, tropical geometry has established itself as an important new field bridging algebraic geometry and combinatorics whose tools and techniques have been used successfully to attack problems in both fields. Tropical geometry also has important connections to areas as diverse as geometric group theory, mirror symmetry, operations research, and phylogenetics.

There are several different ways to describe tropical geometry. On the one hand, it is a "combinatorial shadow" of algebraic geometry [MS09]. Let us start with the following definition. Let K be a field, which we will assume throughout to be algebraically closed and complete with respect to a nonarchimedean valuation $\mathrm{val} : K^* \to \mathbb{R}$ on it. Suppose X is an algebraic subvariety of the torus $(K^*)^n$, that is, the solution set to a system of Laurent polynomials over K. Then the *tropicalization* of X is the set

$$\mathrm{Trop}(X) = \{(\mathrm{val}(x_1), \ldots, \mathrm{val}(x_n)) \in \mathbb{R}^n : (x_1, \ldots, x_n) \in X\}.$$

By the theory of initial degenerations (see [Stu96], [MS09, Theorem 3.2.4]), these tropical varieties are made of polyhedral pieces and have many nice combinatorial properties. Furthermore, they remember information about classical varieties, for example, their dimensions [BG84]. So, if X was a plane curve, then $\mathrm{Trop}(X)$ would be made of 1-dimensional polyhedra, i.e. line segments and rays, in $\mathbb{R}^2$. A tropical plane curve of degree 3 is shown below. Note that different embeddings of a variety X can yield very different tropicalizations.



There is a complementary perspective from which tropical geometry is a tool for taking finite snapshots of Berkovich analytifications. This perspective has been made explicit in the very recent paper [BPR11] in the case of curves. Suppose X is a smooth curve over the field K. Then X has a certain metric space X^{an} intrinsically associated to it called its *Berkovich analytification* [Ber90], which contains the original points of the curve X infinitely far away. The space X^{an} is a very useful object to study, because it has some key desirable properties that X lacks: it admits a good notion of an analytic function on it, along the lines of Tate's pioneering work on rigid geometry; but in addition, it has a more desirable topology than X, which is a totally disconnected space since the nonarchimedean field K itself is. Furthermore, X^{an} has a canonical deformation retract down to a finite metric graph Γ sitting inside it, called its *Berkovich skeleton*. This combinatorial core, decorated with some nonnegative integer weights, is visible in sufficiently nice tropicalizations [BPR11, Theorem 6.20], so we call this decorated metric graph an (abstract) *tropical curve.* More precisely:

Definition 1. An *abstract tropical curve* is a triple (G, w, ℓ), where G is a connected multigraph, $\ell : E(G) \to \mathbb{R}_{>0}$ is a length function on the edges of G, and $w : V(G) \to \mathbb{Z}_{\geq 0}$ is a weight function on the vertices of G such that if $w(v) = 0$, then v has valence at least 3. The *genus* of the curve is $\dim H_1(G, \mathbb{R}) + \sum w(v)$.

Tropical curves play a central role in my research directions, which I will discuss next.

2. Current and future research

I will now discuss four main research topics in the areas of tropical curves, determinantal varieties, and connections to harmonicity and Outer Space. Some of the suggested problems constitute joint projects with S. Backman, M. Baker, D. Maclagan, M. Melo, B. Sturmfels, and F. Viviani.

(1) The map from algebraic to tropical curves.
(2) Harmonic morphisms and tropical morphisms.
(3) Connections to Culler-Vogtmann Outer Space, and level structures.
(4) Tropical bases and determinantal varieties.

2.1. **The map from algebraic to tropical curves.**

Tropical curves have been studied by many authors [MZ07], [GK08], [HMY09]. For a fixed genus g, they can be collected into a $(3g-3)$-dimensional moduli space denoted M_g^{tr}, as predicted in [MZ07] and constructed explicitly in [BMV11] and [Cap11a]. In [Cha11a], I gained a thorough combinatorial understanding of the spaces M_g^{tr} for small g. For example:

Theorem 2. *Using an algorithm implemented in* Mathematica*, we computed:*

(i) The moduli space M_3^{tr} *has* 42 *cells and* f*-vector* $(1, 2, 5, 9, 12, 8, 5)$.
(ii) The moduli space M_4^{tr} *has* 379 *cells and* f*-vector* $(1, 3, 7, 21, 43, 75, 89, 81, 42, 17)$.
(iii) The moduli space M_5^{tr} *has* 4555 *cells and* f*-vector*

$$(1, 3, 11, 34, 100, 239, 492, 784, 1002, 926, 632, 260, 71).$$

These results were also obtained in [MP10]. The cells of M_g^{tr} are in bijection with all possible dual graphs to stable curves in $\overline{\mathcal{M}_g}$.

