

Practice Midterm 2 Solutions

1.

a) Using the chain rule

$$\frac{dy}{dx} = e^{x^3+1} \cdot 3x^2$$

b) Using the change of base formula,

$$y = \frac{\ln \cos x}{\ln x}$$

so from the quotient and chain rules,

$$\frac{dy}{dx} = \frac{\ln x \cdot \frac{1}{\cos x} (-\sin x) - \frac{1}{x} \ln \cos x}{(\ln x)^2}$$

c) Applying $\frac{d}{dx}$ to each side and using the product and chain rules

$$2y \frac{dy}{dx} + 6x \frac{dy}{dx} + 6y + 2x = 0 \Rightarrow \frac{dy}{dx} = \frac{-3y - x}{y + 3x}$$

2. Let ℓ be the distance from the origin to the point q . By the Pythagorean theorem, $\ell^2 = x^2 + y^2$. Applying $\frac{d}{dt}$ to each side and using the chain rule gives

$$2\ell \frac{d\ell}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \Rightarrow \frac{d\ell}{dt} = \frac{x}{\ell} \cdot \frac{dx}{dt} + \frac{y}{\ell} \cdot \frac{dy}{dt}$$

Since $y = x^2$, by the chain rule $\frac{dy}{dt} = 2x \frac{dx}{dt}$, so

$$\frac{d\ell}{dt} = \frac{(x + 2x^3)}{\sqrt{x^2 + x^4}} \cdot \frac{dx}{dt} = \text{sign}(x) \frac{1 + 2x}{\sqrt{1 + x^2}} \cdot \frac{dx}{dt}$$

3. Approximating \sin by its tangent line at 0,

$$\sin 1^\circ = \sin \frac{\pi}{180} \approx \sin 0 + (\cos 0) \left(\frac{\pi}{180} - 0 \right) = \frac{\pi}{180}$$

4.

a) **Method 1** Recognizing the limit as a derivative,

$$\lim_{b \rightarrow 2} \frac{b^{2010} - 2^{2010}}{b - 2} = \frac{d}{db} (b^{2010}) \Big|_{b=2} = 2010 \cdot 2^{2009}$$

Method 2 Factoring b^{2010} out of the numerator and b out of the denominator and recognizing the result as the formula for summing a geometric series

$$\frac{b^{2010} - 2^{2010}}{b - 2} = b^{2009} \frac{1 - \frac{2^{2010}}{b}}{1 - \frac{2}{b}} = b^{2009} \left(1 + \frac{2}{b} + \left(\frac{2}{b}\right)^2 + \cdots + \left(\frac{2}{b}\right)^{2009} \right)$$

so

$$\lim_{b \rightarrow 2} \frac{b^{2010} - 2^{2010}}{b - 2} = 2^{2009} (1 + \cdots + 1) = 2010 \cdot 2^{2009}$$

b) Finding the logarithm of the limit and using the continuity of the logarithm function,

$$\begin{aligned} \ln \left(\lim_{n \rightarrow \infty} \left(1 + \frac{5}{n} \right)^n \right) &= \lim_{n \rightarrow \infty} \ln \left(\left(1 + \frac{5}{n} \right)^n \right) \\ &= \lim_{n \rightarrow \infty} n \cdot \ln \left(1 + \frac{5}{n} \right) = \lim_{n \rightarrow \infty} \frac{\ln \left(1 + \frac{5}{n} \right)}{\frac{1}{n}} \end{aligned}$$

Recognizing the limit as a derivative, this is equal to

$$\left. \frac{d}{dx} \ln(1 + 5x) \right|_{x=0} = \left. \frac{5}{1 + 5x} \right|_{x=0} = 5$$

so

$$\ln \left(\lim_{n \rightarrow \infty} \left(1 + \frac{5}{n} \right)^n \right) = 5 \Rightarrow \lim_{n \rightarrow \infty} \left(1 + \frac{5}{n} \right)^n = e^5$$

c) Multiplying by the conjugate

$$\begin{aligned} \frac{x^2 - 4}{\sqrt{x} - \sqrt{4 - x}} &= \frac{x^2 - 4}{\sqrt{x} - \sqrt{4 - x}} \cdot \frac{\sqrt{x} + \sqrt{4 - x}}{\sqrt{x} + \sqrt{4 - x}} \\ &= \frac{(x - 2)(x + 2)(\sqrt{x} + \sqrt{4 - x})}{x - (4 - x)} = \frac{(x + 2)(\sqrt{x} + \sqrt{4 - x})}{2} \end{aligned}$$

so the limit is $4\sqrt{2}$.

5. Suppose there were two solutions. Then since $f(x) = x^5 - 6x + c$ is differentiable, by Rolle's theorem there is a point between the solutions where the derivative is zero. However, $f'(x) = 5x^4 - 6$ is never zero on the interval, so there is a contradiction. Therefore, there is at most one solution in the interval.

6. There is a horizontal asymptote at $y = 0$. Using the quotient rule,

$$y' = \frac{1 + x^2 - 2x^2}{(1 + x^2)^2} = \frac{1 - x^2}{(1 + x^2)^2}$$

so $\{-1, 1\}$ are the critical points. Using $-2, 0, 2$ as trial points, the function is decreasing on $(-\infty, 1) \cup (1, \infty)$ and increasing on $(-1, 1)$, so $x = -1$ is a local minimum and $x = 1$ is a local maximum. Using the quotient rule again,

$$\begin{aligned} y'' &= \frac{(1+x^2)^2(-2x) - (1-x^2) \cdot 2(1+x^2) \cdot 2x}{(1+x^2)^4} \\ &= \frac{(1+x^2)(-2x) - (1-x^2) \cdot 4x}{(1+x^2)^3} = \frac{-6x + 2x^3}{(1+x^2)^3} \end{aligned}$$

so $\{-\sqrt{3}, 0, \sqrt{3}\}$ are the inflection points. Using $-2, -1, 1, 2$ as trial points, the function is concave on $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$ and convex on $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$.

7. Using completing-the-square,

$$x^4 - 8x^2 + 17 = (x^2)^2 - 8(x^2) + 17 = (x^2 - 4)^2 + 1$$

which is always positive. Squaring is a monotonically increasing function on positives, while taking the reciprocal is a monotonically decreasing function on positives, so we need to find the minimum value of

$$x^4 - 8x^2 + 17 = (x^2 - 4)^2 + 1$$

which is clearly 1 when $x^2 - 4 = 0 \Rightarrow x = \pm 2$. Therefore the maximum value of f is $\frac{1}{1^2} = 1$.