Abstract

Log Algebraic Stacks and Moduli of Log Schemes

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In an effort to incorporate the log smooth deformation theory developed by K. Kato and F. Kato into the study of moduli, we develop a theory of log algebraic stacks and an analog of M. Artin's method for determining representability by an algebraic stack. We also study the moduli of log structures themselves and prove that there exist universal log structures in the sense of algebraic stacks. Using this result we translate results from the theory of stacks into the language of log geometry and thereby lay the foundations for the study of moduli of log schemes.

In the second half of the thesis, we apply our foundational work to the study of moduli of stable curves. We construct the Deligne-Mumford compactification of the moduli space of curves of genus $g \geq 2$ using log geometry as well as the Deligne-Rapoport compactifications of moduli stacks of elliptic curves with level structure. We use K. Kato's work on log finite flat group schemes instead of generalized elliptic curves to define level structure on degenerate elliptic curves. In the last chapter we use K. Kato and S. Usui's work on log Hodge structures to interpret the compactified moduli stack of elliptic curves from the analytic point of view.

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Acknowledgements

In the course of this work I have had conversations with a number of people and I am grateful to them all. I would especially like to thank my advisor A. Ogus, A. Abbes, J. Borger, and S. Ünver. Throughout my graduate studies I have learned a great deal from frequent conversations with J. Borger and S. Ünver. Some of these conversations have contributed directly to this thesis but many more are hidden in my understanding of foundational material necessary for this work. A. Abbes gave very valuable suggestions after reading an earlier version of this thesis. In particular, he first suggested that the stacks LOG_S of chapter 3 might be algebraic. Most of all I want to thank my advisor Professor A. Ogus whose help can be seen throughout what follows. He was the one who first suggested that I look at log algebraic stacks and has continually helped me with my work on this thesis. Most of all, however, I am grateful to him for introducing me to the wondrous subject which is modern algebraic geometry. Indeed my first exposure to the subject came in a course he taught in Fall 1997 and since then he has patiently guided me in my studies and answered innumerable questions.

Chapter 0

Introduction

The subject of this thesis is a theory of log algebraic stacks and its applications to compactification of moduli spaces. The development of such a theory is motivated by the following two phenomena:

1. Compactifying a moduli stack M in a modular fashion essentially amounts to understanding degenerations of the objects that M classifies. In many examples, these degenerate objects which are singular in the category of schemes become smooth objects in the category of log schemes when endowed with suitable log structure.

2. Compactifications of moduli spaces sometimes have a natural log structure. If $M \subset \overline{M}$ is a compactification, the substack $\overline{M} - M$ classifying singular objects is often a divisor with normal crossings and hence defines a log structure on $\overline{M}$.

Examples of these phenomena are the Deligne-Mumford compactification of the moduli space of curves of genus $g \geq 2$ ([8]) and the Deligne-Rapoport compactifications of modular curves ([9]).

An algebraic stack with a log structure defines in a natural way a stack over the category of log schemes. In this thesis we study two closely related problems about such stacks.

The first problem is to give a method for determining when a fibered category over the category of log schemes arises from an algebraic stack with a log structure. In this thesis we develop a version of M. Artin's theorem ([5], 5.3) for determining representability by a log algebraic stack.

Recall that given a stack F over the category of S-schemes (for some fixed S) M. Artin's theorem gives necessary and sufficient conditions for F to be an algebraic stack. The key ingredients in his theorem are:

1. Deformation theory of objects of F. In applications the deformations are usually controlled by some type of cohomology groups.

2. Effectivity of formal objects of F. Here the Grothendieck existence theorem ([10], III.5) almost always plays a crucial role.

Our version of M. Artin's theorem will have the same main ingredients except we work in the logarithmic category. Let S be a fixed log scheme and let $FLog_S$ be the category of fine log schemes over S with the strict étale topology (see the text for the definition). Given a stack $F/FLog_S$ we prove a theorem giving sufficient conditions for F to be equivalent to the stack defined by a log algebraic stack. The key ingredients are as follows:

1. Log deformation theory. In the applications we use the log smooth deformation theory of K. Kato and F. Kato ([18], [20]).

2. Effectivity of formal objects of F. In the log setting this usually involves an effectivity statement about certain types of formal log schemes, and hence a version of the Grothendieck existence theorem for log schemes is needed. We study versions of such a theorem in chapter 4, but the problem of effectivity of formal log schemes in general seems quite subtle.

3. Identification of special objects in F. If F is a log algebraic stack then there is a distinguished class of objects in F which are those obtained by pulling back the log structure on the stack. We assume that a characterization of these special objects is known. In section 4.3 where we prove a theorem which is useful for identifying the special objects when working with varieties with normal crossings.

For more details about our log version of Artin's theorem see 3.5.13.

The second problem discussed in this thesis is to give modular interpretations of the log algebraic stacks which arise in classical contexts. For example, which morphisms of log schemes $f : C \to X$ are obtained by pullback from the universal morphism of log schemes over the Deligne-Mumford compactification of the moduli stack of curves of genus $g \geq 2$? In this thesis we give modular interpretations of the log stacks defined by the moduli stacks of stable curves of genus $g \geq 2$ and some modular curves arising in the theory of elliptic curves.

We now discuss the contents of the individual chapters in more detail.

In chapter 1 we introduce the language of log algebraic stacks. The main technical issue in this chapter arises from the fact that we wish to work with Artin stacks and then have to consider log structures on the smooth site. We could just work with the smooth site for the rest of the thesis, but in order to conform with the more standard practice of considering log structures on the étale site we instead prove a descent theorem for log structures. This also allows us to consider stacks over the category of fine log schemes with the strict étale topology as opposed to the smooth topology.

Chapter 2 is concerned with a review of M. Artin's theorem ([5], 5.3) and a slight extension of his results to the case when the base is an algebraic stack instead of a scheme.

The third chapter discusses the moduli of log structures themselves. Fix a fine log scheme S with underlying scheme $\underline{S}$ and define a stack LOG_S over the category of $\underline{S}$-schemes as follows. If $f : \underline{X} \rightarrow \underline{S}$ is an $\underline{S}$-scheme then the fiber of LOG_S over $\underline{X}$ consists of pairs $(\mathcal{M}_X, f^b)$ where $\mathcal{M}_X$ is a fine log structure on $\underline{X}$ and

$$f^b : f^*\mathcal{M}_S \rightarrow \mathcal{M}_X$$

is a morphism of log structures. Pullback and morphisms in LOG_S are defined in the natural way. The main result of chapter 3 is that the stack LOG_S is algebraic. We have included two proofs of this result. The first is based on local constructions using charts, whereas the second is based on deformation theory and M. Artin's theorem.

Combining the results of chapter 2 with the fact that LOG_S is algebraic we obtain the version of M. Artin's theorem for log stacks discussed above.

We hope that chapter 3 might have wider applications in log geometry than just to the theory of moduli. In particular, in this chapter we explain how the theory of log smoothness and log flatness in log geometry can be viewed as special cases of the theory of smoothness and flatness for algebraic stacks. In addition, using the results of this chapter one can develop the foundations of the theory of log crystalline cohomology from a theory of crystalline cohomology over algebraic stacks ([31]). This last subject, however, will not be touched upon in this thesis.

Chapter 4 gathers together various technical facts about log structures needed in what follows. We prove two theorems which may be viewed as versions of the Grothendieck existence theorem ([10], III.5) for log structures. One of these theorems is due to K. Kato (cited in [28]) though we do not know to what extent our proof agrees with his. In this chapter we also prove a theorem which enables us to determine what the distinguished class of objects alluded to above has to be in the moduli problems we consider. More precisely we prove the following. Suppose $f : X \to S$ is a smooth, proper, integral morphism of noetherian log schemes and that $\mathcal{M}_{X/S} = Coker(f^*\mathcal{M}_S \to \mathcal{M}_X)$ is a sheaf of torsion free groups. Assume that for each geometric point $\bar{s} \to S$ the underlying scheme of the fiber $X_{\bar{s}}$ is scheme with normal crossings. Then we show that X/S is obtained by base change from a morphism $f^\sharp : X^\sharp \to S^\sharp$ of a special form whose underlying morphism of schemes is that of the original $f : X \to S$.

In chapter 5 we study the moduli of so-called stable log curves. For each $g \geq 1$ we define a stack $\mathcal{M}_g^{log}$ over the category of fine log schemes which classifies certain smooth log curves. Combining the results of chapter 4 with the log smooth deformation theory of K. Kato and F. Kato we conclude from our version of M. Artin's theorem that $\mathcal{M}_g^{log}$ is algebraic. It should be noted that the technical results of chapter 4 are foundational in nature and not specific to $\mathcal{M}_g^{log}$. Thus the fact that $\mathcal{M}_g^{log}$ is algebraic follows more or less formally from our foundational work and the log smooth deformation theory.

Once we know that $\mathcal{M}_g^{log}$ is algebraic we can compare it to the Deligne-Mumford compactification $\mathcal{M}_g$ ([8]) of the moduli space of curves of genus g (or in the case of $g = 1$ to the Deligne-Rapoport compactification [9] of the moduli stack of elliptic curves). The universal curve $\mathcal{C}/\mathcal{M}_g$ with its natural log structure coming from the divisor at infinity defines a map $\mathcal{M}_g \to \mathcal{M}_g^{log}$. By comparing the deformation theory of objects of $\mathcal{M}_g^{log}$ with the deformation theory of the underlying schemes we show that this map is an isomorphism, and hence we get a modular interpretation of the log stack defined by $\mathcal{M}_g$.

In chapter 6 we modify the method of chapter 5 to study the moduli stack of log elliptic curves with full level N structure. We do not use generalized elliptic curves but rather consider a group action to rigidify the moduli problem.

Chapter 7 gives an entirely new interpretation of the Deligne-Rapoport compactifications of the moduli stacks of elliptic curves with level structure which only involves irreducible pointed curves. This is made possible by K. Kato's construction of N-torsion points on the Kato-Tate curve as outlined in ([16]). Using Kato's work one can attach a finite flat log group scheme E_N of rank N^2 to any irreducible pointed log curve of genus 1. Using this log group scheme of N-torsion points, one can define moduli problems with level structure just as in the smooth case. In chapter 7 this is done for the stacks $\mathcal{M}_N$, $\mathcal{M}_{\Gamma_{00}(N)}$, and $\mathcal{M}_{\Gamma_0(N)}$ of ([9]) over $\mathbb{Z}[1/N]$. All these stacks turn out to be log étale over $\mathcal{M}_1$, and the ramification structure at infinity of the underlying morphisms of algebraic stacks can be explicitly seen in the structure of the log schemes E_N. Whereas the construction in chapter 6 does not seem to generalize well to other moduli problems the method of chapter 7 should carry through for any moduli problem on elliptic curves involving torsion subgroups and points. In this chapter we also use K. Kato's notion of a valuative log space (described in [16]).

In chapter 8 we study the analytic theory of log elliptic curves. The main result of the chapter is that if $S/\mathbb{C}$ is a log smooth scheme with $\underline{S}/\mathbb{C}$ smooth, then the category of log elliptic curves over S is naturally equivalent to the category of log Hodge structures on S^{log} (notation as in [29]) of rank 2 and type $\{(-1,0),(0,-1)\}$ satisfying a monodromy condition. This chapter is based on the work of K. Kato and S. Usui ([22]) on log Hodge structures.

0.1 Notation

Throughout this thesis S will denote a base log scheme whose underlying scheme is of finite type over an excellent Dedeking ring. Denote by $FLog_S$ the category of fine log scheme over S. Sometimes if the reference to S is clear we write just $FLog$.

For basic definitions and results about log schemes we refer to [20]. In our notation we will however follow that of ([29], [30]). Throughout we denote log spaces (schemes, algebraic spaces, stacks etc.) by unadorned letters (e.g. X) and spaces without log structure by an underlined letter (e.g. $\underline{X}$).

If X is a log scheme with log structure $\mathcal{M}_X$, we write the monoid law on $\mathcal{M}_X$ additively. For this reason, it becomes notationally significant to avoid identifying O_X^* with its inverse image in $\mathcal{M}_X$ under the map $\alpha : \mathcal{M}_X \to O_X$. We denote by λ the inverse to the map

$$\alpha^{-1}(O_X^*) \to O_X^*.$$

With these conventions we have for sections $a \in O_X^*, m \in \mathcal{M}_X$ the formula

$$\alpha(\lambda(a) + m) = a\alpha(m).$$

If $f : X \to Y$ is a morphism of log schemes, $\Omega^1_{X/Y}$ denotes the sheaf of logarithmic differentials and $\Omega^1_{\underline{X}/\underline{Y}}$ denotes the relative differentials of the underlying morphism of schemes. The symbol $\mathcal{M}_{X/Y}$ denotes the relative characteristic. By definition, it is the cokernel in the category of sheaves of integral monoids of the map

$$f^*\mathcal{M}_Y \to \mathcal{M}_X.$$

We also write $\overline{\mathcal{M}}_X = \mathcal{M}_X/O_X^*$ for $\mathcal{M}_{X/\underline{X}}$.