Having constructed M_g^{tr}, we can define a tropicalization map $\mathrm{trop} : \mathcal{M}_g(K) \to M_g^{tr}$ on the level of moduli spaces, sending a curve over K to its Berkovich skeleton [BPR11, Remark 5.51]. Equivalently, the map sends a curve to the dual graph, appropriately metrized, of the special fiber of a semistable model.

Problem 3. *Study the map* $\mathrm{trop} : \mathcal{M}_g(K) \to M_g^{tr}$.

A detailed understanding of the map from algebraic to tropical curves is very important in light of its deep interaction with divisor theory on tropical curves, as follows. There is a very nice way to define a *divisor* on a tropical curve and its *rank* such that a Riemann-Roch formula holds; this was done first for graphs in the ground-breaking work [BN07] and extended in [GK08], [MZ07]. The Specialization Lemma [Bak08] says that the rank of a divisor on an algebraic curve X can only go up under the tropicalization map. (By the *rank of* a divisor D on an algebraic curve, we simply mean the number $\dim_K L(D) - 1$). This remarkable result led to a tropical proof of the Brill-Noether Theorem [CDPR10].

In [Cha11b], I undertook a study of *tropical hyperelliptic curves*, defined as those tropical curves that admit a divisor of degree 2 and rank 1. Hyperellipticity was studied in [BN09] in the case of graphs. I extended many of those results to metric graphs and tropical curves. In particular, I defined harmonic morphisms of metric graphs and proved

Theorem 4. *A tropical curve is hyperelliptic if and only if it admits a harmonic morphism of degree 2 to a tree.*

Theorem 4 allows us to characterize the locus of hyperelliptic curves of genus g in M_g^{tr}. It is natural to restrict to 2-edge-connected graphs, i.e. curves whose semistable reduction has no separating nodes. Extending [Cha11a], I obtained the following results in [Cha11b]:

Theorem 5. *The locus of 2-edge-connected tropical hyperelliptic curves is a $(2g-1)$-dimensional stacky polyhedral fan; thus it has the same dimension as the classical hyperelliptic locus. Its maximal cells are in bijection with trees on $g-1$ vertices having valence ≤ 3.*

Furthermore, if $X \subseteq (K^*)^2$ is a hyperelliptic curve defined by a polynomial $f \in K[x,y]$ whose Newton complex is a unimodular triangulation, and $\mathrm{Trop}(X)$ has no internal bridges, then the Berkovich skeleton of the smooth completion $\hat{X}$ is a standard ladder of genus g.

These theorems are the starting point for several specific problems which I plan to solve.

Problem 6. *Use the theory of admissible covers (in the sense of Beauville, see [HM98]), to characterize the image of the hyperelliptic locus $\mathcal{H}_g$ under* trop.

Admissible covers of $\mathbb{P}^1$ are precisely the tool we can use to study degenerations of hyperelliptic curves. By [Bak08], the tropicalization of the classical hyperelliptic locus $\mathcal{H}_g$ is contained in the locus of tropical hyperelliptic curves. We conjecture:

Conjecture 7. *Every 2-edge-connected tropical hyperelliptic curve lies in* $\mathrm{trop}(\mathcal{H}_g)$.

Note that the hyperelliptic curves of genus 3 are the smallest instance of a *Brill-Noether locus*, that is, the locus of points $[X] \in \mathcal{M}_g$ where X is a curve admitting a divisor of rank r and degree d, for some r, d satisfying $\rho(g,r,d) = g - (r+1)(g-d+r) < 0$. The following problem was posed to me by Joe Harris.

Problem 8. *Characterize the tropical Brill-Noether loci $M_{g,d}^{r,tr}$ inside M_g^{tr}.*

As noted, we solved this problem for all g in the case $r = 1$, $d = 2$ in [Cha11b]. I am working on a complete solution, which would be especially interesting in light of recent progress in tropical Brill-Noether theory [Bak08, Cap11b, CDPR10, LPP11].

