Singular Continuous Measures
by Michael Pejic 5/14/10

Preliminaries

Given a set X, a *σ-algebra* on X is a collection of subsets of X that contains X and $\varnothing$ and is closed under complementation and countable unions (hence, countable intersections as well). A *measure space* is a triplet $(X, \mathcal{A}, \mu)$, where X is a set, $\mathcal{A}$ is a σ-algebra on X and the measure $\mu : \mathcal{A} \to [0, \infty]$ satisfies: (i) $\mu(\varnothing) = 0$; (ii) if $\{A_j\}_{j \in J} \subset \mathcal{A}$ is a countable collection of disjoint sets, then

$$\mu\left(\bigcup_{j \in J} A_j\right) = \sum_{j \in J} \mu(A_j)$$

If $\mu(X)$ is finite, μ is termed a *finite* measure; if X can be expressed as the countable union of sets in $\mathcal{A}$, each of which has finite measure, then μ is termed a *σ-finite* measure. The *support* of a measure is the largest closed set such that any open subset of it has nonzero measure. A function $f : X \to [-\infty, \infty]$ is *$\mathcal{A}$-measurable* if the preimage $f^{-1}((c, \infty])$ is in $\mathcal{A}$ for all $c \in \mathbb{R}$ (hence the preimage of any Borel set is in $\mathcal{A}$). A *simple* $\mathcal{A}$-measurable function is one taking on only finitely many distinct values, all of which are finite. Such a function is *integrable* if its support has finite measure. The integral of a positive $\mathcal{A}$-measurable function with respect to measure μ is defined as

$$\int f \, d\mu = \sup \left\{ \sum_j a_j \mu(A_j) \,\middle|\, \begin{array}{c} \text{integrable simple functions} \\ \sum_j a_j \chi_{A_j} \leq f \text{ pointwise} \end{array} \right\}$$

The integral of a general $\mathcal{A}$-measurable function with respect to measure μ is defined by

$$\int f \, d\mu = \int f_{>0} \, d\mu - \int f_{<0} \, d\mu$$

Note that this excludes any notion of conditional convergence. Finally, a property is said to hold *almost everywhere* with respect to μ if it fails on a set of measure zero.

Examples

i) The trivial measure which assigns measure 0 to every element of the σ-algebra.

ii) As an example of finite measures, probability measures $(\Omega, \mathcal{E}, \pi)$ with $\pi(\Omega) = 1$.
iii) As an example of a σ-finite measure, Lebesgue measure with $X = \mathbb{R}$, $\mathcal{A}$ given by Borel sets, and measure λ given by

$$\lambda(A) = \inf \left\{ \sum_j (b_j - a_j) \,\middle|\, \begin{array}{c} \text{all countable collections} \\ \text{of open intervals} \\ \text{such that } \bigcup_j (a_j, b_j) \supset A \end{array} \right\}$$

Then continuous functions $f : [a, b] \to \mathbb{R}$ are measurable and for them the definition of integration agrees with the standard Riemann integral,

$$\int_a^b f(x)\, dx = \int f \cdot \chi_{[a,b]} d\lambda = \int_{[a,b]} f\, d\lambda$$

Stieltjes Integral

For $X = \mathbb{R}$ and $\mathcal{A}$ given by Borel sets, there is an equivalent formulation. Let $\varphi : \mathbb{R}$ be nondecreasing and continuous from the left. Let the Lebesgue-Stieltjes measure μ_φ be the measure induced by $\mu_\varphi([a, b)) = \varphi(b) - \varphi(a)$. Conversely, given any measure on Borel subsets of $\mathbb{R}$ that is finite on bounded sets, μ, define $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(x) = \begin{cases} -\mu([x, 0)) & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \mu([0, x)) & \text{if } x > 0 \end{cases}$$

Then $\mu = \mu_\varphi$. For continuous functions $f : [a, b] \to \mathbb{R}$, this agrees with the standard Riemann-Stieltjes integral,

$$\int_{[a,b]} f\, d\mu_\varphi = \int_a^b f(x)\, d\varphi(x) = \lim_{\max_k (x_{k+1} - x_k) \to 0} \sum_{j=0}^{n-1} f\left(x_j^*\right) \left(\varphi(x_{j+1}) - \varphi(x_j)\right)$$

where the limit is taken over all partitions $a = x_0 \leq x_0^* \leq x_1 \leq \ldots \leq x_n = b$.

