Fractional Derivatives
by Michael Pejic 2/19/10

Motivation

Looking at the basic definition of the second derivative of a function at x,

$$\frac{d^2 f}{dx^2}(x) = \lim_{\delta \to 0} \frac{1}{\delta} \left(\lim_{\sigma \to 0} \frac{f(x+\delta+\sigma) - f(x+\delta)}{\sigma} - \lim_{\zeta \to 0} \frac{f(x+\zeta) - f(x)}{\zeta} \right)$$

If this exists, it necessarily coincides with

$$\lim_{\delta \to 0^+} \frac{f(x+2\delta) - 2f(x+\delta) + f(x)}{\delta^2}$$

although the opposite does not always hold. (Consider a function that is 0 for $x \leq 0$ and x for $x > 0$. This is not even differentiable at $x = 0$, but the second expression exists with value 0.) Similarly, if f is n-times differentiable at x,

$$\frac{d^n f}{dx^n}(x) = \lim_{\delta \to 0^+} \frac{1}{\delta^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x+k\delta)$$

One could also go back rather than forward,

$$\frac{d^n f}{dx^n}(x) = \lim_{\delta \to 0^+} \frac{1}{\delta^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{k} f(x-k\delta)$$

Generalizing from the binomial theorem, one may attempt to define a forward fractional derivative for $\alpha \in \mathbb{C}$ by

$$D_F^\alpha f(x) = \lim_{\delta \to 0^+} \frac{1}{\delta^\alpha} \sum_{k=0}^{\infty} \frac{(-\alpha)_k}{k!} f(x+k\delta)$$

(where Pochhammer's notation is used: $(z)_0 = 1$ and for $n \in \mathbb{N}$, $(z)_n = z(z+1)(z+2)\cdots(z+n-1)$) and a backward fractional derivative by

$$D_B^\alpha f(x) = \lim_{\delta \to 0^+} \frac{1}{\delta^\alpha} \sum_{k=0}^{\infty} \frac{(-\alpha)_k}{k!} f(x-k\delta)$$

so one would expect for $n \in \mathbb{N}$

$$\frac{d^n f}{dx^n}(x) = (-1)^n D_F^n f(x) = D_B^n f(x)$$

Without taking the limit as $\delta \to 0^+$, one may investigate the properties of these for sequences, such as for long-time memory time series.

Basic Definitions and Properties

Since the above expressions are difficult to work with, a more promising approach is to note that the expressions are akin to Riemann sums. Using

$$\frac{(-\alpha)_k}{\delta^\alpha k!} = \frac{\Gamma(k-\alpha)}{\delta^\alpha \Gamma(k+1)\Gamma(-\alpha)} = \frac{\delta\left(1+O\left(\frac{1}{k}\right)\right)}{\Gamma(-\alpha)(\delta k)^{\alpha+1}}$$

motivates the definition

$$D_F^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_x^\infty \frac{f(t)\,dt\,(t)}{(t-x)^{\alpha+1}}$$

for all α in the strip $a < \Re\alpha < b$ for which the integral converges, with extension to other α by analytic continuation. Similarly,

$$D_B^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_{-\infty}^x \frac{f(t)\,dt}{(t-x)^{\alpha+1}}$$

with the corresponding condition on α. Note the relation to the Mellin transformation,

$$\mathcal{M}f(s) = \Gamma(s)\, D_F^{-s} f(0)$$

Using the uniqueness of analytic continuation then allows properties of the integral to be extended to the fractional derivative. Firstly, it commutes with scalar multiplication. If two functions, f and g, have overlapping strips for α where both $D_F^\alpha f(x)$ and $D_F^\alpha g(x)$ are defined by the integrals, then

$$D_F^\alpha (f+g)(x) = D_F^\alpha f(x) + D_F^\alpha g(x)$$

for all α for which analytic continuation is possible. The backward fractional derivative has the same property. If the integral for $D_F^\alpha f(y)$ exists for all $y \in [x,\infty)$ and the integral for $D_F^\beta (D_F^\alpha f)(x)$ exists with $\Re\alpha < 0, \Re\beta < 0$, then

$$D_F^\alpha \left(D_F^\beta f\right)(x) = D_F^\alpha \left(\frac{1}{\Gamma(-\beta)} \int_\bullet^\infty \frac{f(t)}{(t-\bullet)^{\beta+1}} dt\right)(x)$$

$$= \frac{1}{\Gamma(-\alpha)\Gamma(-\beta)} \int_x^\infty \frac{1}{(y-x)^{\alpha+1}} \left(\int_y^\infty \frac{f(t)}{(t-y)^{\beta+1}} dt \right) dy$$

$$= \frac{1}{\Gamma(-\alpha)\Gamma(-\beta)} \int_x^\infty f(t) \left(\int_x^t \frac{dy}{(y-x)^{\alpha+1}(t-y)^{\beta+1}} \right) dt$$