To study the map $\mathrm{trop} : \mathcal{M}_g(K) \to M_g^{tr}$, it is also important to be able to compute it. Given a curve X over K, we have seen that its Berkovich skeleton is visible in sufficiently nice tropicalizations of X. But how can we find such a tropicalization? In a current joint project with Spencer Backman, Matt Baker, and Bernd Sturmfels, we propose to answer the following concrete question which was raised by Arne Buchholz and Hannah Markwig:

Question 9. *Given a homogeneous polynomial $f \in K[x,y,z]$ of degree 3 that defines an elliptic plane curve with $\mathrm{val}\, j(f) < 0$, does there exist a matrix $A \in GL_3(K)$ such that $\mathrm{trop}(f \circ A)$ is dual to a unimodular triangulation, so that in particular $\mathrm{trop}(f \circ A)$ has a copy of the Berkovich skeleton inside it? If so, find an algorithm to produce such an A.*

A starting point for Question 9 is Theorem 7.2 in [BPR11]. Finally, we mention the following

Question 10. *What is the correct definition of a tropical stack?*

This question is broad and likely quite difficult. One might hope that the category of stacky fans defined in [Cha11a] is the right notion; for example, M_g^{tr} and A_g^{tr} are objects. However, M_g^{tr} acts more like a coarse moduli space than a moduli stack in some ways; for example, it does not seem to admit $M_{g,1}^{tr}$ as a universal space. We propose specifically to construct a moduli stack of tropical curves.

2.2. **Harmonic morphisms and tropical morphisms.** There is a substantial literature on combinatorial harmonic morphisms [Ura00, ES64] that predates tropical geometry. They are a natural analogue of holomorphic maps of Riemann surfaces. The papers [BN09] and [Cha11b] are the first occurrences of harmonic morphisms of graphs and metric graphs, respectively, in the tropical geometry literature. However, there is a more general notion of harmonicity, due to Wang [Wan98], here of maps from simplicial complexes to Riemannian manifolds of nonpositive curvature (see [ES64]). I believe that harmonic morphisms in this sense could be the correct notion of a tropical morphism in a much more general setting.

Problem 11. *Establish a precise connection between harmonic morphisms in the sense of* [Wan98] *and maps between tropical algebraic varieties.*

For example, in [Cha11a], I undertook a thorough study of the *tropical Torelli map* $t_g^{tr} : M_g^{tr} \to A_g^{tr}$, first constructed and studied in [BMV11]. The tropical Torelli map is a map from the moduli space of tropical curves to the moduli space of principally polarized tropical abelian varieties (see Definition 4.9 in [Cha11a]), sending a curve to its *tropical Jacobian.* The tropical Jacobian [MZ07] is a flat torus of dimension g. It is quite amazing to note that the following tropical theorem was in fact proved in an entirely different community, in the paper [KS01], which is primarily concerned with equilibrium configurations of crystal lattices:

Theorem 12. *The tropical Abel-Jacobi map, taking a metric graph into its Jacobian, is harmonic. Indeed, it is the unique, up to translation, harmonic map between these spaces in its homotopy class.*

To continue to make precise connections between these communities, I shall attack the following specific problem, taking advantage of my particular expertise gained from [Cha11a].

Problem 13. *Prove that the tropical Torelli map is harmonic, for an appropriate definition of harmonicity.*

Here one cannot immediately apply the existing theory of harmonicity because although A_g^{tr} is indeed nonpositively curved, M_g^{tr} is not quite a simplicial complex. The material presented in §2.2, and especially Problem 13, was inspired by a conversation with Benson Farb.

2.3. **Connections with Culler-Vogtmann Outer Space and level structures.** The space X_g, constructed by Culler and Vogtmann in [CV86] and dubbed *outer space* by Shalen, is extremely important in geometric group theory. It parametrizes genus g metric graphs marked with a fixed isomorphism of the fundamental group to the free group F_g. The space X_g is contractible and the group $\mathrm{Out}(F_g)$ acts with finite stabilizers on X_g [CV86]. It is well-known that X_g can be viewed as an analogue of Teichmüller space. I believe that making that analogy more precise by employing the tropical perspective would be beneficial both to tropical geometers and geometric group theorists. In the forthcoming paper [CMV11] with Margarida Melo and Filippo Viviani, we prove the following precise statement:

Proposition 14. *The moduli space of tropical curves M_g^{tr} is the stratified quotient of the simplicial closure of outer space X_g by the action of the outer automorphism group* $\mathrm{Out}(F_g)$.