We denote by $X^\sharp$ the log scheme whose underlying scheme is $\underline{X}$ with the log structure $f^*\mathcal{M}_Y$. There is a canonical factorization of f

$$X \longrightarrow X^\sharp \xrightarrow{f^\sharp} Y.$$

Regarding monoids, we write the monoid law additively. If M is a monoid, M^* denotes the group of units in M and M^+ is the unique maximal ideal $M - M^*$. We denote the projection map $M \to M/M^*$ by q_M. A morphism $\phi : M_1 \to M_2$ of monoids is called logarithmic if the map $\phi^{-1}(M_2^*) \to M_2^*$ is bijective. ϕ is called local if $\phi^{-1}(M_2^*) = M_1^*$.

If P is a sharp monoid, we define a partial ordering on P by $p_1 \leq p_2$ if there exists $p \in P$ such that

$$p_1 + p = p_2.$$

We use the notion of algebraic space (locally of finite type) freely. For definitions and results about algebraic spaces see ([3], [25]). Throughout this paper, all algebraic spaces (and hence also schemes) will be assumed locally quasi-separated. Also we use the notion of a log algebraic space which by definition is an algebraic space A with a log structure on A_{et}.

If $\mathcal{S}$ is a ringed site we denote by $LS(\mathcal{S})$ the category of integral log structures on $\mathcal{S}$. It contains as a full subcategory the category $FLS(\mathcal{S})$ of fine log structures on $\mathcal{S}$.

If $\underline{X}$ is a scheme, we denote by $\underline{X}_{et}$ (resp. $\underline{X}_{fl}$, $\underline{X}_{sm}$) the small étale (resp. fppf, smooth) site on $\underline{X}$.

Chapter 1

Log Algebraic Stacks

1.1 Flat descent of log structures

1.1.1. The goal of this section is prove some technical results about log structures comparing the flat, smooth, and étale topologies. Some of the results of this section apply in greater generality to log structures which are not necessarily integral but on which the units O^* act freely. However, in keeping with the restrictions necessary for the results used later we restrict our attention to integral log structures.

Lemma 1.1.2. *Let $\underline{X}$ be a scheme, and let $\phi : \mathcal{N} \to \mathcal{M}$ be a morphism of log structures on $\underline{X}$. Then ϕ is an isomorphism if and only if $\overline{\phi} : \overline{\mathcal{N}} \to \overline{\mathcal{M}}$ is an isomorphism.*

Proof. The only if direction is immediate.

For the other direction, consider the diagram

$$\begin{array}{ccc} \mathcal{N} & \longrightarrow & \overline{\mathcal{N}} \\ {\scriptstyle\phi}\downarrow & & {\scriptstyle\overline{\phi}}\downarrow \\ \mathcal{M} & \longrightarrow & \overline{\mathcal{M}}. \end{array}$$

Suppose $n_1, n_2 \in \mathcal{N}$ are section such that $\phi(n_1) = \phi(n_2)$. Then

$$\overline{n}_1 = \overline{\phi}^{-1}(\phi(n_1)) = \overline{\phi}^{-1}(\phi(n_2)) = \overline{n}_2.$$

Thus $n_1 = \lambda(a) + n_2$ for some $a \in O^*_{\underline{X}}$. But then

$$\phi(\lambda(a) + n_2) = \lambda(a) + \phi(n_2) = \phi(n_2)$$

and so $a = 1$ and ϕ is injective. Surjectivity is immediate. $\square$

Lemma 1.1.3. *Let $\underline{X}$ be a scheme and $\pi : \underline{X}' \to \underline{X}$ a faithfully flat cover. Suppose $\mathcal{N}$ is a log structure on $\underline{X}$, and that $\underline{U} \to \underline{X}$ is an étale morphism. Then the sequence*

$$\mathcal{N}(\underline{U}) \to \pi^*\mathcal{N}(\underline{U} \times_{\underline{X}} \underline{X}') \rightrightarrows p_1^*\pi^*\mathcal{N}(\underline{U} \times_{\underline{X}} \underline{X}' \times_{\underline{X}} \underline{X}')$$

is exact (here $p_1^*\pi^*\mathcal{N}$ is identified with $p_2^*\pi^*\mathcal{N}$ via the canonical isomorphism).

Proof. We can without loss of generality assume that $\underline{U} = \underline{X}$. Using ([27], 1.5) we reduce to the case when $\underline{X} = Spec(A)$, $\underline{X}' = Spec(B)$ and $A \to B$ is faithfully flat.

The sheafification functor $PreSh(\underline{X}_{et}) \to Sh(\underline{X}_{et})$ is exact so it is enough to prove that the sequence is exact in the category of presheaves. Let N_1 be the presheaf on $\underline{X}'$ which to every $\underline{V} \to \underline{X}'$ associates the pushout in the category of monoids

$$\begin{array}{ccc} \alpha^{-1}(O^*_{\underline{X}'}(\underline{V})) & \longrightarrow & (\pi^{-1}\mathcal{M}_{\underline{X}})(\underline{V}) \\ \downarrow & & \\ O^*_{\underline{X}'}(\underline{V}) & & \end{array}$$

where $\alpha : \pi^{-1}\mathcal{M}_{\underline{X}} \to O_{\underline{X}'}$. By the universal property of pushout and sheafification, $\pi^*\mathcal{M}_{\underline{X}}$ is the sheafification of N_1. Similarly, for $\underline{V} \to \underline{X}' \times_{\underline{X}} \underline{X}'$, define $N_2(\underline{V})$ to be the pushout of

$$\begin{array}{ccc} \alpha^{-1}(O_{\underline{X}' \times \underline{X}'})(\underline{V}) & \longrightarrow & (p_1^{-1}\pi^{-1})\mathcal{M}_{\underline{X}}(\underline{V}) \\ \downarrow & & \\ O^*_{\underline{X}' \times \underline{X}'}(\underline{V}) & & \end{array}$$

so that $p_1^*\pi^*\mathcal{M}_{\underline{X}}$ is the sheafification of N_2. It then suffices to show that

$$\mathcal{M}_{\underline{X}}(\underline{X}) \to N_1(\underline{X}') \rightrightarrows N_2(\underline{X}' \times_{\underline{X}} \underline{X}')$$

is exact.

Since the map $A \to B$ is faithfully flat, it is injective and $A \cap B^* = A^*$ ([27], 2.17). Thus the above sequence can be written as (letting $M = \mathcal{M}_{\underline{X}}(\underline{X})$)

$$M \to M \oplus_{A^*} B^* \rightrightarrows M \oplus_{A^*} (B \otimes_A B)^*.$$

If (m, b) is in the equalizer then there exists $h \in A^*$ such that

$$(m, b \otimes 1) = (mh, 1 \otimes bh^{-1}).$$

This implies that $h = 1$ and that $b \in A \cap B^* = A^*$. $\square$

1.1.4. If $\underline{X}$ is a scheme, and if $\mathcal{M} \in LS(\underline{X}_{et})$, then we can define a presheaf of monoids $G(\mathcal{M})$ on $\underline{X}_{fl}$ by

$$G(\mathcal{M})(g : \underline{X}' \to \underline{X}) = \Gamma(\underline{X}', g^*\mathcal{M}).$$

Lemma 1.1.3 implies that this presheaf is in fact a log structure on $\underline{X}_{fl}$. In other words, we have functors

$$LS(\underline{X}_{et}) \xrightarrow{G} LS(\underline{X}_{fl}) \xrightarrow{res} LS(\underline{X}_{et})$$

and the composite functor $res \circ G$ is the identity functor.

Theorem 1.1.5. *1. Let $\pi : \underline{X}' \to \underline{X}$ be a faithfully flat locally of finite presentation morphism of schemes. Suppose we are given a log structure $\mathcal{M}$ on $\underline{X}'$ together with an isomorphism*

$$\phi : p_1^*\mathcal{M} \simeq p_2^*\mathcal{M}$$

on $\underline{X}' \times_{\underline{X}} \underline{X}'$ such that on $\underline{X}' \times_{\underline{X}} \underline{X}' \times_{\underline{X}} X'$ we have

$$p_{13}^*(\phi) = p_{23}^*(\phi) \circ p_{12}^*(\phi).$$

Then there exists unique pair $(\mathcal{N}, s)$ on $\underline{X}$ where $\mathcal{N}$ is a log structure and s is an isomorphism of log structures with descent datum on $\underline{X}'$

$$s : \pi^*\mathcal{N} \simeq \mathcal{M}.$$

2. Let $\underline{X}$ be a scheme, and let $\mathcal{M}_1 \in FLS(\underline{X}_{et})$. Then for any $\mathcal{M}_2 \in LS(\underline{X}_{et})$ the natural map

$$Hom_{\underline{X}_{et}}(\mathcal{M}_1, \mathcal{M}_2) \to Hom_{\underline{X}_{fl}}(G(\mathcal{M}_1), G(\mathcal{M}_2))$$

is bijective.

Proof. (1) The previous lemma assures uniqueness of $(\mathcal{N}, s)$ if it exists. By ([6], VIII.9.4) the sheaf $\overline{\mathcal{M}}$ descends to $\underline{X}$. Let $\underline{U} \to \underline{X}$ be an étale morphism. We identify $p_1^*\mathcal{M}$ with $p_2^*\mathcal{M}$ using ϕ and thus obtain a commutative diagram as below and we define $\mathcal{N}(\underline{U})$ to be the equalizer of the first row:

$$\begin{array}{ccccc}
\mathcal{N}(\underline{U}) & \to & \mathcal{M}(\underline{U} \times_{\underline{X}} \underline{X}') & \rightrightarrows & p_2^*\mathcal{M}(\underline{U}_{\underline{X}} \times_{\underline{X}} \underline{X}' \times_{\underline{X}} \underline{X}') \\
 & & \downarrow & & \downarrow \\
O_{\underline{X}}(\underline{U}) & \to & O_{\underline{X}}(\underline{U} \times_{\underline{X}} \underline{X}') & \rightrightarrows & O_{\underline{X}}(\underline{U}_{\underline{X}} \times_{\underline{X}} \underline{X}' \times_{\underline{X}} \underline{X}').
\end{array}$$

Then we have a natural morphism of monoids $\alpha_{\mathcal{N}} : \mathcal{N}(\underline{U}) \to O_{\underline{X}}(\underline{U})$. Moreover, both $\mathcal{N}(\underline{U}) \to \mathcal{M}(\underline{U} \times_{\underline{X}} \underline{X}')$ and $O_{\underline{X}}(\underline{U}) \to O_{\underline{X}}(\underline{U} \times_{\underline{X}} \underline{X}')$ are injective, and $O_{\underline{X}}(\underline{U})^* = O_{\underline{X}}(\underline{U} \times_{\underline{X}}$

$\underline{X}')^* \cap O_{\underline{X}}(\underline{U})$ by faithful flatness. From this it follows that $\alpha_{\mathcal{N}}^{-1}(O(\underline{U})^*) \to O_{\underline{X}}(\underline{U})^*$ is an isomorphism. Hence $\mathcal{N}$ is a log structure and comes with a natural morphism of log structures with descent datum

$$\pi^*\mathcal{N} \to \mathcal{M}.$$

We must check that $\pi^*\mathcal{N} \simeq \mathcal{M}$. Using lemma 1.1.2 it is enough to show that the induced map

$$(\pi^*\mathcal{N})/O^*_{\underline{X}'} = \pi^{-1}(\mathcal{N}/O^*_{\underline{X}}) \to \mathcal{M}/O^*_{\underline{X}}$$

is an isomorphism. To prove this it suffices to show that $\mathcal{N}/O^*_{\underline{X}}$ is the equalizer of the maps

$$\mathcal{M}/O^*_{\underline{X}'} \rightrightarrows p_2^{-1}(\mathcal{M}/O^*_{\underline{X}'})$$

since this equalizer is the sheaf obtained by descent from $\overline{\mathcal{M}}$. Let K be this equalizer. Let $\underline{U} \to \underline{X}$ be étale. We have to analyze the commutative diagram

$$\begin{array}{ccccc}
K(\underline{U}) & \to & (\mathcal{M}/O^*_{\underline{X}'})(\underline{X}' \times_{\underline{X}} \underline{U}) & \rightrightarrows & p_2^{-1}(\mathcal{M}/O^*_{\underline{X}'})((\underline{X}' \times_{\underline{X}} \underline{X}' \times_{\underline{X}} \underline{U}) \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{N}(\underline{U}) & \to & \mathcal{M}((\underline{X}' \times_{\underline{X}} \underline{U})) & \rightrightarrows & p_2^*\mathcal{M}(\underline{X}' \times_{\underline{X}} \underline{X}' \times_{\underline{X}} \underline{U}) \\
\uparrow & & \uparrow & & \uparrow \\
O^*_{\underline{X}}(\underline{U}) & \to & O^*_{\underline{X}'}(\underline{X}' \times_{\underline{X}} \underline{U}) & \rightrightarrows & p_2^*O^*_{\underline{X}'}(\underline{X}' \times_{\underline{X}} \underline{X}' \times_{\underline{X}} \underline{U}).
\end{array}$$

The map

$$(\mathcal{N}/O^*_{\underline{X}})(\underline{U}) \to K(\underline{U})$$

is injective for by faithful flatness $O_{\underline{X}}(\underline{U}) \cap O_{\underline{X}}(\underline{U} \times_{\underline{X}} \underline{X}')^* = O_{\underline{X}}(\underline{U})^*$.