Letting

$$y = (t-x)u + x \Leftrightarrow u = \frac{y-x}{t-x} \Rightarrow dy = (t-x)\,du$$

this becomes

$$\frac{1}{\Gamma(-\alpha)\Gamma(-\beta)} \int_x^\infty \frac{f(t)}{(t-x)^{\alpha+\beta+1}} dt \int_0^1 \frac{du}{u^{\alpha+1}(1-u)^{\beta+1}}$$

$$= \frac{B(-\alpha,-\beta)}{\Gamma(-\alpha)\Gamma(-\beta)} \int_x^\infty \frac{f(t)}{(t-x)^{\alpha+\beta+1}} dt = \frac{1}{\Gamma(-\alpha-\beta)} \int_x^\infty \frac{f(t)}{(t-x)^{\alpha+\beta+1}} dt$$

so

$$D_F^\alpha \left(D_F^\beta f \right)(x) = D_F^{\alpha+\beta} f(x)$$

for all $\alpha, \beta, \alpha+\beta$ for which analytic continuation is possible.If β and $\alpha+\beta$ are in the strip where $D_F^\beta f(x)$ is given by the integral and $\Re\alpha < 0, \Re\beta < 0$, then clearly a similar property holds for the backward fractional derivative.

Examples

As a basic example, consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^m & \text{if } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

for $m \in \mathbb{N}$ and $a, b \in \mathbb{R}$ with $a < b$. Using repeated integration by parts for $\Re\alpha < 0$ and then analytic continuation, the backward fractional derivative, $D_B^\alpha f(x)$, is given by

$$0 \qquad \text{if } x \le a$$

$$\sum_{j=0}^m \frac{m! a^{m-j}}{(m-j)!\Gamma(j+1-\alpha)(x-a)^{\alpha-m}} \qquad \text{if } a < x \le b$$

$$\sum_{j=0}^m \frac{m! a^{m-j}}{(m-j)!\Gamma(j+1-\alpha)} \left(\frac{1}{(x-a)^{\alpha-m}} - \frac{1}{(x-b)^{\alpha-m}} \right) \quad \text{if } x > b$$

Using the binomial theorem, this gives the expected result for $\alpha = n \in \mathbb{N}$,

$$D_B^n f(x) = \begin{cases} \frac{m!}{(m-n)!} x^{m-n} & \text{if } x \in (a,b], n \in \{0,1,\ldots,m\} \\ 0 & \text{otherwise} \end{cases}$$

By linearity, this result extends to piecewise polynomial functions of compact support.

As another example, consider the function $f : \mathbb{R} \to \mathbb{C}$ given by $f(x) = e^{-\tau x}$ for some $\tau \in \mathbb{C}$ with $\Re\tau > 0$. The forward fractional derivative is given by

$$D_F^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_x^\infty \frac{e^{-\tau t}}{(t-x)^{\alpha+1}} dt = \frac{e^{-\tau x}}{\Gamma(-\alpha)} \int_0^\infty \frac{e^{-\tau s}}{s^{\alpha+1}} ds$$

which converges for $\Re\alpha < 0$. Performing the integration yields

$$D_F^\alpha f(x) = \frac{e^{-\tau x}}{\Gamma(-\alpha)} \Gamma(-\alpha)\, \tau^\alpha = \tau^\alpha e^{-\tau x}$$

which clearly agrees with the regular derivative. Also note for fixed x and τ, the result is an entire function in α.

Closely related to the previous example, consider the function $f : \mathbb{R} \to \mathbb{C}$ given by $f(x) = e^{i\omega x}$ for some $\omega \in \mathbb{R}\setminus\{0\}$. The forward fractional derivative is given by

$$D_F^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_x^\infty \frac{e^{i\omega t}}{(t-x)^{\alpha+1}} dt = \frac{e^{i\omega x}}{\Gamma(-\alpha)} \int_0^\infty \frac{e^{i\omega s}}{s^{\alpha+1}} ds$$

This never converges absolutely, but does converge conditionally for $-1 < \Re\alpha < 0$, since for $\omega > 0$, considering the contour integral about the square with vertices at 0, R, $R + iR$, iR for $R > 0$, the magnitude of the integral along the top of the square is bounded by

$$\left| e^{-\omega R} \int_0^R \frac{e^{i\omega s}}{(iR+s)^{\alpha+1}} ds \right| \leq \frac{e^{-\omega R}}{R^{\alpha+1}} \cdot R$$

which goes to 0 as $R \to \infty$. The magnitude of the integral along the right of the square is bounded by

$$\left| e^{i\omega R} \int_0^R \frac{e^{-\omega s}}{(R+is)^{\alpha+1}} i\, ds \right| \leq \frac{1 - e^{-\omega R}}{\omega R^{\alpha+1}}$$

which also goes to 0 as $R \to \infty$. Therefore,

$$\lim_{R\to\infty} \int_0^R \frac{e^{i\omega s}}{s^{\alpha+1}} ds = \lim_{R\to\infty} e^{-\frac{i\pi\alpha}{2}} \int_0^R \frac{e^{-\omega t}}{t^{\alpha+1}} dt$$

Performing the integration yields

$$D_F^\alpha f(x) = \frac{e^{i\omega x - \frac{i\pi\alpha}{2}}}{\Gamma(-\alpha)} \Gamma(-\alpha)\, \omega^\alpha = \omega^\alpha e^{i\omega x - \frac{i\pi\alpha}{2}}$$

