Many Cheerful Facts

Presents

Perilous Perturbatory Perihelion Precession Problem

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Perturbation theory has been commonly used in the sciences for calculation of the consequences of small effects. Unfortunately, in periodic dynamical systems, these calculations have been beset by difficulties due to the presence of secular terms in the expansion. The use of the multiple rescaling technique to avoid these problems will be illustrated by the calculation of the effect of Asaph Hall's 1894 suggested correction to Newton's gravitational law on the perihelion precession of the planet Mercury.

Introduction

According to Kepler, the planets travel in elliptical orbits around the sun. With the discovery of universal gravitation by Newton, it was realized this was only a approximation to a far more complex reality. Interaction amongst the various planets creates slight variations in the velocities of the planets as well as in the shapes of their orbits. Most of these changes are oscillatory in nature, and therefore difficult to measure accurately. An exception is the precession of the *perihelion*, the point of closest approach of a planet to the sun. This quantity changes in a cumulative fashion, enabling it to be measured to great accuracy over the course of decades and centuries.

Apply named Mercury is not only the fastest of the planets, it also has by far the most elliptic orbit, so of all the planets it displays this effect best. When detailed orbital calculations were made for it in the nineteenth century, it became evident there was an extra, anomalous precession of 43 arcseconds per century. The search for a perturbing mass in the form of either an interior planet, termed Vulcan, or a swarm of minor bodies, became a popular topic of discussion in scientific journals of the 1870's through the 1910's. When finally explained by Einstein's theory of general relativity, it became one of the classic tests of that theory.

Relegated to the position of minor comment in almost every book on relativity is Asaph Hall's 1894 suggestion of accounting for this anomalous effect by changing Newton's law of gravitation to

$$\mathbf{F} = \frac{Gm_1m_2}{r^{2+\delta}}\hat{\mathbf{r}} \tag{1}$$

where $\delta = 0.0000001612$. However, any trace of how this calculation is made is utterly lacking from these books and from the published literature.

Two-body Problem

Replicating Newton's three-and-a-half century old calculations in modern language, in Newtonian dynamics the action for two rigid, spherical bodies moving under the influence of gravity is given by

$$S = \int \mathcal{L}\,(t)\,dt$$

where the Lagrangian $\mathcal{L}$ is given by

$$\mathcal{L} = \frac{1}{2}m_1\|\dot{\mathbf{q_1}}\|^2 + \frac{1}{2}m_2\|\dot{\mathbf{q_2}}\|^2 + \frac{Gm_1m_2}{\|\mathbf{q_1} - \mathbf{q_2}\|} \tag{2}$$

Introducing new coordinates, the relative displacement, $\mathbf{q} = \mathbf{q_1} - \mathbf{q_2}$, and the center of mass, $\mathbf{Q} = \frac{m_1}{M}\mathbf{q_1} + \frac{m_2}{M}\mathbf{q_2}$, where the total mass $M = m_1 + m_2$ and the reduced mass $m = \frac{m_1m_2}{M}$, the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}M\|\dot{\mathbf{Q}}\|^2 + \frac{1}{2}m\|\dot{\mathbf{q}}\|^2 + \frac{GmM}{\|\mathbf{q}\|} \tag{3}$$

The Euler-Lagrange equations for $\mathbf{Q}$ then become

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{Q}}}\right) = 0 \Longrightarrow \ddot{\mathbf{Q}} = 0 \tag{4}$$

so by translation and Galilean boosts, it is possible to take the coordinate system such that $\mathbf{Q} = 0$ for all time. Then introducing spherical coordinates, the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}mr^2\sin^2\theta\dot{\phi}^2 + \frac{GmM}{r} \tag{5}$$

The Euler-Lagrange equation for θ is

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{\partial \mathcal{L}}{\partial \theta} \Longrightarrow \frac{d}{dt}\left(mr^2\dot{\theta}\right) = mr^2 \sin\theta \cos\theta \dot{\phi}^2 \tag{6}$$

By rotations, it is possible to take the x, y-plane containing both $\mathbf{q}$ and $\dot{\mathbf{q}}$ at any given time, so $\theta = \frac{\pi}{2}$ will solve the equation and the motion will remain in the x, y-plane for all time. Then the Euler-Lagrange equation for ϕ is

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \frac{\partial \mathcal{L}}{\partial \phi} \Longrightarrow \frac{d}{dt}\left(mr^2\dot{\phi}\right) = 0 \Longrightarrow mr^2\dot{\phi} = L \tag{7}$$

where L, the angular momentum, is a constant of the motion. This is Kepler's second law. The Euler-Lagrange equation for r is