The difficulty is in proving surjectivity étale locally on $\underline{X}$. Let $\bar{m} \in K(\underline{U})$. Replacing $\underline{X}$ by $\underline{U}$ we may assume that $\bar{m} \in K(\underline{X})$. Let $\mathcal{N}^{fl}$ be the sheaf on X_{fl} defined by

$$\underline{V} \mapsto Eq(\mathcal{M}(\underline{V} \times_{\underline{X}} \underline{X}') \rightrightarrows \mathcal{M}(\underline{V} \times_{\underline{X}} \underline{X}' \times_{\underline{X}} \underline{X}')).$$

$\mathcal{N}^{fl}$ is a log structure on $\underline{X}_{fl}$. Define a presheaf S on X_{fl} by

$$\underline{V} \mapsto \{\text{liftings of } \bar{m} \text{ to } \mathcal{N}^{fl}(\underline{V})\}.$$

Then S is a $\mathbb{G}_m$-pseudo-torsor on X_{fl}. We show that S is in fact a torsor.

Let $\underline{V} \to \underline{X}'$ be such that $\bar{m} \in \overline{\mathcal{M}}(\underline{V})$ lifts to $\mathcal{M}(\underline{V})$. Then $\pi_2^*\bar{m} \in \overline{\mathcal{M}}(\underline{V} \times_{\underline{X}} \underline{X}')$ also lifts to $\mathcal{M}(\underline{V} \times_{\underline{X}} \underline{X}')$. For we have a diagram

$$\underline{X}' \leftleftarrows \underline{X}' \times_{\underline{X}} \underline{X}' \leftarrow \underline{V} \times_{\underline{X}} \underline{X}'$$

from which we deduce that $\pi_1^*\bar{m} = \pi_2^*\bar{m}$ in $\overline{\mathcal{M}}(\underline{V} \times_{\underline{X}} \underline{X}')$. Now we have

$$\begin{array}{ccc} \overline{\mathcal{M}}(\underline{V} \times_{\underline{X}} \underline{X}') & \longleftarrow & \mathcal{M}(\underline{V} \times_{\underline{X}} \underline{X}') \\ {\scriptstyle \pi_1^*}\Big\uparrow & & \Big\uparrow{\scriptstyle \pi_1^*} \\ \overline{\mathcal{M}}(\underline{V}) & \longleftarrow & \mathcal{M}(\underline{V}) \end{array}$$

and hence $\pi_2^*\bar{m} = \pi_1^*\bar{m}$ lifts.

Choose a lifting $m \in \mathcal{M}(\underline{V} \times_{\underline{X}} \underline{X}')$ of $\pi_2^*\bar{m}$ and consider the diagram

$$\begin{array}{ccccc} \mathcal{M}((\underline{V} \times_{\underline{X}} \underline{X}')) & \rightrightarrows & p_2^*\mathcal{M}(\underline{V} \times_{\underline{X}} \underline{X}' \times_{\underline{X}} \underline{X}') & \substack{\rightarrow\\ \rightarrow\\ \rightarrow} & s_3^*\mathcal{M}(\underline{V} \times_{\underline{X}} (\underline{X}')^{\otimes 3}) \\ \uparrow & & \uparrow & & \uparrow \\ O_{\underline{X}'}^*(\underline{V} \times_{\underline{X}} \underline{X}') & \rightrightarrows & p_2^*O_{\underline{X}'}^*(\underline{V} \times_{\underline{X}} \underline{X}' \times_{\underline{X}} \underline{X}') & \substack{\rightarrow\\ \rightarrow\\ \rightarrow} & s_3^*O_{\underline{X}}^*(\underline{V} \times_{\underline{X}} (\underline{X}')^{\otimes 3}). \end{array}$$

Let p_1 and p_2 denote the double rightarrows in the diagram (so $p_1 = \phi \circ p_1^*$ and $p_2 = p_2^*$). Write

$$p_1(m) = \lambda(t) + p_2(m).$$

It follows from the standard simplicial identities that for all $m \in \mathcal{M}(\underline{V} \times_{\underline{X}} \underline{X}')$

$$\eta_1(p_1(m)) + \eta_2(p_2(m)) + \eta_3(p_1(m)) = \eta_1(p_2(m)) + \eta_2(p_1(m)) + \eta_3(p_2(m))$$

where the η_i are the triple arrows. From this we have

$$\eta_1(\lambda(t)) + \eta_3(\lambda(t)) + \eta_1(p_2(m)) + \eta_2(p_2(m)) + \eta_3(p_2(m)) =$$
$$\eta_2(\lambda(t)) + \eta_1(p_2(m)) + \eta_2(p_2(m)) + \eta_3(p_2(m)).$$

Since our log structures are integral, it then follows that

$$\eta_1(\lambda(t)) + \eta_3(\lambda(t)) = \eta_2(\lambda(t)).$$

Thus t defines a class in $H^1(\underline{V}, \mathbb{G}_m)$. Let $\underline{V}'/\underline{V}$ be an étale cover such that this class is zero in $H^1(\underline{V}', \mathbb{G}_m)$ and replace $\underline{V}$ by $\underline{V}'$. Then we can assume that this class is zero. Therefore, after replacing $\underline{V} \times_{\underline{X}} \underline{X}'$ by a cover $\underline{W}$ we can find $t_0 \in \mathbb{G}_m(\underline{W})$ such that the image of t in $\mathbb{G}_m(\underline{W} \times_{\underline{V}} \times \underline{W})$ can be written as $\pi_1^*(t_0^{-1})\pi_2^*(t_0)$. Then

$$-\lambda(t_0) + m \in \mathcal{M}(\underline{W})$$

maps to $\bar{m} \in \overline{\mathcal{M}}(\underline{W})$ and is in the equalizer of

$$\mathcal{M}(\underline{W}) \rightrightarrows \mathcal{M}(\underline{W} \times_{\underline{V}} \underline{W}).$$

We claim that $\underline{W} \to \underline{X}$ is the desired cover. That is, there exists $m \in \mathcal{N}^{fl}$ mapping to $\bar{m} \in K(\underline{W})$. To see this consider the diagram

$$\begin{array}{ccc} \mathcal{M}(\underline{W} \times_{\underline{X}} \underline{X}') & \rightrightarrows & \mathcal{M}((\underline{W} \times_{\underline{X}} \underline{X}') \times_{\underline{W}} (\underline{W} \times_{\underline{X}} \underline{X}')) \\ \uparrow & & \uparrow \\ \mathcal{M}(\underline{W}) & \rightrightarrows & \mathcal{M}(\underline{W} \times_{\underline{V}} \underline{W}) \end{array}$$

where the vertical arrows are induced from

$$\pi_1 : \underline{W} \times_{\underline{X}} \underline{X}' \to \underline{W}, \quad \pi_1 \times \pi_1 : (\underline{W} \times_{\underline{X}} \underline{X}') \times_{\underline{W}} (\underline{W} \times_{\underline{X}} \underline{X}') \to \underline{W} \times_{\underline{V}} \underline{W}.$$

By construction there exists $m \in \mathcal{N}^{fl}(\underline{W})$ mapping to $\pi_1^* \bar{m} \in \overline{\mathcal{M}}(\underline{W} \times_{\underline{X}} \underline{X}')$. But $\pi_1^* \bar{m} = \pi_2^* \bar{m}$ and hence m maps to the image of $\bar{m}$ in $K(\underline{W})$. We conclude that S is a torsor.

Now any $\mathbb{G}_m$-torsor in the fppf-topology can be trivialized étale locally and hence the map

$$\overline{\mathcal{N}} \to K$$

is surjective.

(2) Let G be the functor defined in 1.1.4. It follows from the formula $res \circ G = id$ that for any two log structures $\mathcal{M}_1, \mathcal{M}_2$ the natural map

$$Hom_{\underline{X}_{et}}(\mathcal{M}_1, \mathcal{M}_2) \to Hom_{\underline{X}_{fl}}(G(\mathcal{M}_1), G(\mathcal{M}_2))$$

is injective.

If $\mathcal{M}_1$ is fine, then by replacing $\underline{X}$ by an étale cover, we may assume that there exists a global chart $P \to \Gamma(\underline{X}_{et}, \mathcal{M}_1)$. It follows that any morphism $\eta : G(\mathcal{M}_1) \to G(\mathcal{M}_2)$ is uniquely determined by the induced map

$$P \to \Gamma(\underline{X}_{fl}, G(\mathcal{M}_2)) = \Gamma(\underline{X}_{et}, \mathcal{M}_2).$$

From this the second part of the theorem follows. $\square$

Corollary 1.1.6. *For any scheme $\underline{X}$, the functor G induces equivalences of categories*

$$G : FLS(\underline{X}_{et}) \to FLS(\underline{X}_{fl})$$

and

$$G : FLS(\underline{X}_{et}) \to FLS(\underline{X}_{sm}).$$

Proof. This will follow from the theorem once we show that any log structure $\mathcal{M}_X$ on $\underline{X}_{et}$ for which $G(\mathcal{M}_X)$ is a fine log structure on X_{fl} is in $FLS(X_{et})$.

Observe first that for all geometric points $\bar{x} \to \underline{X}$ the stalk $\overline{\mathcal{M}}_{X,\bar{x}}$ is a finitely generated monoid. For if

$$f : \underline{X}' \to \underline{X}$$

is an fppf-cover such that $f^*\mathcal{M}_X$ is fine and if $\bar{x}' \to \underline{X}'$ is a geometric point lying over $\bar{x}$ then

$$\overline{\mathcal{M}}_{X,\bar{x}} \simeq \overline{f^*\mathcal{M}_X}_{\bar{x}'}.$$

Therefore, in some étale neighborhood of $\bar{x}$ we can find a finitely generated monoid P and a morphism of sheaves

$$P \to \mathcal{M}_X$$

such that if $\mathcal{N}_X$ denotes the log structure associated to P then the map

$$\mathcal{N}_{X,\bar{x}} \to \mathcal{M}_{X,\bar{x}}$$

is an isomorphism.

Let $f : \underline{X}' \to \underline{X}$ be an fppf cover such that $f^*\mathcal{M}_X$ is fine and let $\bar{x}' \to \underline{X}'$ be a geometric point lying over $\bar{x}$. Then

$$f^*\mathcal{N}_{\bar{x}'} \to f^*\mathcal{M}_{X,\bar{x}'}$$

is an isomorphism, and hence by replacing $\underline{X}'$ by an étale neighborhood of $\bar{x}'$ we can assume that $f^*\mathcal{N} \to f^*\mathcal{M}_X$ is an isomorphism. It now follows from 1.1.5 that the map $\mathcal{N} \to \mathcal{M}_X$ is an isomorphism in some étale neighborhood of $\bar{x}$. □

1.2 Review of algebraic stacks

For the sake of completeness, and because conventions and definitions are not standardized, we review some of the theory of algebraic stacks. References are [8], [11], [26]. For definitions concerning 2-categories we refer to [12].

1.2.1. Let C and F be a categories and let $p : F \to C$ be a functor. For each object $X \in C$, F_X denotes the subcategory of F whose objects are $x \in Ob(F)$ such that $p(x) = X$, and whose morphisms are morphisms in F inducing the identity on X. We call $p : F \to C$ a *groupoid* if the following conditions hold:

1. For any morphism $f : X \to Y$ in C and $y \in Ob(F_Y)$ there exists an object $x \in Ob(F_X)$ together with a morphism $\tilde{f} : x \to y$ such that $p(\tilde{f}) = f$;

2. Given a diagram in C,

$$\begin{tikzcd} Y \arrow[dd, "\alpha"'] \arrow[dr, "\phi'"] & \\ & X \\ Z \arrow[ur, "\phi"'] & \end{tikzcd}$$

   and a partial lifting of the diagram to F

$$\begin{tikzcd} y \arrow[dr, "\tilde{\phi}'"] & \\ & x \\ z \arrow[ur, "\tilde{\phi}"'] & \end{tikzcd}$$

there exists a unique lifting of α to a morphism $\tilde{\alpha} : y \to z$ such that the diagram commutes.

In the presence of condition (2), the data $\tilde{f} : x \to y$ whose existence is assured by (1) is unique up to unique isomorphism. We assume from this point on that when speaking of a groupoid a *splitting* has been chosen. This means that for each f as in (1) a choice of $\tilde{f} : x \to y$ has been chosen. We denote this data by $f^*y \to y$.

1.2.2. Suppose now that our category C comes equipped with a Grothendieck topology, and that products and fiber products exist in C. We say that a groupoid $p : F \to C$ is a *pre-stack* if the following additional conditions holds:

3. If $X \in Ob(C)$, and $x, y \in F_X$, then the functor

$$C/X \to Ens$$

   given by $(f : Z \to X) \mapsto Hom_{F_Z}(f^*x, f^*y)$ is a sheaf for the topology on C. Here C/X denotes the category whose objects are morphisms $f : Z \to X$ in C and whose morphisms are commutative diagrams

$$\begin{tikzcd} Z \arrow[dd] \arrow[dr, "f"] & \\ & X \\ Z' \arrow[ur, "f'"'] & \end{tikzcd}$$

in C.