Similarly, for $\omega < 0$, considering the contour integral about the square with vertices at 0, R, $R - iR$, $-iR$ for $R > 0$,

$$D_F^\alpha f(x) = (-\omega)^\alpha\, e^{i\omega x + \frac{i\pi\alpha}{2}}$$

so for $\omega \in \mathbb{R}\setminus\{0\}$,

$$D_F^\alpha f(x) = |\omega|^\alpha e^{i\omega x - \operatorname{sign}\omega \frac{i\pi\alpha}{2}}$$

Looking at the backward fractional derivative,

$$D_B^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_{-\infty}^x \frac{e^{i\omega t}}{(x-t)^{\alpha+1}} dt = \frac{e^{i\omega x}}{\Gamma(-\alpha)} \int_0^\infty \frac{e^{-i\omega s}}{s^{\alpha+1}} ds$$

so

$$D_B^\alpha f(x) = |\omega|^\alpha e^{i\omega x + \operatorname{sign}\omega \frac{i\pi\alpha}{2}}$$

Complex

The fractional derivative can be generalized for meromorphic functions. For such a function f, pick a point $z \in \mathbb{C}$ and a sufficiently regular path γ, excluding any poles, extending from z to infinity. Then the fractional derivative is given by

$$D_{[\gamma]}^\alpha f(z) = \frac{\Gamma(1+\alpha)}{2\pi i} \int_\Gamma \frac{f(w)}{(w-z)^{\alpha+1}} dw$$

where Γ is a counter-clockwise contour from ∞ to ∞ "hugging" γ sufficiently close to exclude any poles and with the phase of $w - z$ cut along γ. For $z = x$ on the real axis and with γ extending to the left along the real axis, this agrees with the backward fractional derivative if it exists since

$$D_{[\gamma]}^\alpha f(x) = \frac{\Gamma(1+\alpha)}{2\pi i} \int_\Gamma \frac{f(w)}{(w-z)^{\alpha+1}} dw$$

$$= \frac{\Gamma(1+\alpha)}{2\pi i}\left(e^{\imath\pi(\alpha+1)} - e^{-\imath\pi(\alpha+1)}\right)\int_{-\infty}^{x} \frac{f(t)}{e^{-\imath\pi(\alpha+1)}(x-t)^{\alpha+1}}dt$$

$$= \frac{\Gamma(1+\alpha)\sin\pi(\alpha+1)}{\pi}\int_{-\infty}^{x}\frac{f(t)}{(x-t)^{\alpha+1}}dt$$

$$= \frac{1}{\Gamma(-\alpha)}\int_{-\infty}^{x}\frac{f(t)}{(x-t)^{\alpha+1}}dt$$

using the gamma function identity

$$B(z,1-z) = \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}$$

Note that for any $\alpha = n \in \{0,1,2,\ldots\}$ for which the integral converges, the phases match along γ, so that portion of the contour does not contribute and one gets only the contribution from a loop about z, so by Cauchy's formula

$$D_{[\gamma]}^{n} f(z) = \frac{n!}{2\pi i}\int_{\substack{|w-z|=\varepsilon \\ \text{clockwise}}} \frac{f(w)}{(w-z)^{n+1}}dw$$

$$= \frac{n!}{2\pi i}\int_{\substack{|w-z|=\varepsilon \\ \text{counter-clockwise}}} \frac{f(w)}{(w-z)^{n+1}}dw = \frac{d^n f}{dz^n}(z)$$

Note the value of $D_{[\gamma]}^{\alpha} f(z)$ only depends on the homotopy class of γ since given any $\gamma_1 \sim \gamma_2$ with associated Γ_1, Γ_2, one may construct a contour Γ_3 enclosing both γ_1 and γ_2 such that

$$\int_{\Gamma_1}\frac{f(w)}{(z-w)^{\alpha+1}}dw = \int_{\Gamma_3}\frac{f(w)}{(z-w)^{\alpha+1}}dw = \int_{\Gamma_2}\frac{f(w)}{(z-w)^{\alpha+1}}dw$$

Other Real Fractional Derivatives

If both the integrals for the forward and backward derivatives exist and $\Re\alpha < 0, \Re\beta < 0, \Re(\alpha+\beta) > -1$, one may consider the following,

$$D_B^{\alpha}\left(D_F^{\beta} f\right)(x) = D_B^{\alpha}\left(\frac{1}{\Gamma(-\beta)}\int_{\bullet}^{\infty}\frac{f(t)}{(t-\bullet)^{\beta+1}}dt\right)(x)$$

$$= \frac{1}{\Gamma(-\alpha)\Gamma(-\beta)}\int_{-\infty}^{x}\frac{1}{(x-y)^{\alpha+1}}\left(\int_{y}^{\infty}\frac{f(t)}{(t-y)^{\beta+1}}dt\right)dy$$

$$= \frac{1}{\Gamma(-\alpha)\Gamma(-\beta)} \int_{-\infty}^{x} f(t) \left(\int_{-\infty}^{t} \frac{dy}{(x-y)^{\alpha+1}(t-y)^{\beta+1}} \right) dt$$

$$+ \frac{1}{\Gamma(-\alpha)\Gamma(-\beta)} \int_{x}^{\infty} f(t) \left(\int_{-\infty}^{x} \frac{dy}{(x-y)^{\alpha+1}(t-y)^{\beta+1}} \right) dt$$