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = \frac{\partial \mathcal{L}}{\partial r} \Longrightarrow m\ddot{r} = mr\dot{\phi}^2 - \frac{GmM}{r^2} \tag{8}$$

Substituting in from (7)

$$m\ddot{r} = \frac{L^2}{mr^3} - \frac{GmM}{r^2} \tag{9}$$

Multiplying through by $\dot{r}$ and integrating yields another constant of motion, the energy E,

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GmM}{r} \tag{10}$$

Returning to equation (9) and making the substitutions $u = \frac{1}{r}$ and

$$\frac{d}{dt} = \dot{\phi}\frac{d}{d\phi} = \frac{Lu^2}{m}\frac{d}{d\phi}$$

gives

$$\frac{d^2u}{d\phi^2} + u = \frac{Gm^2M}{L^2} \tag{11}$$

which has solution

$$u = \frac{1 + e\cos\phi}{a\left(1 - e^2\right)} \Longrightarrow r = \frac{a\left(1 - e^2\right)}{1 + e\cos\phi} \tag{12}$$

where $L^2 = a\left(1-e^2\right)Gm^2M$, an ellipse with semimajor-axis a and eccentricity e. The last remaining degree of freedom of the choice of coordinates has been employed to align the final constant of motion, the Laplace-Runge-Lenz vector, with the x-axis. This is Kepler's first law.

To get Kepler's third law, equation (7) yields

$$\int_0^{2\pi} mr^2 d\phi = \int_0^T L dt \Longrightarrow ma^2\left(1-e^2\right)^2\int_0^{2\pi}\frac{d\phi}{\left(1+e\cos\phi\right)^2} = LT \tag{13}$$

where T is the period of one orbit. Using contour methods

$$\begin{aligned}\int_0^{2\pi}\frac{d\phi}{\left(1+e\cos\phi\right)^2} &= \frac{2^2}{ie^2}\int_{|z|=1}\frac{zdz}{\left(z^2+\frac{2}{e}z+1\right)^2}\\ &= 2\pi\frac{2^2}{e^2}\frac{d}{dz}\frac{z}{\left(z^2+\frac{2}{e}z+1\right)^2}\Big|_{z=\frac{1}{e}\left(\sqrt{1-e^2}-1\right)}\\ &= \frac{2\pi}{\left(1-e^2\right)^{\frac{3}{2}}}\end{aligned}$$

The same result can also be achieved using the binomial theorem and

$$\int_0^{2\pi}\cos^j\phi d\phi = \begin{cases}0 & \text{if } j \text{ is odd}\\ 2\pi\frac{j!}{2^j\frac{j}{2}!\frac{j}{2}!} & \text{if } j \text{ is even}\end{cases}$$

which is probably how Newton did it. Either way, one has

$$T = \frac{2\pi ma^2\sqrt{1-e^2}}{L} = \frac{2\pi}{\sqrt{GM}}a^{\frac{3}{2}} \tag{14}$$

Asaph Halls Correction–First Attempt

To accommodate Asaph Hall's suggested correction to Newton's law of gravity, (1), the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}m_1\|\dot{\mathbf{q}}_\mathbf{1}\|^2 + \frac{1}{2}m_2\|\dot{\mathbf{q}}_\mathbf{2}\|^2 + \frac{Gm_1m_2}{(1+\delta)\left\|\mathbf{q_1}-\mathbf{q_2}\right\|^{1+\delta}} \tag{15}$$

The steps repeat as before until one comes to the equivalent to equation (11),

$$\frac{d^2u}{d\phi^2}+u = \frac{Gm^2M}{L^2}u^\delta = \frac{1}{a\left(1-e^2\right)}\left(a\left(1-e^2\right)u\right)^\delta \tag{16}$$

This is the "perilous" part of the problem because generations of leading mathematicians from Newton onward mistakingly believed they knew how to solve equations of this sort by expanding $\left(a\left(1-e^2\right)u\right)^\delta$ in a power series

$$\left(a\left(1-e^2\right)u\right)^\delta = 1 + \delta\log\left(a\left(1-e^2\right)u\right) + \frac{1}{2!}\delta^2\left(\log\left(a\left(1-e^2\right)u\right)\right)^2 + \cdots$$

and then iterating to create increasingly better approximations $u_0, u_1, u_2, \ldots$

$$\begin{aligned}(17)\qquad \frac{d^2u_j}{d\phi^2} + u_j = {} & \frac{1}{a\left(1-e^2\right)} + \frac{1}{a\left(1-e^2\right)}\delta\log\left(a\left(1-e^2\right)u_{j-1}\right) \\ & + \frac{1}{a\left(1-e^2\right)}\frac{1}{2!}\delta^2\left(\log\left(a\left(1-e^2\right)u_{j-2}\right)\right)^2 + \cdots\end{aligned}$$

where $u_j = \frac{1}{a(1-e^2)}$ if $j < 0$. To see why this approach fails, it is instructive to look at a simpler problem.