Examples
i) For $\varphi(x) = x$, we recover the Lebesgue measure, $\mu_\varphi = \lambda$.
ii) For $\varphi = \Theta$, the Heaviside function

$$\Theta(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

we get a measure μ_Θ that has an *atom of mass* 1 at $x = 0$, which is typically (for those ignorant of measure theory) denoted $d\varphi = \delta(x)\,dx$ for δ the Dirac delta "function." Note this measure, unlike Lebesgue measure, can be extended from the σ-algebra of Borel sets to the σ-algebra of all subsets of $\mathbb{R}$.

Note that discontinuities in φ correspond to atoms with mass given by the height of the jump.

Absolute Continuity

Given two $\mathcal{A}$-measures, μ, ν, if a set is of measure 0 with respect to ν whenever it is measure 0 with respect to μ, then ν is said to be *absolutely continuous* with respect to μ, denoted $\nu \ll \mu$. If μ, ν are in addition σ-finite, by the Radon-Nikodym theorem there is an $\mathcal{A}$-measurable function, denoted $\frac{d\nu}{d\mu}$, such that for any $A \in \mathcal{A}$,

$$\nu(A) = \int_A \frac{d\nu}{d\mu}\,d\mu$$

This function is unique within a μ-null function.

Mutual Singularity

Given two $\mathcal{A}$-measures, μ, ν, if there are disjoint sets E, F such that for any set $A \in \mathcal{A}$, $A \cap E, A \cap F \in \mathcal{A}$ and $\mu(A \cap E) = \mu(A), \nu(A \cap F) = \nu(A)$, then μ, ν are said to be *mutually singular*, denoted $\mu \perp \nu$.

Decomposition of a Measure

For two σ-finite, $\mathcal{A}$-measures μ, ν, by the Lebesgue decompostion theorem $\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{sing}}$, where $\mu_{\mathrm{ac}} \ll \nu$ and $\mu_{\mathrm{sing}} \perp \nu$. Then the Radon-Nikodym derivative $\frac{d\mu}{d\nu}$ is defined to be $\frac{d\mu_{\mathrm{ac}}}{d\nu}$.

Example

Take λ and μ_Θ. These are mutually singular since for any Borel set A, $\lambda(A) = \lambda(A \setminus \{0\})$ whereas $\mu_\Theta(A) = \mu_\Theta(A \cap \{0\})$.

Specializing to the case where one of the measures is Lebesgue measure, there is a further decomposition of the singular part into atoms and a *singular continuous* part. The first is familiar from physical concepts of point masses or charges; the second is far less so.

First Example of Singular Continuous Measure: Devil's Staircase

Consider the sequence of non-decreasing, continuous functions $\langle f_j \rangle_{j=0}^{\infty}$ from the unit interval onto itself with

$$f_1|_{\left[\frac{1}{3},\frac{2}{3}\right]} = \frac{1}{2}$$

$$f_2|_{\left[\frac{1}{9},\frac{2}{9}\right]} = \frac{1}{4}, f_2|_{\left[\frac{1}{3},\frac{2}{3}\right]} = \frac{1}{2}, f_2|_{\left[\frac{7}{9},\frac{8}{9}\right]} = \frac{3}{4}$$

$$f_3|_{\left[\frac{1}{27},\frac{2}{27}\right]} = \frac{1}{8}, f_3|_{\left[\frac{1}{9},\frac{2}{9}\right]} = \frac{1}{4}, f_3|_{\left[\frac{7}{27},\frac{8}{27}\right]} = \frac{3}{8}, f_3|_{\left[\frac{1}{3},\frac{2}{3}\right]} = \frac{1}{2},$$

$$f_3|_{\left[\frac{19}{27},\frac{20}{27}\right]} = \frac{5}{8}, f_3|_{\left[\frac{7}{9},\frac{8}{9}\right]} = \frac{3}{4}, f_3|_{\left[\frac{25}{27},\frac{26}{27}\right]} = \frac{7}{8}$$