Letting, for $x > t$,

$$y = \frac{t - xu}{1-u} \Leftrightarrow u = \frac{t-y}{x-y} \Rightarrow dy = -\frac{x-t}{(1-u)^2} du$$

$$\int_{-\infty}^{t} \frac{dy}{(x-y)^{\alpha+1}(t-y)^{\beta+1}} = (x-t)^{-\alpha-\beta-1} \int_0^1 (1-u)^{\alpha+\beta} u^{-\beta-1} du$$

$$= (x-t)^{-\alpha-\beta-1} B(\alpha+\beta-1, -\beta)$$

while, for $t > x$,

$$y = \frac{x - tu}{1-u} \Leftrightarrow u = \frac{x-y}{t-y} \Rightarrow dy = -\frac{t-x}{(1-u)^2} du$$

$$\int_{-\infty}^{x} \frac{dy}{(x-y)^{\alpha+1}(t-y)^{\beta+1}} = (t-x)^{-\alpha-\beta-1} \int_0^1 (1-u)^{\alpha+\beta} u^{-\alpha-1} du$$

$$= (t-x)^{-\alpha-\beta-1} B(\alpha+\beta-1, -\alpha)$$

Then

$$D_B^{\alpha}\left(D_F^{\beta} f\right)(x) = \frac{B(\alpha+\beta-1,-\beta)}{\Gamma(-\alpha)\Gamma(-\beta)} \int_{-\infty}^{x} \frac{f(t)}{(x-t)^{\alpha+\beta+1}} dt$$

$$+ \frac{B(\alpha+\beta-1,-\alpha)}{\Gamma(-\alpha)\Gamma(-\beta)} \int_{x}^{\infty} \frac{f(t)}{(x-t)^{\alpha+\beta+1}} dt$$

$$= \frac{\Gamma(\alpha+\beta-1)\Gamma(-\alpha-\beta)}{\Gamma(\alpha-1)\Gamma(-\alpha)} D_B^{\alpha+\beta} f(x)$$

$$+ \frac{\Gamma(\alpha+\beta-1)\Gamma(-\alpha-\beta)}{\Gamma(\beta-1)\Gamma(-\beta)} D_F^{\alpha+\beta} f(x)$$

$$= \frac{\sin\pi\alpha}{\sin\pi(\alpha+\beta)} D_B^{\alpha+\beta} f(x) + \frac{\sin\pi\beta}{\sin\pi(\alpha+\beta)} D_F^{\alpha+\beta} f(x)$$

Similarly,

$$D_F^{\beta}\left(D_B^{\alpha} f\right)(x)$$

$$= \frac{\sin \pi\alpha}{\sin \pi (\alpha+\beta)} D_B^{\alpha+\beta} f(x) + \frac{\sin \pi\beta}{\sin \pi (\alpha+\beta)} D_F^{\alpha+\beta} f(x)$$

so backward and forward fractional derivatives commute.

Looking for meromorphic linear combinations of forward and backward fractional derivatives that maintain themselves under composition,

$$(a(\alpha) D_B^\alpha + b(\alpha) D_F^\alpha)\left(a(\beta) D_B^\beta + b(\beta) D_F^\beta\right)$$

$$= \left(a(\alpha) a(\beta) + \frac{a(\alpha) b(\beta) \sin \pi\alpha + a(\beta) b(\alpha) \sin \pi\beta}{\sin \pi (\alpha+\beta)}\right) D_B^{\alpha+\beta}$$
$$+ \left(b(\alpha) b(\beta) + \frac{a(\beta) b(\alpha) \sin \pi\alpha + a(\alpha) b(\beta) \sin \pi\beta}{\sin \pi (\alpha+\beta)}\right) D_F^{\alpha+\beta}$$
$$= a(\alpha+\beta) D_B^{\alpha+\beta} + b(\alpha+\beta) D_F^{\alpha+\beta}$$

$$\Rightarrow \begin{cases} \sin \pi (\alpha+\beta)(a(\alpha+\beta) - a(\alpha) a(\beta)) \\ = a(\alpha) b(\beta) \sin \pi\alpha + a(\beta) b(\alpha) \sin \pi\beta \\ \\ \sin \pi (\alpha+\beta)(b(\alpha+\beta) - b(\alpha) b(\beta)) \\ = a(\beta) b(\alpha) \sin \pi\alpha + a(\alpha) b(\beta) \sin \pi\beta \end{cases}$$