Model Problem

Consider the equation

$$(18)\qquad \frac{d^2y}{dx^2} + 2\varepsilon\frac{dy}{dx} + y = 0$$

with initial conditions $y(0) = a$ and $\frac{dy}{dx}|_{x=0} = b$. Following the iterative approach, the first approximation is

$$(19)\qquad \frac{d^2y_0}{dx^2} + y_0 = 0$$

with solution $y_0 = a\cos x + b\sin x$. The second approximation is

$$(20)\qquad \frac{d^2y_1}{dx^2} + y_1 = -2\varepsilon\frac{dy_0}{dx}$$

with solution

$$y_1 = y_0 + \varepsilon\left(a\sin x - ax\cos x - bx\sin x\right)$$

The third approximation is

$$(21)\qquad \frac{d^2y_2}{dx^2} + y_2 = -2\varepsilon\frac{dy_1}{dx}$$

with solution

$$y_2 = y_1 + \varepsilon^2 \left(-\frac{1}{2} a x \sin x + \frac{1}{2} a x^2 \cos x + \frac{1}{2} b \sin x - \frac{1}{2} b x \cos x + \frac{1}{2} b x^2 \sin x \right)$$

Comparing to the actual solution of the problem,

$$y = a e^{-\varepsilon x} \cos\left(\sqrt{1-\varepsilon^2}\, x\right) + \frac{b + \varepsilon a}{\sqrt{1-\varepsilon^2}} e^{-\varepsilon x} \sin\left(\sqrt{1-\varepsilon^2}\, x\right) \tag{22}$$

not only do the approximations fail due to presence of unbounded secular terms, but just as importantly, they fail to reveal the important shift in the frequency of the solutions.

The Multiple Rescaling Technique

The correct technique is to seek approximations that respect the various scales for x that occur. This can be done by introducing new variables, $X_j = \varepsilon^j x$, and looking for approximations of the following form:

$$\begin{aligned}
y_0 &= y_{00}(X_0) \\
y_1 &= y_{10}(X_0, X_1) + \varepsilon y_{11}(X_0) \\
y_2 &= y_{20}(X_0, X_1, X_2) + \varepsilon y_{21}(X_0, X_1) + \varepsilon^2 y_{22}(X_0) \\
&\vdots
\end{aligned}$$

where $y_{10}(X_0, 0) = y_{00}(X_0)$, $y_{11}(0) = 0$, $y_{20}(X_0, X_1, 0) = y_{10}(X_0, X_1)$, and so on. Trying this on the model problem, the first approximate equation becomes

$$\frac{\partial^2}{\partial X_0^2} y_{00} + y_{00} = 0 \tag{23}$$

with solution $y_{00} = a \cos X_0 + b \sin X_0$. The second equation is

$$\left(\frac{\partial^2}{\partial X_0^2} + 2\varepsilon \frac{\partial^2}{\partial X_0 \partial X_1} \right) (y_{10} + \varepsilon y_{11}) + 2\varepsilon \frac{\partial}{\partial X_0} y_{10} + y_{10} + \varepsilon y_{11} = 0 \tag{24}$$

By requiring resonant forcing terms that would create secular terms to be zero, the solutions (see appendix) are $y_{10} = a e^{-X_1} \cos X_0 + b e^{-X_1} \sin X_0$ and $y_{11} = a \sin X_0$.
The third equation is

$$(25)\qquad \left(\frac{\partial^2}{\partial X_0^2}+2\varepsilon\frac{\partial^2}{\partial X_0\partial X_1}+\varepsilon^2\frac{\partial^2}{\partial X_1^2}+2\varepsilon^2\frac{\partial^2}{\partial X_0\partial X_2}\right)\left(y_{20}+\varepsilon y_{21}+\varepsilon^2 y_{22}\right)$$

$$+2\varepsilon\left(\frac{\partial}{\partial X_0}+\varepsilon\frac{\partial}{\partial X_1}\right)(y_{20}+\varepsilon y_{21})+y_{20}+\varepsilon y_{21}+\varepsilon^2 y_{22}=0$$