We wish to push this connection even farther. For example, there is a period mapping

$$X_g \longrightarrow \Omega_g$$

from outer space to the cone of $g \times g$ positive semidefinite matrices, precisely analogous to the period map $T_g \to H_g$ from Teichmüller space to the Siegel upper half plane. That period map commutes with the tropical Torelli map in the following diagram:

$$\begin{array}{ccc} X_g & \longrightarrow & \Omega_g \\ \big\downarrow {\scriptstyle /\,\mathrm{Out}(F_g)} & & \big\downarrow {\scriptstyle /GL_g(\mathbb{Z})} \\ M_g^{tr} & \longrightarrow & A_g^{tr} \end{array}$$

Problem 15. *(with Melo, Viviani) Study the diagram above. In particular, study the space of $\mathbb{Z}$-homology marked graphs, a very interesting space which sits in between X_g and M_g^{tr}.*

I am also studying level ℓ structures on the tropical Torelli map, in collaboration with Diane Maclagan. One problem with the spaces M_g^{tr} and A_g^{tr} is that they do not look very tropical: they are not polyhedral fans and do not satisfy a tropical balancing condition, so one cannot yet do tropical intersection theory [AR07] on them. We propose to construct certain finite index covers of these spaces, $M_g^{tr}[\ell]$ and $A_g^{tr}[\ell]$, that do admit a tropical structure. These spaces are obtained as quotients of X_g and Ω_g by the action of the groups $\ker(\mathrm{Out}(F_g) \to GL_g(\mathbb{Z}/\ell\mathbb{Z}))$ and $\ker(GL_g(\mathbb{Z}) \to GL_g(\mathbb{Z}/\ell\mathbb{Z}))$, respectively, so the induced map

$$M_g^{tr}[\ell] \longrightarrow A_g^{tr}[\ell]$$

fits into the middle of the commutative diagram above.

Problem 16. *(with Maclagan) Find realizations of the spaces $M_g^{tr}[\ell]$ and $A_g^{tr}[\ell]$ that are polyhedral fans satisfying a tropical balancing condition.*

2.4. **Tropical bases and determinantal varieties.** This project marks a return to the viewpoint of tropical varieties as embedded polyhedral subsets of $\mathbb{R}^n$. It is important to be able to actually compute tropical varieties, since they in turn are tools to understand classical varieties. Every tropical variety is an intersection of finitely many tropical hypersurfaces [BJSST07]; a set of generators for the ideal $I(X)$ that determines the tropicalization of X is called a *tropical basis*. Hence tropical bases are central to the computational study of tropical varieties. *Determinantal varieties*, defined by the $r \times r$ minors of a matrix of indeterminates and comprising matrices of rank at most $r-1$, constitute an excellent test case. In 2005, Develin, Santos, and Sturmfels posed the question: *for which r, d, n do the $r \times r$ minors of a $d \times n$ matrix form a tropical basis?* They obtained an answer in some cases; the case of the 4×4 minors of a 5×5 matrix was posed as a computational challenge [DSS05]. In 2009, in my joint publication with Anders Jensen and Elena Rubei, we obtained the following result:

Theorem 17. [CJR09] *The 4×4 minors of a $5 \times n$ matrix form a tropical basis.*

We gave two proofs, one combinatorial and one computational. Shitov subsequently solved the remaining cases, settling the question completely [Shi10], [Shi11].

The combined results, shown in Table 1, answer affirmatively the following question of Payne and Osserman in the case of determinantal varieties.

r, d	3	4	5	6	7	8
3	yes [DSS05]	yes [DSS05]	yes [DSS05]	yes [DSS05]	yes [DSS05]	yes [DSS05]
4		yes [DSS05]	yes [CJR09]	yes [Shi11]	no [DSS05]	no [DSS05]
5			yes [DSS05]	no [Shi10]	no [Shi10]	no [DSS05, Shi10]
6				yes [DSS05]	no [Shi10]	no [Shi10]
7					yes [DSS05]	no [Shi10]
8						yes [DSS05]

TABLE 1. Do the $r \times r$ minors of a $d \times n$ matrix of indeterminates, with $d \leq n$, form a tropical basis for the ideal they generate?

Question 18. *Suppose $I = \langle f_1, \ldots, f_k \rangle$ is a prime ideal, and the intersection of the k tropical hypersurfaces of the f_i's has dimension* $\dim V(I)$. *Is $\{f_1, \ldots, f_k\}$ necessarily a tropical basis?*

I propose to answer this fundamental question on tropical bases in the case of symmetric and Hankel matrices. I also propose to make a precise connection between tropical determinantal varieties and tropical Brill-Noether theory, in analogy with the classical case.

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