One family of solutions is parametrized by $\lambda \in \mathbb{C}$ and is gotten by observing

$$\left(D_B^{\lambda\alpha} D_F^{(1-\lambda)\alpha}\right)\left(D_B^{\lambda\beta} D_F^{(1-\lambda)\beta}\right) = D_B^{\lambda\alpha} D_B^{\lambda\beta} D_F^{(1-\lambda)\alpha} D_F^{(1-\lambda)\beta} = D_B^{\lambda(\alpha+\beta)} D_F^{(1-\lambda)(\alpha+\beta)}$$

so since

$$D_\lambda^\alpha = D_B^{\lambda\alpha} D_F^{(1-\lambda)\alpha} = \frac{\sin \pi\lambda\alpha}{\sin \pi\alpha} D_B^\alpha + \frac{\sin \pi (1-\lambda)\alpha}{\sin \pi\alpha} D_F^\alpha$$

taking

$$a(\alpha) = \frac{\sin \pi\lambda\alpha}{\sin \pi\alpha} \qquad b(\alpha) = \frac{\sin \pi (1-\lambda)\alpha}{\sin \pi\alpha}$$

will work. These are the unique (up to a factor of the form η^α for some $\eta \in \mathbb{C}\backslash 0$) solutions with $a(0) = \lambda$ and $b(0) = 1 - \lambda$ finite. There are two more solutions (again up to a factor of the form η^α for some $\eta \in \mathbb{C}\backslash 0$) with $a(0)$ and $b(0)$ infinite,

$$a(\alpha) = \frac{e^{\frac{i\pi\alpha}{2}}}{2i \sin \pi\alpha} \qquad b(\alpha) = -\frac{e^{-\frac{i\pi\alpha}{2}}}{2i \sin \pi\alpha}$$

and its complex conjugate, defining D_L^α and D_R^α respectively,

$$D_L^\alpha = \frac{1}{2i\sin\pi\alpha}\left(e^{\frac{i\pi\alpha}{2}} D_B^\alpha - e^{-\frac{i\pi\alpha}{2}} D_F^\alpha\right)$$

$$D_R^\alpha = \frac{1}{2i\sin\pi\alpha}\left(-e^{\frac{-i\pi\alpha}{2}} D_B^\alpha + e^{\frac{i\pi\alpha}{2}} D_F^\alpha\right)$$

Integration by Parts

If both the integrals for the forward and backward fractional derivatives exist for functions f, g respectively and for α in a strip in $\mathbb{C}$ for almost all $x \in \mathbb{R}$, and if the integral $\int g\, D_F^\alpha f$ exists (absolutely)

$$\int_{-\infty}^{\infty} g(x)\, D_F^\alpha f(x)\, dx = \int_{-\infty}^{\infty} g(x) \frac{1}{\Gamma(-\alpha)} \int_x^{\infty} \frac{f(t)}{(t-x)^{\alpha+1}} dt\, dx$$

$$= \int_{-\infty}^{\infty} f(t) \frac{1}{\Gamma(-\alpha)} \int_{-\infty}^{t} \frac{g(x)}{(t-x)^{\alpha+1}} dx\, dt = \int_{-\infty}^{\infty} f(t)\, D_B^\alpha g(t)\, dt$$

so

$$\int_{-\infty}^{\infty} g(x)\, D_\lambda^\alpha f(x)\, dx = \int_{-\infty}^{\infty} f(t)\, D_{1-\lambda}^\alpha g(t)\, dt$$

and

$$\int_{-\infty}^{\infty} g(x)\, D_L^\alpha f(x)\, dx = \int_{-\infty}^{\infty} f(t)\, D_R^\alpha g(t)\, dt$$

Fourier Transform

If $\hat{f}$ is given by

$$\hat{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} f(t)\, dt$$

and

$$f(x) = \int_{-\infty}^{\infty} e^{-i\omega x} \hat{f}(\omega)\, d\omega$$

then for $-1 < \Re\alpha < 0$, using our previous results, we have, if the resulting integrals converge,

$$D_F^\alpha f(x) = \int_{-\infty}^{\infty} |\omega|^\alpha e^{-i\omega x + \operatorname{sign}\omega \frac{i\pi\alpha}{2}} \hat{f}(\omega)\, d\omega$$

$$D_B^\alpha f(x) = \int_{-\infty}^{\infty} |\omega|^\alpha e^{-i\omega x - \operatorname{sign}\omega \frac{i\pi\alpha}{2}} \hat{f}(\omega)\, d\omega$$

Using

$$\frac{\sin \pi\lambda\alpha}{\sin \pi\alpha} e^{-\operatorname{sign}\omega \frac{i\pi\alpha}{2}} + \frac{\sin \pi(1-\lambda)\alpha}{\sin \pi\alpha} e^{\operatorname{sign}\omega \frac{i\pi\alpha}{2}}$$

$$= \frac{\cos\frac{\pi\alpha}{2}}{\sin \pi\alpha}(\sin \pi\lambda\alpha + \sin \pi(1-\lambda)\alpha) + i \operatorname{sign}\omega \frac{\sin\frac{\pi\alpha}{2}}{\sin \pi\alpha}(\sin \pi(1-\lambda)\alpha - \sin \pi\lambda\alpha)$$

$$= \frac{\cos\frac{\pi\alpha}{2}}{\sin \pi\alpha} \cdot 2\sin\frac{\pi\alpha}{2}\cos\pi\left(\frac{1}{2}-\lambda\right)\alpha + i \operatorname{sign}\omega \frac{\sin\frac{\pi\alpha}{2}}{\sin \pi\alpha} \cdot 2\cos\frac{\pi\alpha}{2}\sin\pi\left(\frac{1}{2}-\lambda\right)\alpha$$

$$= e^{i\pi \operatorname{sign}\omega \left(\frac{1}{2}-\lambda\right)\alpha}$$

we have

$$D_\lambda^\alpha f(x) = \int_{-\infty}^{\infty} |\omega|^\alpha e^{-i\omega x + i\pi \operatorname{sign}\omega \left(\frac{1}{2}-\lambda\right)\alpha} \hat{f}(\omega)\, d\omega$$