Once again requiring resonant forcing terms that would create secular terms to be zero, the solutions (see appendix) are

$$y_{20}=ae^{-X_1}\cos\left(X_0-\frac{1}{2}X_2\right)+be^{-X_1}\sin\left(X_0-\frac{1}{2}X_2\right)$$

$y_{21}=ae^{-X_1}\sin X_0$, and $y_{22}=\frac{1}{2}b\sin X_0$. Not only are secular terms absent, but the shift in frequency is apparent since $1-\frac{1}{2}\varepsilon^2\approx\sqrt{1-\varepsilon^2}$.

Asaph Hall's Correction–Second Attempt

Employing the multiple rescaling approach, the first approximation is

$$(26)\qquad \frac{\partial^2}{\partial\Phi_0^2}u_{00}+u_{00}=\frac{1}{a\left(1-e^2\right)}$$

with solution

$$u_{00}=\frac{1+e\cos\Phi_0}{a\left(1-e^2\right)}$$

The second approximation is

$$(27)\qquad \left(\frac{\partial^2}{\partial\Phi_0^2}+2\delta\frac{\partial^2}{\partial\Phi_0\partial\Phi_1}\right)(u_{10}+\delta u_{11})+u_{10}+\delta u_{11}$$

$$=\frac{1}{a\left(1-e^2\right)}+\frac{1}{a\left(1-e^2\right)}\delta\log\left(a\left(1-e^2\right)u_{10}\right)$$

Looking for a solution of the form

$$u_{10}=\frac{1+A\left(\Phi_1\right)\cos\left(\Phi_0-\Psi\left(\Phi_1\right)\right)}{a\left(1-e^2\right)}$$

equation (27) becomes

$$\frac{\partial^2 u_{11}}{\partial \Phi_0^2} + u_{11} = \frac{1}{a\left(1-e^2\right)} \log\left(1 + A\cos\left(\Phi_0 - \Psi\right)\right) \tag{28}$$

$$+\frac{1}{a\left(1-e^2\right)}\left(2A' \sin\left(\Phi_0 - \Psi\right) - 2A\cos\left(\Phi_0 - \Psi\right)\Psi'\right)$$

Using

$$1 + A\cos\Theta = \frac{\left(1 + \frac{1-\sqrt{1-A^2}}{A} e^{i\Theta}\right)\left(1 + \frac{1-\sqrt{1-A^2}}{A} e^{-i\Theta}\right)}{2\frac{1-\sqrt{1-A^2}}{A^2}}$$

the resonant Fourier mode of the left-hand side of equation (28) is proportional to

$$\frac{1-\sqrt{1-A^2}}{A}\cos\left(\Phi_0 - \Psi\right) + A'\sin\left(\Phi_0 - \Psi\right) - A\cos\left(\Phi_0 - \Psi\right)\Psi' \tag{29}$$

Therefore $A' = 0$ and $\Psi' = \frac{1-\sqrt{1-A^2}}{A^2}$, so $A = e$, $\Psi = \frac{1-\sqrt{1-e^2}}{e^2}\Phi_1$, and

$$u_{10} = \frac{1 + e\cos\left(\Phi_0 - \frac{1-\sqrt{1-e^2}}{e^2}\Phi_1\right)}{a\left(1-e^2\right)}$$

Then the angle of precession is given by

$$\frac{2\pi}{1 - \frac{1-\sqrt{1-e^2}}{e^2}\delta} - 2\pi \approx 2\pi\frac{1-\sqrt{1-e^2}}{e^2}\delta$$

and for small values of e, it is approximately $\pi\delta$.

Checking this against the quoted number for δ of 0.0000001612, since Mercury revolves around the sun in 88 days, 43 arcseconds per century is 0.000000502 radians per revolution, and for eccentricity $e = .21$, $\frac{1-\sqrt{1-e^2}}{e^2} = 0.505$ which gives a value of 0.000000158 for δ, which is close.

Conclusion

The multiple rescaling technique is a fast and effective approach to the problem of calculating perturbations in periodic systems. Not only are secular terms avoided, but interesting phenomenon, such as shifts in frequency, are displayed in the first few iterations.

