Partial Fractions
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For any nth degree polynomial $p(x)$ with distinct roots $\{x_1, \ldots, x_n\}$, it is possible to write

$$\frac{1}{p(x)} = \frac{A_1}{x - r_1} + \frac{A_2}{x - r_2} + \cdots + \frac{A_n}{x - r_n}$$

for some constants $\{A_1, \ldots, A_n\}$. The usual way to calculate these is to put the right hand side over a common denominator, multiply out the resulting terms, and then equate the coefficients of the various powers of x, giving a system of n linear equations in the n unknowns $\{A_1, \ldots, A_n\}$. This is a lot of unnecessary work.

Instead, to find A_k, multiply through by $x - r_k$, giving

$$\frac{x - r_k}{p(x)} = \frac{A_1 (x - r_k)}{x - r_1} + \frac{A_2 (x - r_k)}{x - r_2} + \cdots + A_k + \cdots + \frac{A_n (x - r_k)}{x - r_n}$$

Taking the limit of each side as $x \to r_k$ then gives

$$\frac{1}{p'(r_k)} = A_k$$

where L'Hospital's rule has been used to evaluate the left-hand side.

This technique can be extended to polynomials with multiple roots, but the expressions become more complicated. For example, if x_1 is a root of multiplicity two, then for

$$\frac{1}{p(x)} = \frac{B}{x - r_1} + \frac{C}{(x - r_1)^2} + \frac{A_2}{x - r_2} + \cdots + \frac{A_{n-1}}{x - r_{n-1}}$$

A_k is still given by $\frac{1}{p'(r_k)}$, but C is given by

$$\lim_{x \to r_1} \frac{(x - r_1)^2}{p(x)} = \frac{2}{p''(r_1)}$$

where L'Hospital's rule has been used twice, and B is given by

$$\lim_{x \to r_1} \left(\frac{x - r_1}{p(x)} - \frac{C}{x - r_1} \right) = -\frac{2p'''(r_1)}{3p''(r_1)^2}$$

where L'Hospital's rule has been used three times and the value of C has been substituted in.