If in addition the following holds

4. If $f_i : Z_i \to X$ is a covering family then any descent datum for objects in F relative to the f_i is effective.

then F is a *stack*.

1.2.3. The collection of pre-stacks over C form the objects of a 2-category [12]. If $p_1 : F_1 \to C$ and $p_2 : F_2 \to C$ are two pre-stacks over C, then

$$Hom(F_1, F_2)$$

is the category whose objects are functors $F_1 \to F_2$ inducing the identity functor on C and whose morphisms are natural transformations of functors. The 2-category of pre-stacks contains as a full 2-sub-category the 2-category of stacks.

Lemma 1.2.4. *Let $F \to C$ be a pre-stack. Then there exists a universal pair (F^+, θ) where F^+ is a stack over C and $\theta : F \to F^+$ is a 1-morphism inducing for any stack $G \to C$ an equivalence of categories*

$$Hom(F, G) \to Hom(F^+, G).$$

Proof. F^+ is constructed as follows. If $X \in Ob(C)$, then F^+_X is the category with objects triples $(f_i : X_i \to X, x_i, h)$ where $(f_i : X_i \to X)$ is a covering family in X, $x_i \in F_{X_i}$, and for each i and j, $h_{i,j}$ is an isomorphism

$$h_{i,j} : x_j|_{X_i \times_X X_j} \simeq x_i|_{X_i \times_X X_j}$$

satisfying the standard cocycle condition.

If $(Y_{ij} \to X_i)$ is a refinement of the cover X_i then by restriction the (x_i, h) induce objects with descent datum relative to the cover $Y_{i,j} \to X$. A morphism

$$(f_i : X_i \to X, x_i, h) \to (g_j : Y_j \to X, y_j, l)$$

in F^+ is by definition an isomorphism in F of the objects with descent datum relative to the cover $(X_i \times_X Y_j \to X)$ induced by the x_i and y_j.

The map θ is defined by $x \in F_X \mapsto (id : X \to X, x, id)$. It is straightforward to check that with these definitions the pair (F^+, θ) has the correct universal properties. $\square$

1.2.5. If C is a category such that for $X \in Ob(C)$ the functor h_X associated to X by Yoneda's lemma is a sheaf, then a useful way to construct stacks is using *groupoids in* C [26]. We recall this construction. A groupoid in C consists of two objects $X^0, X^1 \in C$ together with morphisms

$$s, t : X^1 \to X^0, \epsilon : X^0 \to X^1, i : X^1 \to X^1, \mu : X^1 \times_{X^0, s, t} X^1 \to X^1$$

satisfying the following axioms:

1. $s \circ \epsilon = t \circ \epsilon = id_{X^0}$, $s \circ i = t$, $t \circ i = s$, $s \circ \mu = s \circ pr_2$, $t \circ \mu = t \circ pr_1$.
2. The two maps $X^1 \times_{X^0} X^1 \times_{X^0} X^1 \to X^1$ given by $\mu \circ (\mu \times id)$ and $\mu \circ (id \times \mu)$ agree.
3. The two maps
$$X^1 \rightrightarrows X^1 \times_{X^0} X^1 \to X^1$$
given by $\mu \circ (id \times \epsilon)$ and $\mu \circ (\epsilon \times id)$ agree.
4. The two diagrams
$$\begin{array}{ccc} X^1 & \xrightarrow{i \times id} & X^1 \times_{X^0} X^1 \\ s\downarrow & & \downarrow\mu \\ X^0 & \xrightarrow{\epsilon} & X^1 \end{array}$$
$$\begin{array}{ccc} X^1 & \xrightarrow{id \times i} & X^1 \times_{X^0} X^1 \\ t\downarrow & & \downarrow\mu \\ X^0 & \xrightarrow{\epsilon} & X^1 \end{array}$$
commute.

If $(X^0, X^1, s, t, \mu, \epsilon, i)$ is a groupoid, then define a pre-stack $\{X^0/X^1\}$ to be the fibered category over C such that for $U \in Ob(C)$, $\{X^0/X^1\}_U$ is the category whose objects are $h_{X^0}(U)$ and for which a morphism $f \to g$ is an element $\beta \in X^1(U)$ such that $s(\beta) = f$ and $t(\beta) = g$. Composition of morphisms is defined using μ. By 1.2.4, there is a canonical stack associated to $\{X^0/X^1\}$ which we denote by $[X^0/X^1]$.

1.2.6. Let C be the category of $\underline{S}$-schemes with the smooth topology. If $\underline{X}$ is an $\underline{S}$-scheme, then $\underline{X}$ defines a stack over C. The objects of this stack over $\underline{Y} \in Ob(C)$ are $\underline{S}$-morphisms $\underline{Y} \to \underline{X}$, and the morphisms in the stack are the identity morphisms. If F is another stack over C, then a 1-morphism from the stack associated to $\underline{X}$ to F is equivalent to an object of $F_{\underline{X}}$. For this reason we do not distinguish between $\underline{X}$ and the associated stack, and will often think of an object of $F_{\underline{X}}$ as a 1-morphism

$$\underline{X} \to F.$$

Similarly, an algebraic space $\underline{X}/\underline{S}$ defines a stack over C in which the only morphisms are the identity maps.

Definition 1.2.7. *We say that a stack $F \to C$ is* **representable by an algebraic space** *if it is equivalent to the stack associated to an algebraic space $\underline{X}/\underline{S}$.*

Definition 1.2.8. *Let C be the category of $\underline{S}$-schemes with the smooth topology, and let $p : F \to C$ be a stack. We say that F is an* **algebraic stack** *if the following conditions hold:*

1. *F is* **limit preserving.** *This means that for any filtering direct system of noetherian $\underline{S}$-rings $\{A_i\}$ the natural functor*

$$\varinjlim F_{Spec(A_i)} \to F_{Spec(\varinjlim A_i)}$$

is an equivalence of categories.

2. *F is* **relatively representable**. *More precisely, if $\underline{X}$ and $\underline{Y}$ are schemes locally of finite type over $\underline{S}$ and if $x \in F_X$, $y \in F_Y$, then the fibered product stack $X \times_F Y$ is representable by an algebraic space over $\underline{S}$ locally of finite type.*

3. *F* **admits a smooth cover**. *This means that there exists an $X \in C$ and $x \in F_X$ such that for every $Y \in C$ and $y \in F_Y$ the induced morphism of algebraic spaces $X \times_F Y \to Y$ is smooth and surjective.*

1.2.9. From here on, we will suppress the reference to the base category C, and simply write algebraic stack with the implicit assumption that the base category C has been given. The collection of algebraic stacks forms a 2-category. 1-morphisms and 2-morphisms between algebraic stacks are just 1-morphisms and 2-morphisms of stacks.

1.2.10. Let P be a property of morphisms of algebraic spaces which is local for the smooth topology on source and target. For example, P could be the property of being smooth, flat, surjective, locally of finite presentation, or locally of finite type. If F_1, F_2 are algebraic stacks, then a 1-morphism $f : F_1 \to F_2$ has property P if for some (and hence all) 2-commutative diagram of 1-morphisms

$$\begin{array}{ccc} X & \xrightarrow{\tilde{f}} & Y \\ \downarrow & & \downarrow \\ F_1 & \xrightarrow{f} & F_2 \end{array}$$

where $X \to F_1$ and $Y \to F_2$ are smooth and surjective, the morphism $\tilde{f}$ has property P.

1.2.11. If F is an algebraic stack, we define the smooth site F_{sm} on F as follows. Objects in F_{sm} are smooth 1-morphisms of algebraic stacks $f : G \to F$. A morphism in F_{sm}

$$(G_1, f_1) \to (G_2, f_2)$$

is a 1-morphism $\phi : G_1 \to G_2$ of algebraic stacks together with a 2-morphism $f_1 \to \phi \circ f_2$. If $(G, f) \in F_{sm}$, then a covering family of (G, f) is a collection of morphisms in F_{sm}, $(G_i, f_i) \to (G, f)$ such that the induced 1-morphism of algebraic stacks

$$\coprod G_i \to G$$

is surjective. Note that there is a base for this topology given by $f : G \to F$ where G is a scheme. Thus there is a natural pre-sheaf O_F defined on this base and by sheafification a sheaf of rings O_F on F_{sm} making the category of sheaves $Sh(F_{sm})$ a ringed topos. Given a morphism of stacks $f : F_1 \to F_2$ there is a natural induced morphism of ringed topoi $Sh(F_{1,sm}) \to Sh(F_{2,sm})$. In particular, it makes sense to speak of log structures on an algebraic stack, and the usual pullback and pushforward operations on log structures are valid in this context as well.

1.2.12. If $f : F_1 \to F_2$ is a 1-morphism between algebraic stacks, then f induces a functor

$$f^{-1} : Sh(F_{2,sm}) \to Sh(F_{1,sm}).$$

If $f, g : F_1 \to F_2$ are two 1-morphisms, and $r : f \to g$ is a 2-morphism between them, then r induces an isomorphism of functors $f^{-1} \to g^{-1}$. By chasing through the definitions, one can show that the diagram

$$\begin{tikzcd}
f^{-1}O_{F_2} \arrow[dd, "r"'] \arrow[dr] & \\
& O_{F_1} \\
g^{-1}O_{F_2} \arrow[ur] &
\end{tikzcd}$$

commutes. In particular, if $\mathcal{M}$ is a log structure on F_2, then r induces an isomorphism of log structures

$$r : f^*\mathcal{M} \simeq g^*\mathcal{M}.$$

1.2.13. If $(X^0, X^1, s, t, \epsilon, \mu, i)$ is a groupoid in algebraic spaces with s and t smooth, then the stack $[X^0/X^1]$ given by 1.2.5 is an algebraic stack. Moreover, given any algebraic stack F, choose a smooth surjective cover $X \to F$ and let $X^1 = X \times_F X$. Then there are natural maps $s, t : X^1 \to X$ and ϵ, i, μ giving a groupoid in algebraic spaces. Moreover, the natural morphism of pre-stacks $\{X/X^1\} \to F$ induces by the universal property a morphism of stacks

$$[X/X^1] \to F$$

and it is in fact an equivalence of stacks.

Remark 1.2.14. Our definition of algebraic stack is slightly more general than that found in ([26]). Their definition of algebraic stack is equivalent to an algebraic stack in the sense of 1.2.8 in which the algebraic spaces $\underline{X} \times_F \underline{Y}$ occurring in 1.2.8.2 are assume to be separated and of finite type over $\underline{X} \times_{\underline{S}} \underline{Y}$.

1.3 The 2-category of log algebraic stacks

The most direct definition of log algebraic stack is the following:

Definition 1.3.1 (First Version). *A* **log algebraic stack** *is a pair $F = (\underline{F}, \mathcal{M}_F)$ where $\underline{F}$ is an algebraic stack and $\mathcal{M}_F$ is a log structure on $\underline{F}_{sm}$. A 1-morphism of log algebraic stacks $f : (\underline{F}_1, \mathcal{M}_{F_1}) \to (\underline{F}_2, \mathcal{M}_{F_2})$ is a pair (f, f^b) consisting of a 1-morphism of stacks $f : \underline{F}_1 \to \underline{F}_2$ together with a morphism of log structures $f^b : f^*\mathcal{M}_{F_2} \to \mathcal{M}_{F_1}$ on $\underline{F}_{1,sm}$. A 2-morphism $h : (f, f^b) \to (g, g^b)$ is a natural transformation of functors $h : f \to g$ such that the induced diagram of morphisms of sheaves (see 1.2.12)*

$$\begin{tikzcd}
f^*\mathcal{M}_{F_2} \arrow[dd] \arrow[dr, "f^b"] & \\
& \mathcal{M}_{F_1} \\
g^*\mathcal{M}_{F_2} \arrow[ur, "g^b"'] &
\end{tikzcd}$$

commutes. We denote the 2-category of log algebraic stacks by $\mathcal{LAS}$.

For the study of moduli of log schemes it seems more natural to consider stacks over the category of log schemes. The rest of this section is devoted to describing log algebraic stacks in this manner.

1.3.2. Let $FLog$ be the category of fine S-log schemes. By theorem 1.1.5, there is a natural fully faithful 2-functor from the subcategory of $FLog$ of log schemes locally of finite type over S to $\mathcal{LAS}$. We denote this 2-functor by G.

1.3.3. If $F = (\underline{F}, \mathcal{M}_F) \in Ob(\mathcal{LAS})$, we define a fibered category $\eta(F) \to FLog$ as follows. For any $X \in FLog$, the category $\eta(F)_X$ is defined to be

$$Hom_{\mathcal{LAS}}(G(X), F).$$

Give $FLog$ the *strict, smooth* topology in which covering families are collections of morphisms

$$f_i : X_i \to X$$

where each f_i is strict and smooth, and such that $\coprod X_i \to X$ is surjective. It then follows tautologically that $\eta(F)$ is a stack. From this point on we will assume that $FLog$ is given this smooth topology.