$$\vdots$$

with intervening values determined by linear interpolation. This converges in supremum norm to a non-decreasing, continuous function $f : [0,1] \to [0,1]$ that clearly has derivative 0 almost everywhere with respect to Lebesgue measure since the Lebesgue measure of the middle-third Cantor set C is 0, so the Radon-Nikodym derivative $\frac{d\mu_f}{d\lambda} = 0$, yet μ_f is certainly not 0 since $\mu_f([0,1]) = 1$. At the same time, since f is continuous, μ_f contains no atoms. Therefore, μ_f is purely singular continuous. Note μ_f is equal to the Hausdorff measure $h_{\log_3 2}$ defined by

$$h_{\log_3 2}(A) = \lim_{\varepsilon \to 0^+} \inf \left\{ \sum_{B \in \mathcal{B}} (\text{diam } B)^{\log_3 2} \;\middle|\; \begin{array}{c} \text{open covers } \mathcal{B} \text{ of } A \\ \text{by balls of} \\ \text{diameter less than } \varepsilon \end{array} \right\}$$

Doing integrals with respect to μ_f by taking the limit as $n \to \infty$ of doing them with μ_{f_n} is tedious. In some cases, this can be made easier by interpreting the graph of f as an affine *iterated fractal system* with base $B = ((0,0),(1,0),(0,1))$ and generators $G_1 = \left((0,0), \left(\frac{1}{3},0\right), \left(0,\frac{1}{2}\right)\right)$, $G_2 = \left(\left(\frac{1}{3},\frac{1}{2}\right), \left(\frac{2}{3},\frac{1}{2}\right), \left(\frac{1}{3},\frac{1}{2}\right)\right)$, and $G_3 = \left(\left(\frac{2}{3},\frac{1}{2}\right), \left(1,\frac{1}{2}\right), \left(\frac{2}{3},1\right)\right)$ with corresponding affine transformations

$$A_1(x,y) = \left(\frac{1}{3}x, \frac{1}{2}y\right)$$

$$A_2(x,y) = \left(\frac{1}{3}x + \frac{1}{3}, \frac{1}{2}\right)$$

$$A_3(x,y) = \left(\frac{1}{3}x + \frac{2}{3}, \frac{1}{2}y + \frac{1}{2}\right)$$

so graph $f = A_1(\text{graph } f) \cup A_2(\text{graph } f) \cup A_3(\text{graph } f)$, which implies for $x \in [0,1]$,

$$\left(\frac{1}{3}x, \frac{1}{2}f(x)\right) \in \text{graph } f$$

$$\left(\frac{1}{3}x + \frac{1}{3}, \frac{1}{2}\right) \in \text{graph } f$$

$$\left(\frac{1}{3}x + \frac{2}{3}, \frac{1}{2}f(x) + \frac{1}{2}\right) \in \text{graph } f$$

which in turn imply

$$f(x) = \begin{cases} \frac{1}{2}f(3x) & \text{if } x \in \left[0, \frac{1}{3}\right] \\ \frac{1}{2} & \text{if } x \in \left[\frac{1}{3}, \frac{2}{3}\right] \\ \frac{1}{2}f(3x-2) + \frac{1}{2} & \text{if } x \in \left[\frac{2}{3}, 1\right] \end{cases}$$

$$\Rightarrow d\mu_f(x) = \begin{cases} \frac{1}{2}d\mu_f(3x) & \text{if } x \in \left[0, \frac{1}{3}\right] \\ 0 & \text{if } x \in \left[\frac{1}{3}, \frac{2}{3}\right] \\ \frac{1}{2}d\mu_f(3x-2) & \text{if } x \in \left[\frac{2}{3}, 1\right] \end{cases}$$