Appendix

The second approximate equation is

$$\left(\frac{\partial^2}{\partial X_0^2}+2\varepsilon\frac{\partial^2}{\partial X_0\partial X_1}\right)y_1+2\varepsilon\frac{\partial}{\partial X_0}y_1+y_1+\varepsilon y_1=0 \tag{30}$$

Rearranging terms by order in ε and ignoring terms of order ε^2,

$$\frac{\partial^2 y_{10}}{\partial X_0^2}+y_{10}+\varepsilon\left(\frac{\partial^2 y_{11}}{\partial X_0^2}+y_{11}+2\frac{\partial^2 y_{10}}{\partial X_0\partial X_1}+2\frac{\partial y_{10}}{\partial X_0}\right)=0 \tag{31}$$

To cancel out the leading term, take

$$y_{10}=A\left(X_1\right)\cos X_0+B\left(X_1\right)\sin X_0$$

where $A\left(0\right)=a, B\left(0\right)=b$. Then the equation becomes

$$\frac{\partial^2 y_{11}}{\partial X_0^2}+y_{11}=2A'\sin X_0-2B'\cos X_0+2A\sin X_0-2B\sin X_0 \tag{32}$$

To cancel out the resonant forcing terms that would lead to secular terms, it is necessary that $A+A'=B+B'=0$, so $A=ae^{-X_1}, B=be^{-X_1}$; then

$$\frac{\partial^2 y_{11}}{\partial X_0^2}+y_{11}=0 \tag{33}$$

and to fit the initial conditions, the solution is $y_{11}=a\sin X_0$.

The third approximate equation is

$$\left(\frac{\partial^2}{\partial X_0^2}+2\varepsilon\frac{\partial^2}{\partial X_0\partial X_1}+\varepsilon^2\frac{\partial^2}{\partial X_1^2}+2\varepsilon^2\frac{\partial^2}{\partial X_0\partial X_2}\right)y_2 \\ +2\varepsilon\left(\frac{\partial}{\partial X_0}+\varepsilon\frac{\partial}{\partial X_1}\right)y_2+y_2=0 \tag{34}$$

Rearranging terms by order in ε and ignoring terms of order ε^3,

$$\frac{\partial^2 y_{20}}{\partial X_0^2}+y_{20}+\varepsilon\left(\frac{\partial^2 y_{21}}{\partial X_0^2}+y_{21}+2\frac{\partial^2 y_{20}}{\partial X_0\partial X_1}+2\frac{\partial y_{20}}{\partial X_0}\right) \tag{35}$$

$$+\varepsilon^2\left(\frac{\partial^2 y_{22}}{\partial X_0^2}+y_{22}+2\frac{\partial^2 y_{21}}{\partial X_0\partial X_1}+2\frac{\partial y_{21}}{\partial X_0}+\frac{\partial^2 y_{20}}{\partial X_1^2}+2\frac{\partial^2 y_{20}}{\partial X_2\partial X_0}+2\frac{\partial y_{20}}{\partial X_1}\right)=0$$

To cancel out the leading terms, take

$$y_{20}=A\left(X_2\right)e^{-X_1}\cos X_0+B\left(X_2\right)e^{-X_1}\sin X_0$$

where $A\left(0\right)=a, B\left(0\right)=b$ and

$$y_{21}=C\left(X_1\right)\cos X_0+D\left(X_1\right)\sin X_0$$

where $C\left(0\right)=0, D\left(0\right)=a$. Then the equation becomes

$$\frac{\partial^2 y_{22}}{\partial X_0^2}+y_{22}=2\left(C'+C\right)\sin X_0-2\left(D'+D\right)\cos X_0 \tag{36}$$

$$+\left(2A'+B\right)e^{-X_1}\cos X_0-\left(2B'-A\right)e^{-X_1}\sin X_0$$

To cancel out the resonant forcing terms that would create secular terms, it is necessary that

$$C+C'=\left(2B'-A\right)e^{-X_1}$$
$$D+D'=-\left(2A'+B\right)e^{-X_1}$$

Now, in order to cancel out out forcing terms creating secular terms, it is necessary to take $A=2B', B=-2A'$, so

$$A=a\cos\frac{1}{2}X_2-b\sin\frac{1}{2}X_2$$

$$B=b\cos\frac{1}{2}X_2+a\sin\frac{1}{2}X_2$$

Then $C+C'=0\Rightarrow C=0$ and $D+D'=0\Rightarrow D=ae^{-X_1}$. Then equation (36) becomes

$$\frac{\partial^2 y_{22}}{\partial X_0^2}+y_{22}=0 \tag{37}$$

which, with the initial conditions, leads to

$$y_{20}=\frac{1}{2}b\sin X_0$$