1.3.4. If X is a fine log algebraic space, it follows from theorem 1.1.5.2 that X defines a sheaf on $FLog$ which we denote by h_X. For any $U \in FLog$, $h_X(U)$ is the set of morphisms of log algebraic spaces $U \to X$. By classical descent theory, morphisms of algebraic spaces glue in the smooth topology, and by 1.1.5.2 morphisms of log structures also glue.

Definition 1.3.5. *A stack $F \to FLog$ is* **representable by a log algebraic space** *if there exists a log algebraic space X and an equivalence of stacks $h_X \to F$. Note that by our conventions about algebraic spaces, $\underline{X}$ is locally of finite type.*

Definition 1.3.6. *Let $F \to Flog$ be a stack, and denote by $Aff \subset FLog$ the subcategory of fine log schemes whose underlying scheme is isomorphic to the spectrum of a noetherian ring.*

1. *F is* **limit preserving** *if F is compatible with filtering strict inverse limits in Aff. More precisely, if I is a filtering category and $h : I^{op} \to$ Aff is a functor such that for any arrow $i \to j$ in I the induced morphism of log schemes $h(j) \to h(i)$ is strict, then $\varprojlim_I h$ exists in Aff, and our requirement is that the induced functor*

$$\varinjlim_I F_{h(i)} \to F_{(\varprojlim_I h)}$$

is an equivalence of categories.

2. *F is* **relatively representable** *if for any pair of 1-morphisms $x : X \to F$, $y : Y \to F$ with X and Y log schemes locally of finite type over S, the product functor $X \times_F Y$ is representable by a log algebraic space locally of finite type over S.*

3. *If $X \in FLog$ and $x : X \to F$ is a 1-morphism, then we say that x is* **relatively strict** *if for all $Y \in FLog$ locally of finite type over S and $y : Y \to F$ the product $X \times_F Y$ is representable by a log algebraic space locally of finite type over X, and the induced morphism of log algebraic spaces*

$$X \times_F Y \to Y$$

is strict.

4. *We say that F* **admits a smooth, strict cover** *if F is relatively representable, and if there exists $X \in FLog$ and $x : X \to F$ with X locally of finite type over S such that for all $Y \in FLog$ and $y : Y \to F$ with Y locally of finite type over S the induced map of log algebraic spaces*

$$X \times_F Y \to Y$$

is smooth, strict and surjective.

5. *Denote by $\mathcal{LAS}'$ the full 2-subcategory of the 2-category of stacks over $FLog$ whose objects are limit preserving, relatively representable, and admit a smooth cover.*

1.3.7. Suppose $F \to FLog$ is a limit preserving, relatively representable stack which admits a smooth cover. Choose such a cover $X^0 \to F$ and let $X^1 = X^0 \times_F X^0$. Then as in the case of schemes, we get a groupoid in log algebraic spaces $(X^0, X^1, s, t, \epsilon, i, \mu)$ and hence an associated stack $[X^0/X^1]$ which comes with a natural morphism of stacks

$$[X^0/X^1] \to F.$$

It follows from the definition of the stack associated to a pre-stack that this is in fact an equivalence of stacks.

Theorem 1.3.8. *1. If $F \in \mathcal{LAS}$ then $\eta(F) \to FLog$ is an object of $\mathcal{LAS}'$;*

2. η induces an equivalence of 2-categories $\mathcal{LAS} \to \mathcal{LAS}'$.

Proof. (1) That $\eta(F)$ is limit preserving follows from the assumption (1.2.8) that $\underline{F}$ is limit preserving and from the following lemma:

Lemma 1.3.9. *Let $\{A_i\}$ be a filtering direct system of noetherian rings and let $A = \varinjlim A_i$. Then the natural functor*

$$\beta : \varinjlim FLS(Spec(A_i))_{sm}) \to FLS((Spec(A))_{sm})$$

is an equivalence of categories.

Proof. Let $\mathcal{M}$ be a fine log structure on $(Spec(A))_{sm}$. Choose $A \to R$ smooth and surjective such that $\mathcal{M}|_R$ admits a global chart $P \to \mathcal{M}$. Since R is of finite presentation over A there exists for some i an A_i-algebra R_i such that $Spec(R_i) \to Spec(A_i)$ is smooth and surjective together with an isomorphism

$$R_i \otimes_{A_i} A \simeq R.$$

If $i \to j$ is a morphism in our index category we let $R_j = R_i \otimes_{A_i} A_j$. Since P is a finitely presented monoid we can lift the map $P \to R$ to R_i after perhaps choosing bigger i. This gives a log structure on $Spec(R_i)$ inducing $\mathcal{M}|_R$. To descend this log structure to some A_i it suffices to lift the descent data given for $\mathcal{M}$ relative to $A \to R$ to some $A_i \to R_i$. On the other hand, morphisms of fine log structures can locally be approximated by morphisms between finitely presented monoids and by using a similar argument to the above we find that morphisms also can be approximated. $\square$

To see that $\eta(F)$ is relatively representable, let $x : X \to F$ and $y : Y \to F$ be two 1-morphisms with $X, Y \in FLog$ locally of finite type over S. If $Z \in FLog$, then giving a morphism $Z \to X \times_{\eta(F)} Y$ is equivalent to giving two morphisms of log schemes $f : Z \to X$, $g : Z \to Y$ together with a morphism of algebraic spaces $s : \underline{Z} \to \underline{X} \times_{\underline{F}} \underline{Y}$ such that the induced diagram

$$\begin{array}{ccc} f^*x^*\mathcal{M}_F \simeq g^*y^*\mathcal{M}_F & \longrightarrow & f^*\mathcal{M}_X \\ \downarrow & & \downarrow \\ g^*\mathcal{M}_Y & \longrightarrow & \mathcal{M}_Z \end{array}$$

commutes. Note that by 1.1.9.2 we can abuse notation and not distinguish between log structures on $\underline{Z}_{et}$ and log structures on $\underline{Z}_{sm}$. It follows that $X \times_{\eta(F)} Y$ is represented by

$$(\underline{X} \times_{\underline{F}} \underline{Y}, pr_1^*\mathcal{M}_X \oplus_{pr_1^* f^* x^* \mathcal{M}_F} pr_2^*\mathcal{M}_Y).$$

From this it also follows that $\eta(F)$ admits a smooth, strict surjective cover.

(2) Recall ([12]) that η being an equivalence of 2-categories means that

1. Every $T \in Ob(\mathcal{LAS}')$ is equivalent to $\eta(F)$ for some $F \in Ob(\mathcal{LAS})$.

2. For any two $F_1, F_2 \in Ob(\mathcal{LAS})$ the natural functor

   $$\beta : Hom_{\mathcal{LAS}}(F_1, F_2) \to Hom_{\mathcal{LAS}'}(\eta(F_1), \eta(F_2))$$

   is an equivalence of categories.

To verify 1, note that any $T \in Ob(\mathfrak{SL}')$ is equivalent to the stack obtained from a groupoid $(X^0, X^1, s, t, \epsilon, i, \mu)$. Forgetting the log structures defines a groupoid in algebraic spaces and hence a stack, and the log structures on X^0 and X^1 induce in a natural way a log structure on $[\underline{X}^0/\underline{X}^1]$. Applying η to the resulting object of $\mathcal{LAS}$ gives back a stack equivalent to T.

That β is an equivalence of categories also follows from looking at representations of stacks as groupoids. If $F_1, F_2 \in \mathcal{LAS}$, and if $h : \underline{F}_1 \to \underline{F}_2$, $h^b : h^*\mathcal{M}_{F_2} \to \mathcal{M}_{F_2}$ is a 1-morphism in $\mathcal{LAS}$, then one can construct smooth surjections $\underline{X}^0 \to \underline{F}_1$, and $\underline{Y}^0 \to \underline{F}_2$ such that the log structures $\mathcal{M}_{F_1}$ and $\mathcal{M}_{F_2}$ restrict to fine log structures on $\underline{X}^0_{et}$ and $\underline{Y}^0_{et}$ respectively, and morphisms of log schemes giving a model for (h, h^b)

$$h_0 : X^0 = (\underline{X}^0, \mathcal{M}_{F_1}|_{X^0}) \to (\underline{Y}^0, \mathcal{M}_{F_2}|_{Y^0}) = Y^0$$

$$h_1 : X^1 = (\underline{X}^0 \times_{F_1} \underline{X}^0, \mathcal{M}_{F_1}|_{X^0 \times_{F_1} X^0}) \to (\underline{Y}^0 \times_{F_2} \underline{Y}^0, \mathcal{M}_{F_2}|_{Y^0 \times_{F_2} Y^0}) = Y^1.$$

These morphisms satisfy $h_0 \circ s_X = s_y \circ h_1$ and $h_0 \circ t_X = t_y \circ h_1$ and in addition satisfy the axioms necessary for a morphism of groupoids in log schemes. The induced functor $\eta(h)$ is simply the morphism of stacks induced by this morphism of groupoids. From this it follows that β is faithful and essentially surjective. Given a morphism $\eta(F_1) \to \eta(F_2)$ we can choose a presentation for it as above. Then forgetting the log structure gives a morphism of stacks $h : \underline{F}_1 \to \underline{F}_2$ and on the chosen cover there is a natural morphism of log structures h^b with descent datum. In the case when we apply this to $\eta(h, h^b)$ we simply recover h and h^b.

If $T_1, T_2 \in Ob(\mathcal{LAS}')$ and if $f, g : T_1 \rightrightarrows T_2$ are two functors, then one can find a common presentation for the two functors. That is, groupoids $X^1 \rightrightarrows X^0$, $Y^1 \rightrightarrows Y^0$ defining stacks equivalent to T_1 and T_2 together with morphisms of log schemes $f_0, g_0 : X^0 \to Y^0$, $f_1, g_1 : X^1 \to Y^1$ such that (f_0, f_1) induce the functor f and (g_0, g_1) induce the functor g. A natural transformation $f \to g$ is induced from a morphism of log schemes $r : X^0 \to Y^1$ such that certain diagrams commute. Forgetting the log structures defines a 2-morphism in $\mathcal{LAS}$ which induces $f \to g$. □

Thus we have a second equivalent definition of log algebraic stack in terms of fibered categories over $FLog$.

Definition 1.3.10 (Second Version). *Let $F \to FLog$ be a limit preserving stack. Then F is a* **log algebraic stack** *if*

1. *F is relatively representable.*

2. *There exists a smooth strict cover $x : X \to F$.*

1.3.11. If $(X^0, X^1, s, t, \epsilon, i, \mu)$ is a groupoid in log algebraic spaces, then it makes sense to take the associated stack with respect to either the strict smooth or étale topology on $FLog$. However, the existence of a quasi-section of a smooth morphism ([10] IV, 17.16.3) implies that taking the associated stack with respect to either topology gives equivalent stacks. Thus we may work with the strict étale topology on $FLog$.

Definition 1.3.12 (Third Version). *Let $F \to FLog$ be a limit preserving fibered category and suppose F is a stack with respect to the strict étale topology on $FLog$. Then F is a* **log algebraic stack** *if*

1. *F is relatively representable.*

2. *There exists a smooth strict cover $x : X \to F$.*

1.3.13. Suppose $F \to FLog$ is a log algebraic stack. Then under the equivalence of 2-categories 1.3.8 there is an associated algebraic stack $\underline{F}$ over the category of schemes. This stack can be described as follows. Given F let $\tilde{F}$ be the fibered category over the category of schemes locally of finite type over $\underline{S}$ such that for a scheme $\underline{X}$, $\tilde{F}_{\underline{X}}$ is the category with objects pairs $(x, \mathcal{M}_X)$ such that $\mathcal{M}_X$ is a log structure on $\underline{X}$ and x is a 1-morphism

$$x : (\underline{X}, \mathcal{M}_X) \to F$$

which is relatively strict. A morphism $(x, \mathcal{M}_X) \to (x', \mathcal{M}'_X)$ is a morphism of schemes

$$s : \underline{X} \to \underline{(\underline{X}, \mathcal{M}_X) \times_F (\underline{X}, \mathcal{M}'_X)}$$

such that $s \circ pr_1 = s \circ pr_2 = id_{\underline{X}}$. Note that since we have only defined the notion of relatively strict 1-morphisms for log schemes locally of finite type, $\tilde{F}$ is a fibered category over the category of $\underline{S}$-schemes locally of finite type. The above arguments show that $\tilde{F}$ is in fact equivalent to $\underline{F}$ restricted to the category of schemes locally of finite type. However, since $\underline{F}$ is limit preserving, its restriction to the category of schemes locally of finite type determines $\underline{F}$ up to equivalence. The log structure on $\underline{F}$ can be recovered as the sheaf

$$(x, \mathcal{M}_X) \mapsto \Gamma(\underline{X}, \mathcal{M}_X).$$

Chapter 2

Artin's method

2.1 Review of [5]

2.1.1. Fix a scheme $\underline{S}$ locally of finite type over an excellent Dedekind ring and let F be a limit preserving stack over the category of $\underline{S}$-schemes with the étale topology. In this section we review the results of ([5]).