This can be used to find the moments:

$$m_n = \int x^n d\mu_f(x) = \int_{\left[0,\frac{1}{3}\right)} x^n d\mu_f(x) + \int_{\left[\frac{2}{3},1\right)} x^n d\mu_f(x)$$

$$= \int_{\left[0,\frac{1}{3}\right)} x^n \frac{1}{2} d\mu_f(3x) + \int_{\left[\frac{2}{3},1\right)} x^n \frac{1}{2} d\mu_f(3x-2)$$

$$= \frac{1}{2}\int \left(\frac{x}{3}\right)^n d\mu_f(x) + \frac{1}{2}\int \left(\frac{x+2}{3}\right)^n d\mu_f(x)$$

$$= \frac{1}{2\cdot 3^n} m_n + \frac{1}{2\cdot 3^n}\left(m_n + \sum_{j=1}^{n} \binom{n}{j} 2^j m_{n-j}\right)$$

so for $n \geq 1$,

$$m_n = \frac{1}{3^n - 1}\sum_{j=1}^{n} \binom{n}{j} 2^{j-1} m_{n-j}$$

Using this and $m_0 = 0$, the first few are given by $m_1 = \frac{1}{2}, m_2 = \frac{3}{8}, m_3 = \frac{5}{16}, m_4 = \frac{87}{320}$.

Second Example of Singular Continuous Measure: Ferenc Riesz

The previous example is not really that different from the atomic case except the support of the singular part of the measure is an uncountable fractal set rather than a countable collection of points. To see how strange singular measures can be, consider the following construction due to Ferenc Riesz:

Pick a $t \in (0,1)$. Start with the graph of F_0 being the line segment from $\overline{(0,0) \text{ to } (1,1)}$. To construct the graph of F_{j+1}, replace each line segment $\overline{(x_0, y_0)\,(x_1, y_1)}$ in the graph of F_j with the pair of line segments

$$\overline{(x_0, y_0)\left(\frac{x_0 + x_1}{2}, \frac{1-t}{2} y_0 + \frac{1+t}{2} y_1\right)}$$

and

$$\overline{\left(\frac{x_0 + x_1}{2}, \frac{1-t}{2} y_0 + \frac{1+t}{2} y_1\right)(x_1, y_1)}$$

In the limit as $j \to \infty$, the sequence of increasing, continuous functions $\langle F_j \rangle_{j=0}^{\infty}$ from the unit interval onto itself coverges in sup norm to a non-decreasing (actually increasing), continuous function $F : [0,1] \to [0,1]$.

This can also be interpreted as the affine iterated fractal system given by base $B = ((0,0), (1,0), (0,1))$ and generators $G_1 = \left((0,0), \left(\frac{1}{2}, 0\right), \left(0, \frac{1+t}{2}\right)\right)$ and $G_2 = \left(\left(\frac{1}{2}, \frac{1+t}{2}\right), \left(1, \frac{1+t}{2}\right), \left(\frac{1}{2}, 1\right)\right)$ with corresponding affine transformations

$$A_1(x, y) = \left(\frac{1}{2} x, \frac{1+t}{2} y\right)$$

and

$$A_2(x, y) = \left(\frac{1}{2} x + \frac{1}{2}, \frac{1-t}{2} y + \frac{1+t}{2}\right)$$

By repeated application of the chain rule, for $x = .b_1 b_2 b_3 \ldots$ in binary

$$\left.\frac{dF}{dx}\right|_x = \prod_{j=0}^{\infty} \begin{cases} (1+t) & \text{if } b_j = 0 \\ (1-t) & \text{if } b_j = 1 \end{cases}$$

At binary fractions, the two different binary expansions correspond to taking the respective one-sided derivatives. If

$$\limsup_{n \to \infty} \frac{\text{number of 0's in first } n \text{ binary digits of } x}{n} < \frac{-\log(1-t)}{\log(1+t) - \log(1-t)}$$

then $\left.\frac{dF}{dx}\right|_x$ exists and is equal to 0. Yet, by applying the central limit theorem to the binomial distribution it can be shown (see below for details) that, with respect to Lebesgue measure, for almost every $x \in [0,1]$,

$$\lim_{n\to\infty} \frac{\text{number of 0's in first } n \text{ binary digits of } x}{n} = \frac{1}{2}$$

so $\frac{dF}{dx} = 0$ almost everywhere, so the Radon-Nikodym derivative $\frac{d\mu_F}{d\lambda} = 0$. Since F is continuous, μ_F has no atoms, so it is purely singular continuous. Unlike in the previous example, however, the support of μ_F is all of $[0,1]$ since those x with

$$\limsup_{n\to\infty} \frac{\text{number of 0's in first } n \text{ binary digits of } x}{n} \geq \frac{-\log(1-t)}{\log(1+t) - \log(1-t)}$$

are a dense subset; hence, F is increasing.

