How to Do ε, δ Proofs for Continuous Functions Using the Difference Quotient

In order to prove f is continuous at $x = a$ we need to find a $\delta > 0$ for each $\varepsilon > 0$ such that $|f(x) - f(a)| < \varepsilon$ for all x satisfying $|x - a| < \delta$.

Step 1: Check to make sure the function is actually continuous at $x = a$.

Step 2: Pick a value δ_{max} such that the function does not have an infinite discontinuity in the interval $(a - \delta_{max}, a + \delta_{max})$. Setting $\delta_{max} = 1$ usually works and makes the math simpler.

Step 3: Form the difference quotient (this is the slope of the secant line through the points $(a, f(a))$ and $(x, f(x))$.)

Step 4: Find a number m that is larger than the maximum value the absolute value of the difference quotient achieves for x in the interval $(a - \delta_{max}, a + \delta_{max})$.

Step 5: The answer is given by

$$\delta = \begin{cases} \delta_{max} \cdot \dfrac{\varepsilon}{m} & \text{if } \varepsilon < m \\ \delta_{max} & \text{if } \varepsilon \geq m \end{cases}$$

Example 1: Prove $f(x) = x^3 - 2x^2 + 5$ is continuous at $x = 3$.

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Example 2: Prove $f(x) = \dfrac{1}{x-2}$ is continuous at $x = \frac{5}{2}$.

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Example 3: Why does this method fail if used to prove $f(x) = \sqrt{|x|}$ is continuous at $x = 0$? Prove this using another approach.