Definition 2.1.2. *A* **deformation situation** *is a diagram in the category of noetherian* $\underline{S}$*-rings*

$$\begin{array}{ccccc} & & B & & \\ & & \downarrow & & \\ A' & \longrightarrow & A & \longrightarrow & A_0 \end{array}$$

such that

1. A_0 *is reduced.*
2. $A' \to A$ *and* $A \to A_0$ *are surjections with nilpotent kernel.*
3. *The composite* $B \to A_0$ *is surjective.*
4. *The kernel* $I = Ker(A' \to A)$ *is an* A_0*-module of finite type.*

2.1.3. If A is a ring, $a \in F_A$, and $A' \to A$ a morphism of rings, we denote by $F_{a,A'}$ the category whose objects are pairs (a', σ) where $a' \in F_{A'}$ and σ is an isomorphism in F_A between the pullback of a' to A and a. A morphism $(a', \sigma) \to (b', \tau)$ is a morphism $\phi : a' \to b'$ in $F_{A'}$ such that $\sigma = \tau \circ \phi$. We denote by $\overline{F}_{a,A'}$ the isomorphism classes of objects in $F_{a,A'}$. The following conditions (S1) and (S2) on F will be referred to as *Schlessinger's Conditions*:

2.1.4 *(S1).* Suppose given a deformation situation and $a \in F_A$. Then the canonical functor

$$F_{a,A'\times_A B} \to F_{a,A'} \times F_{a,B}$$

is an equivalence of categories (this condition is called (S1′) in [5]).

2.1.5. If A is a ring, and I an A-module we denote by $A[I]$ the ring with underlying abelian group $A \oplus I$ and multiplication defined by

$$(a,i)(b,j) = (ab, aj + bi).$$

If (S1) holds then $\overline{F}_{a_0,A_0[I]}$ has a natural structure of an A_0-module. In addition, if $a_0 \in F_{A_0}$ we denote by $Aut_{a_0}(A_0[I])$ the set of automorphisms of a_0 restricted to $A_0[I]$ which reduce to the identity on a_0. (S1) implies that $Aut_{a_0}(A_0[I])$ also has a natural A_0-module structure.

These facts are fundamental in what follows so we explain them a little more carefully. Let $\mathcal{C}$ be the category of noetherian A-algebras R with a surjection $R \to A_0$ such that the induced map $A \to A_0$ is the given one. Morphisms in $\mathcal{C}$ are defined as morphisms of A-algebras compatible with the maps to A_0. The category $\mathcal{C}$ has fiber products defined using fiber products in the category of rings. Now the rings $A_0[I]$ with the maps $A_0[I] \to A_0$ are in a natural way A_0-module objects in $\mathcal{C}$. This amounts to saying that there is a map (addition)

$$+ : A_0[I] \times_{A_0} A_0[I] \to A_0[I], \quad (a+t, a+t') \mapsto a+t+t'$$

and for each $a_0 \in A_0$ a map (multiplication by a_0)

$$a_0 : A_0[I] \to A_0[I], \quad (a,t) \mapsto (a, a_0 t)$$

such that for all $R \in \mathcal{C}$ these maps induce a structure of an A_0-module on

$$Hom(R, A_0[I]).$$

Now suppose D is a functor

$$D : \mathcal{C} \to Set$$

such that $D(A_0)$ consists of a single element and which commutes with fiber products (in our case D is $\overline{F}_{a_0}$ or Aut_{a_0}). Then $D(A_0[I])$ has a natural A_0-module structure with addition law given by

$$D(A_0[I]) \times D(A_0[I]) \xrightarrow{\simeq} D(A_0[I] \times_{A_0} A_0[I]) \xrightarrow{D(+)} D(A_0[I])$$

and the action of an element $a_0 \in A_0$ given by

$$D(a_0) : D(A_0[I]) \to D(A_0[I]).$$

Condition (S2) is the following:

2.1.6 *(S2).* $\overline{F}_{a_0,A_0[I]}$ and $Aut_{a_0}(A_0[I])$ are A_0-modules of finite type.

2.1.7 *(Obstruction Theory).* By an *obstruction theory for* F we mean the following:

A. For each surjection $A \to A_0$ with nilpotent kernel and element $a \in F_A$, a functor

$$\mathcal{O}_a : (A_0\text{-modules of finite type}) \to (A_0\text{-modules of finite type}).$$

B. For each diagram $A' \to A \to A_0$ as in 2.1.2, and $a \in F_A$, an element $o_a(A') \in \mathcal{O}_a(I)$ such that $\overline{F}_{a,A'}$ is non-empty if and only if $o_a(A') = 0$.

$\mathcal{O}_a$ and $o_a(A')$ should be functorial and linear in I.

2.1.8. In fact the existence of an obstruction theory need only hold étale locally. More precisely, let *a local obstruction theory* for F at $a_{00} \in F_{A_{00}}$ (A_{00} reduced) be data as follows:

A. For every diagram

$$\begin{array}{ccc} a & \longrightarrow & a_0 \\ & & \uparrow \\ & & a_{00} \end{array}$$

in F lying over

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A_0 \\ & & \beta\uparrow \\ & & A_{00} \end{array}$$

where β is étale and α is surjective with nilpotent kernel, a functor

$$\mathcal{O}_a : (A_0\text{-modules of finite type}) \to (A_0\text{-modules of finite type}).$$

B. For each diagram $A' \to A \to A_0$ and $a \in F_A$ where $A \to A_0$ and a are as in (A) and $A' \to A$ is surjective with kernel I (an A_0-module of finite type), an element $o_a(A') \in \mathcal{O}_a(I)$ such that $\overline{F}_{a,A'}$ is non-empty if and only if $o_a(A') = 0$.

This data should be functorial and linear in A_0 and I.

Remark 2.1.9. The point of distinguishing between local and global obstruction theories is that the functor $\mathcal{O}_a$ in 2.1.8 may be defined in terms of some additional structure on a_{00}. The main theorem (2.1.11) requires only that local obstruction theories exist for a collection of a_{00}'s which form a base for the étale topology on elements of F. In particular, the local obstruction theories may be defined in terms of some additional structure on elements of F which exists only étale locally. This seems particularly relevant in the context of log geometry (see remark 3.5.14).

2.1.10 *(Additional Conditions).* If F is to be algebraic, the following conditions must also hold. Let $a \in F_A$ be an object with A of finite type over S, and let the subscript $A_0, B_0, etc.$ denote the objects obtained by tensor product with A_{red}.

A. Compatibility with étale localization: If $\phi : A \to B$ is étale, and if $b \in F_B$ is the image of $a \in F_A$, then

$$\overline{F}_{b_0,B_0[I\otimes B_0]} \simeq \overline{F}_{a_0,A_0[I]} \otimes B_0,$$

$$Aut_{b_0}(B_0[I \otimes B_0]) \simeq Aut_{a_0}(A_0[I]) \otimes B_0,$$

and

$$\mathcal{O}_b(I \otimes B_0) \simeq \mathcal{O}_a(I) \otimes B_0.$$

B. Effectivity of formal objects: If $\mathfrak{m} \subset A_0$ is a maximal ideal and $\hat{A}$ the completion of A_0 along $\mathfrak{m}$, then

$$\overline{F}_{a_0,A_0[I]} \otimes \hat{A} \simeq \varprojlim \overline{F}_{a_0,A_0[I/\mathfrak{m}^n I]},$$

and

$$Aut_{a_0}(A_0[I]) \otimes \hat{A} \simeq \varprojlim Aut_{a_0}(A_0[I/\mathfrak{m}^n I]).$$

C. Constructibility: There exists an open dense set of points p in $Spec(A_0)$ with $k(p)$ of finite type over $\underline{S}$ such that

$$\overline{F}_{a_0,A_0[I]} \otimes k(p) \simeq \overline{F}_{a_0,A_0[I\otimes k(p)]},$$

$$Aut_{a_0}(A_0[I]) \otimes k(p) \simeq Aut_{a_0}(A_0[I \otimes k(p)]),$$

and

$$\mathcal{O}_a(I) \otimes k(p) \subset \mathcal{O}_a(I \otimes k(p)).$$

One form of Artin's theorem is the following:

Theorem 2.1.11 (Artin, [5]). *Let F be a limit preserving stack over the category of S-schemes. Then F is an algebraic stack locally of finite type over S if and only if the following conditions hold:*

1. *The conditions (S1), (S2), and the conditions on $\overline{F}_{a_0,A_0[I]}$ and $Aut_{a_0}(A_0[I])$ in 2.1.10 hold.*

2. *If $\hat{A}$ is a complete local ring with residue field of finite type over S, then the natural functor*
$$F_{\hat{A}} \to \varprojlim F_{\hat{A}/\mathfrak{m}^n}$$
is faithful and has dense image. Recall that having dense image means that the functor
$$F_{\hat{A}} \to F_{\hat{A}/\mathfrak{m}^n}$$
is essentially surjective onto the image of
$$\varprojlim F_{\hat{A}/\mathfrak{m}^n} \to F_{\hat{A}/\mathfrak{m}^n}.$$

3. *There exists local obstruction theories $\mathcal{O}$ satisfying the conditions in 2.1.10 for a base for the étale topology on elements of F of finite type.*

4. *If A_0 is of finite type over S, $a_0 \in F_{A_0}$ an object, and ϕ an automorphism of a_0 which induces the identity in F_k for a dense set of points $A_0 \to k$ with k of finite type over S, then ϕ is the identity over some dense open subset of $Spec(A_0)$.*

Remark 2.1.12. The fact that the conditions of the theorem are also necessary follows from ([5], remark following 5.3).

Remark 2.1.13. If conditions 2.1.11.1, 2.1.11.2, and 2.1.11.4 hold then F is relatively representable (i.e. condition 1.2.8.2 holds). In fact to conclude that F is relatively representable condition 2.1.11.2 can be replaced by the condition that the functor is faithful and that given $a_1, a_2 \in F_{\hat{A}}$, $\xi_n : a_{1,n} \simeq a_{2,n}$ a compatible family of isomorphisms in $F_{\hat{A}/\mathfrak{m}^n}$, there exists for all n an isomorphism $\xi : a_1 \simeq a_2$ in $F_{\hat{A}}$ inducing ξ_n. These statement follow from the proof of the theorem. When F is known to be relatively representable condition 2.1.11.3 can be modified. In order to conclude that F is an algebraic stack it suffices to prove a statement of openness of versality: if R is a ring of finite type over S and if $v : Spec(R) \to F$ is a one morphism which is formally smooth at some point $p \in Spec(R)$ of finite type then there exists an open set containing p such that v is formally smooth everywhere in that open set. In fact, the above theorem is proved by showing that the assumptions imply relative representability and openness of versality.

2.2 A relative version of Artin's theorem.

2.2.1. Let $\underline{S}$ be as before, and let $B \to \underline{S}$ be an algebraic stack locally of finite type.

Definition 2.2.2. *A* B**-stack** *is a stack* F *over the category of* $\underline{S}$*-schemes with the étale topology together with a 1-morphism of stacks*

$$m : F \to B.$$

A morphism of B*-stacks* $F_1 \to F_2$ *is a 1-morphism of stacks* $\phi : F_1 \to F_2$ *together with an isomorphism of functors*

$$\psi : m_2 \circ \phi \simeq m_1.$$

Viewing schemes as stacks we get notions of B-schemes and morphisms of B-schemes.

2.2.3. Fix a B-stack (F, m). If X is a B-scheme define F_X as follows. Objects are morphisms of B-stacks

$$X \to F.$$

More precisely this means that objects of F_X are pairs (f, f^b) where $f : X \to F$ is a morphism of stacks and f^b is an isomorphism between the structural morphism

$$x : \underline{X} \to B$$

and $m \circ f$. A morphism

$$(f', f'^b) \to (f, f^b)$$

in F_X is an isomorphism of functors $\alpha : f' \to f$ such that the two isomorphisms

$$x \to f' \circ m$$

given by f'^b and $\alpha \circ f^b$ agree. Then F_X is a groupoid. If $h : X' \to X$ is a morphism of B-schemes then there is a notion of pullback

$$h^* : F_X \to F_{X'}$$

and so F defines a fibered category over the category of B-schemes. If $a \in F_X$ we denote by $F_{a,X'}$ the category whose objects are pairs (a', α) where $a' \in F_{X'}$ and $\alpha : h^*a' \simeq a$ is an isomorphism in F_X. Morphisms in $F_{a,X'}$ are defined to be morphisms in $F_{X'}$ which are compatible with the morphisms to a. We denote by $\overline{F}_{a,X'}$ the isomorphism classes of objects in $F_{X'}$ mapping to the isomorphism class of a in F_X.

2.2.4. The category of B-schemes has an étale topology. Covering families $\{U_i \to U\}$ are collections of morphisms of B-schemes $U_i \to U$ such that the underlying family of morphisms of schemes $\{\underline{U}_i \to \underline{U}\}$ is an étale cover of $\underline{U}$. The condition that F is a stack with respect to the étale topology on $\underline{S}$-schemes is equivalent to the statement that the fibered category over the category of B-schemes is a stack with respect to this topology.

2.2.5. By a B-ring $\mathcal{A}$ we mean a ring A together with a structure of a B-scheme on $Spec(A)$. It follows from the fact that B is limit preserving that the stack F is limit preserving if and only if for all filtering inductive limits of B-rings $\{\mathcal{A}_i\}$ the natural functor

$$\varinjlim F_{\mathcal{A}_i} \to F_{\varinjlim \mathcal{A}_i}$$

is an equivalence of categories.