The formulation as an iterative fractal system can also be used to do certain integrals with respect to μ_F. Since

$$\text{graph } F = A_1(\text{graph } F) \cup A_2(\text{graph } F)$$

which implies for $x \in [0,1]$,

$$\left(\frac{1}{2}x, \frac{1+t}{2}F(x)\right) \in \text{graph } F$$

and

$$\left(\frac{1}{2}x + \frac{1}{2}, \frac{1-t}{2}F(x) + \frac{1+t}{2}\right) \in \text{graph } F$$

which in turn imply

$$F(x) = \begin{cases} \frac{1+t}{2}F(2x) & \text{if } x \in \left[0, \frac{1}{2}\right] \\ \frac{1-t}{2}F(2x-1) + \frac{1+t}{2} & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

we have

$$d\mu_F(x) = \begin{cases} \frac{1+t}{2}d\mu_F(2x) & \text{if } x \in \left[0, \frac{1}{2}\right] \\ \frac{1-t}{2}d\mu_F(2x-1) & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Using this, we can calcute the moments by getting a relation,

$$m_n = \int_0^1 x^n d\mu_F(x)\,dx = \int_0^{\frac{1}{2}} x^n \frac{1+t}{2} d\mu_F(2x) + \int_{\frac{1}{2}}^1 x^n \frac{1-t}{2} d\mu_F(2x-1)$$

$$= \frac{1+t}{2} \int_0^1 \left(\frac{x}{2}\right)^n d\mu_F(x) + \frac{1-t}{2} \int_0^1 \left(\frac{x+1}{2}\right)^n d\mu_F(x)$$

$$\Rightarrow m_n = \frac{1+t}{2^{n+1}} m_n + \frac{1-t}{2^{n+1}} \left(m_n + \sum_{j=1}^{n} \binom{n}{j} m_{n-j} \right)$$

Then for $n \geq 1$

$$m_n = \frac{1-t}{2^{n+1}-2} \sum_{j=1}^{n} \binom{n}{j} m_{n-j}$$

Using $m_0 = 1$, the first few are given by:

$$m_1 = \frac{1-t}{2}, m_2 = \frac{1-t}{2}\left(1 - \frac{1+t}{3}\right), m_3 = \frac{1-t}{2}\left(1 + \frac{(1+t)(t-7)}{14}\right)$$

$$m_4 = \frac{1-t}{2}\left(1 - \frac{(1+t)(t^2 - 15t + 63)}{105}\right)$$

and for $t = \frac{1}{2}$: $m_1 = \frac{1}{4}, m_2 = \frac{1}{8}, m_3 = \frac{17}{224}, m_4 = \frac{57}{1120}$.

Proof of Assertion

In order to show that, with respect to Lebesgue measure, for almost every $x \in [0,1]$,

$$\lim_{n\to\infty} \frac{\text{number of 0's in first } n \text{ binary digits of } x}{n} = \frac{1}{2}$$

it is only necessary to show that for any $p \in \left(\frac{1}{2}, 1\right]$, the set of $x \in [0,1]$ such that

$$\limsup_{n\to\infty} \frac{\text{number of 0's in first } n \text{ binary digits of } x}{n} \geq p$$

has Lebesgue measure 0 since by symmetry between 0's and 1's in the binary expansion, the corresponding set of $x \in [0,1]$ such that

$$\liminf_{n\to\infty} \frac{\text{number of 0's in first } n \text{ binary digits of } x}{n} \leq 1 - p$$

will also have Lebesgue measure 0, and then, since countable unions of measure 0 sets are of measure 0, we can simply apply the result to the sequence $\left\langle \frac{1}{2} + \frac{1}{2^n} \right\rangle_{n=1}^{\infty}$ for the values of p.