Also, playing the role of the basic fractional derivatives in the frequency realm,

$$D_L^\alpha f(x) = \int_{-\infty}^{0} |\omega|^\alpha e^{-i\omega x} \hat{f}(\omega)\, d\omega$$

and

$$D_R^\alpha f(x) = \int_{0}^{\infty} \omega^\alpha e^{-i\omega x} \hat{f}(\omega)\, d\omega$$

Inverting the Fractional Derivative

If f is sufficiently well behaved, it is possible to invert the fractional derivative in a similar fashion to the Laplace transform. If the integral for $D_F^\alpha f(x)$ converges for α in a vertical strip, $a < \Re\alpha < b$, then consider the contour integral along the vertical line $\gamma = \{z \in \mathbb{C} : \Re z = c\}$ for some $a < c < b$ and for $y > 0$

$$\frac{1}{2\pi i}\int_\gamma D_F^\alpha f(x)\,\Gamma(-\alpha)\, y^\alpha d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_x^\infty \frac{f(t)}{(t-x)^{c+iv+1}} dt\right) y^{c+iv} dv$$

$$= \frac{1}{2\pi}\int_x^\infty f(t)\frac{y^c}{(t-x)^{c+1}}\left(\int_{-\infty}^{\infty}\left(\frac{y}{t-x}\right)^{iv} dv\right) dt$$

$$= \int_x^\infty f(t) \frac{y^c}{(t-x)^{c+1}} \delta\left(\log \frac{y}{t-x}\right) dt$$

$$= \int_x^\infty f(t) \frac{y^c}{(t-x)^{c+1}} \frac{\delta(t-x-y)}{\left|-\frac{1}{y}\right|} dt$$

$$= f(x+y)$$

For example, for $f(x) = e^{i\omega x}$ with $\omega > 0$,

$$D_F^\alpha f(x) = \omega^\alpha e^{i\omega x - \frac{i\pi\alpha}{2}}$$

so

$$\frac{1}{2\pi i} \int_\gamma D_F^\alpha f(x) \Gamma(-\alpha) y^\alpha d\alpha = \frac{1}{2\pi i} \int_\gamma \omega^\alpha e^{i\omega x - \frac{i\pi\alpha}{2}} \Gamma(-\alpha) y^\alpha d\alpha$$

$$= \frac{(\omega y)^c e^{i\omega x - \frac{i\pi c}{2}}}{2\pi} \int_{-\infty}^\infty e^{\frac{\pi v}{2}} \Gamma(-c-iv) (\omega y)^{iv} dv$$

where $y > 0$ and γ is the contour given by the vertical line with real part c between $-\frac{1}{2}$ and 0. (Note this does not include the region with real part between -1 and $-\frac{1}{2}$ where the integral for the fractional derivative still converges.) This is conditionally convergent since, for large $|v|$,

$$|\Gamma(u+iv)| = |v|^u \sqrt{\frac{\pi}{|v|}} e^{-\frac{\pi|v|}{2}} \left(1 + O\left(\frac{1}{|v|}\right)\right)$$

Now consider the clockwise contour integral about the rectangle Γ with vertices at $c-iR$, $c+iR$, $N+\frac{1}{2}+iR$, and $N+\frac{1}{2}-iR$ for $R > 0$ and $N \in \{0, 1, 2 \ldots\}$,

$$\frac{1}{2\pi i} \int_\Gamma e^{-\frac{i\pi\alpha}{2}} \Gamma(-\alpha) (\omega y)^\alpha d\alpha$$

On the top of the rectangle, the magnitude of the integral is bounded by

$$\left| \frac{e^{\frac{\pi R}{2}} (\omega y)^{iR}}{2\pi i} \int_c^{N+\frac{1}{2}} e^{-\frac{i\pi u}{2}} \Gamma(-u-iR) (\omega y)^u d\alpha \right|$$

$$\leq \frac{e^{\frac{\pi R}{2}}}{2\pi} \int_c^{N+\frac{1}{2}} |\Gamma(-u-iR)| (\omega y)^u du$$

$$= \frac{1}{2\sqrt{\pi R}} \int_c^{N+\frac{1}{2}} \left(\frac{\omega y}{R}\right)^u du \left(1 + O\left(\frac{1}{R}\right)\right)$$

$$= \frac{1}{2\sqrt{\pi R}} \frac{\left(\frac{\omega y}{R}\right)^c - \left(\frac{\omega y}{R}\right)^{N+\frac{1}{2}}}{\log \frac{\omega y}{R}} \left(1 + O\left(\frac{1}{R}\right)\right)$$

which goes to 0 as $R \to \infty$ regardless of N as long as $c \geq -\frac{1}{2}$. Similarly, on the bottom of the rectangle the magnitude of the integral is bounded by