2.2.6. Since B is an algebraic stack, for any deformation situation as in 2.1.2 and $a \in B_A$ the functor

$$B_{a,A' \times_A B} \to B_{a,A'} \times B_{a,B}$$

is an equivalence of categories. It follows that products exist in the category of B-rings.

Definition 2.2.7. *A* **B-deformation situation** *is a diagram in the category of B-rings*

$$\begin{array}{ccccc} & & \mathcal{B} & & \\ & & \downarrow & & \\ \mathcal{A}' & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{A}_0 \end{array}$$

such that the underlying diagram of rings is a deformation situation in the sense of 2.1.2.

The following condition will be referred to as $(S1^B)$:

2.2.8 *$(S1^B)$.* Given a B-deformation situation as above and $a \in F_{\mathcal{A}}$ the natural functor

$$F_{a,\mathcal{A}' \times_{\mathcal{A}} \mathcal{B}} \to F_{a,\mathcal{A}'} \times F_{a,\mathcal{B}}$$

is an equivalence of categories.

2.2.9. If $\mathcal{A}$ is a B-ring as above, and if I is an A_0-module of finite type then the natural retraction

$$Spec(A[I]) \to Spec(A)$$

gives $A[I]$ the structure of a B-ring which we denote by $\mathcal{A}[I]$. With this definition

$$\mathcal{A}[I] \times_{\mathcal{A}} \mathcal{A}[I] \simeq \mathcal{A}[I \oplus I]$$

and so just as in 2.1.5 $\mathcal{A}[I]$ has the structure of an A_0-module object in the category of B-rings. It therefore follows from $(S1^B)$ that given $a_0 \in F_{\mathcal{A}_0}$ the isomorphism classes of objects in $F_{a_0, \mathcal{A}_0[I]}$ has a natural structure of an A_0-module. We denote this module by $E_{a_0}(I)$. Similarly, if $Aut_{a_0}(\mathcal{A}_0[I])$ denotes the set of automorphisms of $a_0|_{\mathcal{A}_0[I]}$ which induce the identity on a_0 then $Aut_{a_0}(\mathcal{A}_0[I])$ also has a natural structure of an A_0-module. 2.1.6 in this context is the following:

2.2.10 *$(S2^B)$.* $E_{a_0}(I)$ and $Aut_{a_0}(\mathcal{A}_0[I])$ are A_0-modules of finite type.

Conditions $(S1^B)$ and $(S2^B)$ will be referred to as *Schlessinger's conditions.*

2.2.11. By an **obstruction theory** for $F \to B$ we mean the following:

A. For every morphism of B-rings $\mathcal{A} \to \mathcal{A}_0$ as in 2.2.7 and $a \in F_{\mathcal{A}}$ a functor

$$\mathcal{O}_a : (A_0\text{-modules of finite type}) \to (A_0\text{-modules of finite type}).$$

B. For every diagram $\mathcal{A}' \to \mathcal{A} \to \mathcal{A}_0$ as in 2.2.7 an element $o_a(\mathcal{A}') \in \mathcal{O}_a(I)$ which is zero if and only if there exists a lifting of a to $\mathcal{A}'$.

C. This data is to be functorial and linear in $\mathcal{A}_0$ and I.

2.2.12 *(Additional Conditions).* The modules E_{a_0}, $\mathcal{O}_a(I)$, and $Aut_{a_0}(\mathcal{A}_0[I])$ are required to satisfy the following additional conditions:

A. Compatibility with étale localization: With notation as above and as in 2.1.10, if $\mathcal{A} \to \mathcal{C}$ is an étale morphism of B-rings (i.e. $A \to C$ is étale) then for any $a \in F_{\mathcal{A}}$ with image $c \in F_{\mathcal{C}}$

$$E_{c_0}(I \otimes C_0) \simeq E_{a_0}(I) \otimes C_0,$$

$$Aut_{c_0}(\mathcal{C}_0[I \otimes C_0]) \simeq Aut_{a_0}(\mathcal{A}_0[I]) \otimes C_0,$$

and

$$\mathcal{O}_c(I \otimes C_0) \simeq \mathcal{O}_a(I) \otimes C_0.$$

B. Effectivity of formal objects: If $\mathfrak{m} \subset A_0$ is a maximal ideal and $\hat{A}$ the completion of A_0 along $\mathfrak{m}$, then

$$E_{a_0}(I) \otimes \hat{A} \simeq \varprojlim E_{a_0}(I/\mathfrak{m}^n I),$$

$$Aut_{a_0}(\mathcal{A}_0[I]) \otimes \hat{A} \simeq \varprojlim Aut_{a_0}(\mathcal{A}_0[I/\mathfrak{m}^n I]),$$

and

$$\mathcal{O}_{a_0}(I) \otimes \hat{A} \hookrightarrow \varprojlim O_{a_0}(I/\mathfrak{m}^n I).$$

C. Constructibility: There exists an open dense set of points $p \in Spec(A_0)$ with $k(p)$ of finite type over $\underline{S}$ such that

$$E_{a_0}(I) \otimes k(p) \simeq E_{a_0}(I \otimes k(p)),$$

$$Aut_{a_0}(\mathcal{A}_0[I]) \otimes k(p) \simeq Aut_{a_0}(\mathcal{A}_0[I \otimes k(p)]),$$

and

$$\mathcal{O}_a(I) \otimes k(p) \subset \mathcal{O}_a(I \otimes k(p)).$$

Theorem 2.2.13. *Let $F \to B$ be a limit preserving B-stack, and suppose an obstruction theory $\mathcal{O}$ is given as above. Then F is an algebraic stack if the following conditions hold:*

1. *The conditions $(S1^B), (S2^B)$, and the conditions on $E_{a_0}(I)$ and $Aut_{a_0}(\mathcal{A}_0[I])$ in 2.2.12 hold.*

2. *If $\hat{\mathcal{A}}$ is a complete local B-ring with residue field of finite type over $\underline{S}$, then the natural functor*

$$F_{\hat{\mathcal{A}}} \to \varprojlim F_{\hat{\mathcal{A}}/\mathfrak{m}^n}$$

is faithful and has dense image.

3. *The obstruction theory $\mathcal{O}$ satisfies the conditions in 2.2.12. In addition, for every reduced noetherian ring A_0, $a_0 \in F_{A_0}$ and I an A_0-module of finite type the map*

$$B_{m(a_0), A_0[I]} \to \mathcal{O}_{a_0}(I)$$

which sends a lifting b of $m(a_0)$ to $A_0[I]$ to the obstruction $o_{a_0}((A_0[I], b)) \in \mathcal{O}_{a_0}(I)$ is an A_0-module homomorphism. Here $(A_0[I], b)$ denotes the B-ring induced by b which comes with a natural morphism of B-rings

$$(A_0[I], b) \to \mathcal{A}_0$$

where $\mathcal{A}_0$ denotes the B-ring obtained from A_0 via $m(a_0)$.

4. *If $\mathcal{A}_0$ is of finite type over $\underline{S}$, $a_0 \in F_{\mathcal{A}_0}$ an object, and ϕ an automorphism of a_0 which induces the identity in F_k (k viewed as a B-ring) for a dense set of points $A_0 \to k$ with k of finite type over $\underline{S}$, then ϕ is the identity over some dense open subset of $Spec(A_0)$.*

Proof. Let $\underline{B}' \to B$ be a smooth cover over B where $\underline{B}'$ is a scheme locally of finite type over $\underline{S}$. To prove the theorem, it suffices to show that F is relatively representable and that the stack $F' = F \times_B \underline{B}'$ is an algebraic stack. For if $\underline{T} \to F'$ is a smooth surjective cover then for any $\underline{X} \to F$ all the arrows in

$$\underline{T}' \times_{F'} (F' \times_F \underline{X}) \to F' \times_F \underline{X} \simeq \underline{B}' \times_B \underline{X} \to \underline{X}$$

are smooth surjective morphisms of schemes. Hence $\underline{T}'$ also gives a smooth surjective cover of F.

To prove that F is relatively representable it suffices by remark 2.1.13 to prove that conditions 2.1.11.1, 2.1.11.4, and the weakened version of 2.1.11.2 discussed in 2.1.13 hold for F. Let us begin by verifying condition $(S1)$. Given a deformation situation (see 2.1.2, except we write C for B so as not to confuse the notation) and $a \in F_A$ we have to show that

$$F_{a,A' \times_A C} \to F_{a,A'} \times F_{a,C}$$

is an equivalence of categories. Suppose given $a' \in F_{a,A'}$ and $c \in F_{a,C}$. The map

$$m \circ a : Spec(A) \to B$$

gives A the structure of a B-ring $\mathcal{A}$ and similarly a' and c define B-rings $\mathcal{A}'$ and $\mathcal{C}$. Moreover, the isomorphisms $a'|_A \simeq a$ and $c|_A \simeq a$ define morphisms of B-rings

$$\mathcal{A}' \to \mathcal{A}, \quad \mathcal{C} \to \mathcal{A}$$

and hence we have a B-deformation situation and an object in

$$F_{a,\mathcal{A}'} \times F_{a,\mathcal{C}}.$$

By $(S1^B)$ there exists an object in $F_{a,\mathcal{A}' \times_\mathcal{A} \mathcal{C}}$ inducing this object and hence the functor in $(S1)$ is essentially surjective.

For the full faithfulness, suppose $a_1, a_2 \in F_{a,A' \times_A C}$. Given a pair of isomorphisms $(f', f_C) \in F_{a,A'} \times F_{a,B}$ we get a pair of isomorphisms $(m(f'), m(f_C)) \in B_{m(a),A'} \times B_{m(a),C}$ and hence an isomorphism $f : m(a_1) \simeq m(a_2)$ in $B_{a,A' \times_A C}$. The element a_1 gives in the same manner as in the preceding paragraph A', A, and C the structure of B-rings $\mathcal{A}'$, $\mathcal{A}$, and $\mathcal{C}$ and the morphism f allows us to view a_2 as an object in

$$F_{a,\mathcal{A}' \times_\mathcal{A} \mathcal{C}}$$

and the morphisms (f', f_C) as morphisms in

$$F_{a,\mathcal{A}'} \times F_{a,\mathcal{C}}.$$

From $(S1^B)$ it therefore follows that the functor is full. Similarly, given two isomorphisms $f_1, f_2 : a_1 \rightrightarrows a_2$ in $F_{a,A'\times_A C}$ which define the same morphism in $F_{a,A'} \times F_{a,C}$ we must by $(S1)$ for B have $m(f_1) = m(f_2)$. Therefore we can view f_1 and f_2 as morphisms in $F_{a,\mathcal{A}'\times_{\mathcal{A}}\mathcal{C}}$ where we give A', A, C the structure of B-rings via a_1. Therefore, $(S1^B)$ also implies that the functor is faithfull. This proves that $(S1)$ holds.

Now let A_0 be a reduced noetherian $\underline{S}$-ring, and let $a_0 \in F_{A_0}$. Denote by $Aut_{a_0}(A_0[I])$ the module of infinitesimal automorphisms of the image of a_0 in $A_0[I]$ (the module structure follows from the fact that $(S1)$ holds). $m \circ a_0$ gives A_0 the structure of a B-ring which we denote by $\mathcal{A}_0$. We claim that there is a natural 8-term exact sequence of A_0-modules

$$\begin{array}{ccccccccc} 0 & \to & Aut_{a_0}(\mathcal{A}_0[I]) & \to & Aut_{a_0}(A_0[I]) & \to & Aut_{m(a_0)}(A_0[I]) & \to & \\ & \to & E_{a_0}(I) & \to & \overline{F}_{a_0,A_0[I]} & \to & B_{m(a_0),A_0[I]} & \to & \mathcal{O}_{a_0}(I). \end{array} \tag{2.1}$$

Here $Aut_{a_0}(\mathcal{A}_0[I])$ is the module of 2.2.10 for $\mathcal{A}_0$, $Aut_{a_0}(A_0[I])$ is the module of infinitesimal automorphisms of the image of a_0 in $A_0[I]$ (ignoring the map to B), $Aut_{m(a_0)}(A_0[I])$ is the module of 2.1.6 for B, and $E_{a_0}(I)$ is as in 2.2.10. The maps in the sequence are the ones induced from functoriality. The only non-trivial ones are the maps $Aut_{m(a_0)}(A_0[I]) \to E_{a_0}(I)$ and the map $B_{m(a_0),A_0[I]} \to \mathcal{O}_{a_0}(I)$. The map $Aut_{m(a_0)}(A_0[I]) \to E_{a_0}(I)$ is defined as follows. Let $h : Spec(A_0[I]) \to B$, $f : Spec(A_0[I]) \to F$, $\eta : h \simeq m \circ f = m(a_0)$ be the data defining the element over $A_0[I]$ defined by a_0. The map sends an automorphism $\phi : m(a_0) \to m(a_0)$ to the element in $E_{a_0[I]}$ defined by the maps h, f together with the isomorphism

$$\phi \circ \eta.$$

It follows from unwinding the definition of the module structures that this in fact is an A_0-module homomorphism. This amounts to saying that if $I' \to I$ is a morphism of A_0-modules then the induced diagram

$$\begin{array}{ccc} Aut_{m(a_0)}(A_0[I]) & \longrightarrow & E_{a_0}(I) \\ \uparrow & & \uparrow \\ Aut_{m(a_0)}(A_0[I']) & \longrightarrow & E_{a_0}(I') \end{array}$$

commutes which is immediate. The map $B_{m(a_0),A_0[I]} \to \mathcal{O}_{a_0}(I)$ is described in 2.2.13.3. The exactness of the sequence is tautological.