Now take any $\delta > 0$ and let $q \in \left(\frac{1}{2}, p\right)$. Then the proportion of binary fractions with denominator 2^n that have proportion of 0's in first n binary digits greater than or equal to q is given by

$$\frac{1}{2^n} \sum_{j=\lceil q\cdot n \rceil}^{n} \binom{n}{j}$$

For large n, by applying the central limit theorem to the binomial distribution, for j of order $\sqrt{n}$ from $\frac{n}{2}$,

$$\frac{1}{2^n} \binom{n}{j} \approx \frac{1}{\sqrt{2\pi n \cdot \frac{1}{4}}} e^{-\frac{\left(j-n\cdot\frac{1}{2}\right)^2}{2n\cdot\frac{1}{4}}} = \sqrt{\frac{2}{\pi n}} e^{-\frac{2\left(j-\frac{n}{2}\right)^2}{n}}$$

since for $n = 1$ the mean is $\frac{1}{2}$ and the variance $\frac{1}{4}$. (The same result also follows using Stirling's formula.) Unfortunately, we are interested in j of order n from $\frac{n}{2}$; however, by numerical computation the Gaussian is actually an (rather drastic) overestimate in this region, so

$$\frac{1}{2^n} \sum_{j=\lceil q\cdot n \rceil}^{n} \binom{n}{j} < \sum_{j=\lceil q\cdot n \rceil}^{n} \sqrt{\frac{2}{\pi n}} e^{-\frac{2\left(j-\frac{n}{2}\right)^2}{n}} < \int_{\lceil q\cdot n \rceil - 1}^{n} \sqrt{\frac{2}{\pi n}} e^{-\frac{2\left(t-\frac{n}{2}\right)^2}{n}} dt$$

$$= \int_{\sqrt{2n}\left(\frac{\lceil q\cdot n \rceil - 1}{n} - \frac{1}{2}\right)}^{\sqrt{\frac{n}{2}}} \frac{1}{\sqrt{\pi}} e^{-x^2} dx < \int_{\sqrt{2n}\left(q-\frac{1}{2}-\frac{2}{n}\right)}^{\infty} \frac{1}{\sqrt{\pi}} e^{-x^2} dx$$

letting $x = \sqrt{\frac{2}{n}}\left(t - \frac{n}{2}\right)$. Therefore, for n sufficiently large,

$$\frac{1}{2^n} \sum_{j=\lceil q\cdot n \rceil}^{n} \binom{n}{j} < \frac{1}{2} \cdot \operatorname{erfc}\left(\sqrt{2n}\left(q - \frac{1}{2} - \frac{2}{n}\right)\right) \approx \frac{e^{-2n\left(q-\frac{1}{2}\right)^2}}{2\sqrt{2\pi n}\left(q - \frac{1}{2}\right)}$$

from the asymptotic form for the complimentary error function. Hence, there is a sequence of numbers, $\langle a_n \rangle_{n=0}^{\infty}$ with the following properties: (i) the sequence has limit ∞; (ii) the sum

$$\sum_{n=0}^{\infty} a_n \cdot \frac{1}{2^n} \sum_{j=\lceil q\cdot n \rceil}^{n} \binom{n}{j}$$

is finite. For example, from the preceding asymptotic expression, $a_j = r^j$ for $r \in \left(1, e^{2\left(q-\frac{1}{2}\right)^2}\right)$ will certainly work by the limit comparison test. Let the sum be denoted by M.

Now consider the subset B given by the union of balls of diameter $\frac{a_n \delta}{2^n M}$ centered at binary fractions with denominator 2^n that have proportion of 0's in their first n binary digits greater than or equal to q. The Lebesgue measure of B is less than or equal to

$$\sum_{n=0}^{\infty} \frac{a_n \delta}{2^n M} \sum_{j=\lceil q \cdot n \rceil}^{n} \binom{n}{j} = \delta$$

Since $a_j \to \infty$, there is a N such that for all $m > N$, $\frac{a_m \cdot \delta}{2M} > 1$. Let x be any number such that

$$\limsup_{n \to \infty} \frac{\text{number of 0's in first } n \text{ binary digits of } x}{n} \geq p$$

so by definition there is a $n > N$ such that

$$\frac{\text{number of 0's in first } n \text{ binary digits of } x}{n} \geq q$$

Then x is within $\frac{1}{2^n} < \frac{1}{2} \cdot \frac{a_n \delta}{2^n M}$ of a binary fraction with denominator 2^n that has proportion of 0's in first n binary digits greater than or equal to q, so x is in B. Since x was arbitrary, all such numbers are in B. Since δ was arbitrary, we can conclude that the set of all such numbers is of Lebesgue measure 0.