$$\left| \frac{e^{\frac{-\pi R}{2}} (\omega y)^{-iR}}{2\pi i} \int_c^{N+\frac{1}{2}} e^{-\frac{i\pi u}{2}} \Gamma(-u + iR) (\omega y)^u du \right|$$

$$\leq \frac{e^{-\pi R}}{2\sqrt{\pi R}} \frac{\left(\frac{\omega y}{R}\right)^c - \left(\frac{\omega y}{R}\right)^{N+\frac{1}{2}}}{\log \frac{\omega y}{R}} \left(1 + O\left(\frac{1}{R}\right)\right)$$

which goes to 0 regardless of N or c. Since

$$|\Gamma(u + iv)| \leq |\Gamma(u)|$$

for $u, v \in \mathbb{R}$, the magnitude of the integral along the left of the rectangle is bounded by

$$\left| \frac{e^{-i\pi\left(\frac{N}{2}+\frac{1}{4}\right)} (\omega y)^{N+\frac{1}{2}}}{2\pi} \int_{-R}^{R} e^{\frac{\pi v}{2}} \Gamma\left(-N + \frac{1}{2} + iv\right) (\omega y)^{iv} dv \right|$$

$$\leq \frac{(\omega y)^{N+\frac{1}{2}}}{2\pi} \int_{-R}^{R} e^{\frac{\pi v}{2}} \left|\Gamma\left(-N + \frac{1}{2} + iv\right)\right| dv$$

$$\leq \frac{(\omega y)^{N+\frac{1}{2}} \left|\Gamma\left(-N + \frac{1}{2}\right)\right|}{2\pi} \int_{-R}^{R} e^{\frac{\pi v}{2}} dv = \frac{(\omega y)^{N+\frac{1}{2}}}{\pi \Gamma\left(N + \frac{1}{2}\right)} \left(e^{\frac{\pi R}{2}} - e^{-\frac{\pi R}{2}}\right)$$

which goes to 0 as $N \to \infty$ as long as R grows more slowly than $N \log N$. Therefore, by letting $R \to \infty$ and $N \to \infty$, R growing more slowly than $N \log N$,

$$\frac{1}{2\pi i} \int_\gamma e^{i\omega x - \frac{i\pi\alpha}{2}} \Gamma(-\alpha) (\omega y)^\alpha d\alpha$$

$$= -\sum_{n=0}^{\infty} \left(\text{Residue of } e^{i\omega x - \frac{i\pi\alpha}{2}} \Gamma(-\alpha) (\omega y)^\alpha \text{ at } \alpha = n\right)$$

$$= -\sum_{n=0}^{\infty} e^{i\omega x - \frac{i\pi n}{2}} (\omega y)^n \left(\text{Residue of } \Gamma(-\alpha) \text{ at } \alpha = n\right)$$

$$= -\sum_{n=0}^{\infty} e^{i\omega x - \frac{i\pi n}{2}} (\omega y)^n \frac{(-1)^{n+1}}{n!} = e^{i\omega(x+y)}$$

Fractional Second Derivative

Concentrating attention on

$$D^{\alpha}_{\frac{1}{2}} = D^{\frac{\alpha}{2}}_{B} D^{\frac{\alpha}{2}}_{F} = \frac{1}{2\cos\frac{\pi\alpha}{2}} \left(D^{\alpha}_{B} + D^{\alpha}_{F}\right)$$

note it generalizes the second derivative in the same way as the forward fractional derivative generalizes the first, so for $n \in \mathbb{N}$,

$$D^{2n}_{\frac{1}{2}} f(x) = (-1)^n \frac{d^{2n} f}{dx^{2n}}(x)$$

In terms of the integral form,

$$D^{\alpha}_{\frac{1}{2}} f(x) = \frac{1}{2\cos\frac{\pi\alpha}{2}\Gamma(-\alpha)} \int_{-\infty}^{\infty} \frac{f(t)}{|x-t|^{\alpha+1}} dt$$

and in terms of the Fourier transform,

$$D^{\alpha}_{\frac{1}{2}} f(x) = \int_{-\infty}^{\infty} |\omega|^{\alpha} e^{-i\omega x} \hat{f}(\omega)\, d\omega$$

Fractional Laplacian

In $\mathbb{R}^n$, the Laplacian is given by

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

with the solution to $-\triangle\varphi = \rho$ given by

$$\frac{\Gamma\left(\frac{n-2}{2}\right)}{4\sqrt{\pi}^n} \int_{\mathbb{R}^n} \frac{\rho(y)\,\Omega(y)}{\|x-y\|^{n-2}} \qquad \text{if } n \neq 2$$

$$-\frac{1}{2\pi}\int_{\mathbb{R}^2}\log\|x-y\|\,\rho(y)\,\Omega(y) \qquad\qquad \text{if } n=2$$

A fractional form of the Laplacian is given by

$$\begin{aligned}(-\triangle)^{\frac{\alpha}{2}}\rho(x) &= \int_{y\in\mathbb{R}^n}\frac{1}{(2\pi)^n}\int_{k\in\mathbb{R}^n} e^{\imath k\cdot(x-y)}\|k\|^{\alpha}\Omega(k)\,\rho(y)\,\Omega(y)\\ &= \frac{2^{\alpha}\Gamma\left(\frac{n+\alpha}{2}\right)}{\sqrt{\pi}^n\Gamma\left(\frac{-\alpha}{2}\right)}\int_{y\in\mathbb{R}^n}\frac{\rho(y)\,\Omega(y)}{\|x-y\|^{n+\alpha}}\end{aligned}$$

Substituting $\alpha=-2$ recovers the previous solutions for $n\neq 2$, with the solution for $n=2$ differing by the infinite constant $\frac{1}{2\pi(2+\alpha)}$.

The fractional Laplacian can also be written in terms of a forward fractional derivative,

$$(-\triangle)^{\frac{\alpha}{2}}\rho(x) = \frac{\Gamma\left(\frac{n+\alpha}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)}{2\sqrt{\pi}^{n+1}}D_F^{\alpha}\left(\int_{s\in\mathbb{S}^{n-1}}\rho(x+\bullet s)\,\Omega_{\mathbb{S}^{n-1}}(s)\right)(0)$$

For $n=1$, using

$$\int_{s\in\mathbb{S}^0}\rho(x+rs)\,\Omega_{\mathbb{S}^0}(s) = \rho(x+r)+\rho(x-r)$$

we recover $(-\triangle)^{\frac{\alpha}{2}}\rho(x) = D^{\alpha}_{\frac{1}{2}}\rho(x)$.

Fractional d'Alembertian

In Minkowski space-time ($\simeq\mathbb{R}^n$), a fractional form of the d'Alembertian

$$\Box = \frac{\partial^2}{\partial x_0^2}-\frac{\partial^2}{\partial x_1^2}-\cdots-\frac{\partial^2}{\partial x_{n-1}^2}$$

is given by

$$\begin{aligned}\Box_B^{\frac{\alpha}{2}}\rho(x) &= \int_{y\in\Lambda_-}\frac{1}{(2\pi)^n}\int_{k\in\mathbb{R}} e^{\imath\left(k_0(x_0-y_0)+\vec{k}\cdot(\vec{x}-\vec{y})\right)}\left(\vec{k}\cdot\vec{k}-k_0^2\right)^{\frac{\alpha}{2}}\Omega(k)\,\rho(y)\,\Omega(y)\\ &= \frac{2^{\alpha+1}}{\sqrt{\pi}^{n-2}\Gamma\left(\frac{2-n-\alpha}{2}\right)\Gamma\left(\frac{-\alpha}{2}\right)}\int_{y\in\Lambda_-}\frac{\rho(y)\,\Omega(y)}{(\text{interval}(x,y))^{n+\alpha}}\end{aligned}$$

where Λ_- is the interior of the past light-cone and the k_0 integral is interpreted as a contour integral in $\mathbb{C}$ below branch cuts for $\left(\vec{k}\cdot\vec{k}-k_0^2\right)^{\frac{\alpha}{2}}$. $\Box_F^{\frac{\alpha}{2}}$ can be defined similarly using the forward light-cone and a contour above the branch cuts. Note that simply plugging in $\alpha=-2$ to reconstruct a solution to $\Box\varphi=\rho$ as in the Laplacian gives

$$\frac{1}{2\sqrt{\pi}^{n-2}\Gamma\left(\frac{4-n}{2}\right)}\int_{\Lambda_-}\frac{\rho\left(y\right)\Omega\left(y\right)}{\left(\text{interval}\left(x,y\right)\right)^{n-2}}$$

which makes no sense for $n>3$; it is identically 0 for $n\in\{4,6,8,\ldots\}$ and badly divergent for $n\in\{5,7,9,\ldots\}$ on the past light-cone itslf. The correct procedure is to interpret the integral as a fractional derivative, calculate the relevant quantity for α for which the expression makes sense, and analytically continue to $\alpha=-2$.

$$\Box_B^{\frac{\alpha}{2}}\rho\left(x\right)=\frac{2^{\alpha}}{\sqrt{\pi}^{n-2}\Gamma\left(\frac{-\alpha}{2}\right)}$$

$$\times\int_0^{\infty}D_F^{\frac{n-\alpha-2}{2}}\left(\frac{1}{\sqrt{\bullet}}\int_{s\in\mathbb{S}^{n-2}}\rho\left(x_0-\sqrt{\bullet},\vec{x}+rs\right)\Omega_{\mathbb{S}^{n-2}}\left(s\right)\right)\left(r^2\right)r^{n-2}dr$$

Note that for $\alpha=-2$, one has the interesting behavior that for $n\in\{4,6,8,\ldots\}$, the result only depends on values of ρ and its derivatives on the past light-cone itself, whereas for other n, the values of ρ within the light-cone are also relevant.

Also note that by going back to the original expression, if ρ is a function only of time, then $\Box_B^{\frac{\alpha}{2}}\rho=D_B^{\alpha}\rho$, with a similar relation holding for the forward derivatives.

Dimensional Regularization

Note the way that n and α enter into the expressions for the fractional Laplacian and d'Alembertian in a complementary fashion. This explains the success of otherwise absurd dimensional regularization techniques: what is really doing, within some prefactors, is a fractional derivative.