This sequence implies that $(S2)$ holds since all the modules except $Aut_{a_0}(A_0[I])$ and $\overline{F}_{a_0,A_0[I]}$ are known to be of finite type.

Similarly, a diagram chase shows that the conditions in 2.1.10 are satisfied. This shows that 2.1.11.1 is satisfied.

The faithfulness of the functor in 2.1.11.2 can be shown as follows. If $a_1, a_2 \in F_{\hat{A}}$ and $f_1, f_2 : a_1 \rightrightarrows a_2$ are morphisms which define the same morphism in $\varprojlim F_{\hat{A}/\mathfrak{m}^n}$ we see that $m(f_1), m(f_2)$ must be equal. View $\hat{A}$ as a B-ring via $m(f_1)$. Then f_1 and f_2 are morphisms in $F_{\hat{\mathcal{A}}}$ which induce the same map in $\varprojlim F_{\hat{\mathcal{A}}/\mathfrak{m}^n}$ and hence must be equal.

Next suppose $a_1, a_2 \in F_{\hat{A}}$ and that a compatible family of of isomorphisms $\xi_n : a_{1,n} \simeq a_{2,n}$ are given. Since B is an algebraic stack, $m(a_1) \times_B m(a_2)$ is representable by a scheme and hence there exists a unique isomorphism $m(\xi) : m(a_1) \simeq m(a_2)$ in $B_{\hat{A}}$ inducing the isomorphisms $m(\xi_n)$. It follows that we can view a_1 and a_2 as objects in $F_{\hat{\mathcal{A}}}$ where $\hat{\mathcal{A}}$ is the B-ring obtained via $m(a_1)$. It now follows from 2.2.13.2 that for all n there exists an isomorphism $\xi : a_1 \simeq a_2$ inducing ξ_n.

Finally condition 2.1.11.4 can be verified as follows. If A_0 is reduced and of finite type over $\underline{S}$, $a_0 \in F_{A_0}$, ϕ an automorphism of a_0 which induces the identity for a dense set of points, then there exists a dense open set of A_0 such that $m(\phi)$ is the identity automorphism of $m(a_0)$ over this open set. Replacing A_0 by the ring of a basic open set inside this open set we can assume that $m(\phi)$ is the identity map. Thus viewing A_0 as a B-ring via $m(a_0)$ we are reduced to 2.2.13.4.

This proves that F is relatively representable.

Now fix a smooth surjective cover $b : \underline{B}' \to \underline{B}$ and let $F' = F \times_B \underline{B}'$. If $s : Spec(A) \to \underline{B}'$ is a morphism then an object in F'_A is equivalent to a pair (f, h) where $f : Spec(A) \to F$ is a 1-morphism and $h : s^*(b) \simeq m(f)$ is an isomorphism in B. A morphism $(f', h') \to (f, h)$ in F'_A is the same as a morphism $\eta : f' \to f$ in F_A such that $m(\eta) \circ h' = h$. In other words, if we view A as a B-ring $\mathcal{A}$ via the map b, the category F'_A is naturally equivalent to the category $F_{\mathcal{A}}$.

This implies that all the hypothesis of 2.1.11 hold except possibly 2.1.11.3. To construct an obstruction theory $\mathcal{O}'$ for F' proceed as follows. An infinitesimal extension $A \to A_0$ over $\underline{B}'$ can be viewed via b as an infinitesimal extension of B-rings

$$\mathcal{A} \to \mathcal{A}_0.$$

Moreover, an element $a \in F'_A$ defines in a natural way an element in $F_{\mathcal{A}}$ which we also denote by a. Define the functor $\mathcal{O}'_a$ to be the functor $\mathcal{O}_a$ of 2.2.11 coresponding to this data. Given a deformation situation $A' \to A \to A_0$ over $\underline{B}'$ we can view it as a B-deformation situation, and hence get an element $o_a(I) \in \mathcal{O}'_a(I)$. It is immediate that $(\mathcal{O}'_a, o_a)$ satisfies the axioms of an obstruction theory. This proves that F' is an algebraic stack, and hence that F is also algebraic. □

Remark 2.2.14. From the proof it follows that it suffices to construct local obstruction theories $\mathcal{O}_a$ in the sense of 2.1.8 translated into the language of B-schemes.

Chapter 3

Universal Log Structures

3.1 Direct construction

The main result of this section is the following theorem suggested by A. Abbes:

Theorem 3.1.1. *Let $\underline{S}$ be a scheme locally of finite type over an excellent Dedekind ring and let $\mathcal{M}_S$ be a fine log structure on $\underline{S}_{fl}$. Let LOG_S be the fibered category over the category of $\underline{S}$-schemes for which $LOG_S(f : \underline{X} \to \underline{S})$ is the category of fine log structures $\mathcal{M}_X$ on $\underline{X}_{fl}$ with a morphism*

$$f^*\mathcal{M}_S \to \mathcal{M}_X.$$

Then LOG_S is an algebraic stack.

We refer to the objects of LOG_S as S**-log structures**.

Remark 3.1.2. Note that by 1.1.7 we can view LOG_S as classifying log structures on the étale site.

The proof of the theorem will be in two steps (theorems 3.1.3 and 3.1.11).

Theorem 3.1.3. *Let $\underline{X}$ be a scheme locally of finite type over an excellent Dedekind ring and let*

$$\begin{array}{ccc} \mathcal{M}_S & \xrightarrow{s_1} & \mathcal{M}_1 \\ {\scriptstyle s_2}\downarrow & & \\ \mathcal{M}_2 & & \end{array} \tag{3.1}$$

be a diagram of fine log structures on $\underline{X}$. Then the functor I on $\underline{X}$-schemes defined by

$$(f : \underline{Z} \to \underline{X}) \mapsto \text{set of isomorphisms } i : f^*\mathcal{M}_1 \simeq f^*\mathcal{M}_2 \text{ such that } i \circ s_1 = s_2$$

is representable by a locally separated algebraic space.

Remark 3.1.4. The functor I of 3.1.3 is not separated in general. For an explicit example where the valuative criterion for separation fails, let A be a discrete valuation ring with uniformizer π, and let $\mathcal{M}$ be the log structure associated to the chart

$$\mathbb{N} \oplus \mathbb{N} \to A, \quad (i, j) \mapsto \pi^{i+j}.$$

Then the identity map and the map induced by

$$(i, j) \mapsto (j, i)$$

are two automorphisms of $\mathcal{M}$ both of which induce the identity map on the generic fiber.

We begin the proof of 3.1.3 by considering a special case:

Lemma 3.1.5. *If s_1 and s_2 are surjective then the functor I is represented by a locally closed subscheme of $\underline{X}$.*

Proof. For $i = 1, 2$ let

$$K^i = Ker(s_i^{gp} : \mathcal{M}_S^{gp} \to \mathcal{M}_i^{gp}), \quad \overline{K}^i = Ker(\bar{s}_i^{gp} : \overline{\mathcal{M}}_S^{gp} \to \overline{\mathcal{M}}_i^{gp}).$$

From the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & O_X^* & \longrightarrow & \mathcal{M}_S^{gp} & \longrightarrow & \overline{\mathcal{M}}_S^{gp} & \longrightarrow & 0 \\ & & {\scriptstyle =}\downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & O_X^* & \longrightarrow & \mathcal{M}_i^{gp} & \longrightarrow & \overline{\mathcal{M}}_i^{gp} & \longrightarrow & 0 \end{array}$$

and the snake lemma we see that the natural map $K^i \to \overline{K}^i$ is an isomorphism of sheaves. If $f : \underline{Z} \to \underline{X}$ is a morphism of schemes then we denote by f^*K^i and $f^*\overline{K}^i$ the sheaves

$$Ker(s_i^{gp} : f^*\mathcal{M}_S^{gp} \to f^*\mathcal{M}_i^{gp}) \text{ and } Ker(\bar{s}_i^{gp} : f^{-1}\overline{\mathcal{M}}_S^{gp} \to f^{-1}\overline{\mathcal{M}}_i^{gp}).$$

Let F be the functor on $\underline{X}$-schemes for which $F(\underline{Z})$ consists of a single element if $f^*K^1 = f^*K^2$ as subsheaves of $f^*\mathcal{M}_S^{gp}$ and the empty set otherwise. There is a natural morphism of functors

$$I \to F$$

which we claim is an isomorphism. To see this suppose $K^1 = K^2$. Then we can define an isomorphism $j : \mathcal{M}_1 \to \mathcal{M}_2$ be sending an element $m \in \mathcal{M}_1$ to $s_2(\tilde{m})$ where $\tilde{m} \in \mathcal{M}_S$

is any lifting of m. This does not depend on the choice of $\tilde{m}$. For if $\tilde{n}$ is a second such lifting then $\tilde{m} - \tilde{n} \in K^1 = K^2$ and hence maps to zero on $\mathcal{M}_2^{gp}$. Conversely, if we have an isomorphism $j : \mathcal{M}_1 \simeq \mathcal{M}_2$ then we have

$$K^2 = Ker(s_2^{gp}) = Ker(j^{gp} \circ s_1^{gp}) = Ker(s_1^{gp}) = K^1.$$

It therefore suffices to show that F is representable by a locally closed subscheme of $\underline{X}$.

Let F' be the functor which to any $f : \underline{Z} \to \underline{X}$ associates the unital set if $f^*\overline{K}^1 = f^*\overline{K}^2$ in $f^{-1}\overline{\mathcal{M}}_S^{gp}$ and the empty set otherwise. We have a diagram of functors

$$F \to F' \to \underline{X}.$$

We claim that F' is represented by an open subset of $\underline{X}$ and that F is represented by a closed subscheme of F'.

To prove these statements we may pass to a cover of $\underline{X}$. We claim that locally there exists a chart

$$P \to H^0(\underline{X}, \mathcal{M}_S)$$

such that the image P^i of P in

$$H^0(\underline{X}, \mathcal{M}_i)$$

is a chart for $\mathcal{M}_i$. To see this it suffices to construct P and P^i at the stalk of some geometric point $\bar{x} \to \underline{X}$ since charts can always be extended to some étale neighborhood. Fix a chart

$$P^i \to \mathcal{M}_{i,\bar{x}}$$

which is exact and injective. Note that an exact chart $\lambda : P^{i'} \to \mathcal{M}_{i,\bar{x}}$ exists by ([17], II.2.2.10). If P^i denotes the image of $P^{i'}$ in $\mathcal{M}_{i,\bar{x}}$ then the inclusion

$$P^i \to \mathcal{M}_{i,\bar{x}}$$

is again an exact chart. It is exact because

$$P^{i'} = P^{i',gp} \times_{\mathcal{M}_{i,\bar{x}}^{gp}} \mathcal{M}_{i,\bar{x}}$$

and hence the image of $P^{i'}$ in $\mathcal{M}_{i,\bar{x}}$ (which by definition is P^i) is equal to

$$P^{i,gp} \times_{\mathcal{M}_{i,\bar{x}}^{gp}} \mathcal{M}_{i,\bar{x}} \hookrightarrow \mathcal{M}_{i,\bar{x}}.$$

To see that it is a chart, it suffices to show that the map

$$P^i/(P^i \cap O_{X,\bar{x}}^*) \to \overline{\mathcal{M}}_{i,\bar{x}}$$

is an isomorphism.

Since λ is exact,

$$P^{i'} = P^{i',gp} \times_{P^{i,gp}} P^i$$

and hence if $G = Ker(P^{i',gp} \to P^{i,gp})$ then $G \subset P^{i'}$ and

$$\overline{\mathcal{M}}_{i,\bar{x}} \simeq P^{i'}/\lambda^{-1}(O^*_{X,\bar{x}}) = (P^{i'}/G)/(\lambda^{-1}(O^*_{X,\bar{x}})/G) \simeq P^i/(P^i \cap O^*_{X,\bar{x}}).$$

It follows that $P^i \to \mathcal{M}_{i,\bar{x}}$ is also a chart.

Given our injective and exact chart $P^i \hookrightarrow \mathcal{M}_i$, define P by

$$P = P^{i,gp} \times_{\mathcal{M}^{gp}_{i,\bar{x}}} \mathcal{M}_{S,\bar{x}}.$$

Then by ([20], 2.10) P is a chart for $\mathcal{M}_{S,\bar{x}}$ and the image of P in $\mathcal{M}_{i,\bar{x}}$ is by the exactness equal to P^i.

Let $\tilde{K}^i$ be the kernel of the map

$$P^{gp} \to P^{i,gp}.$$

We claim that for all geometric points $\bar{x} \to \underline{X}$ the map

$$\tilde{K}^i \to \overline{K}^i_{\bar{x}}$$

is surjective. To see this consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{K}^i & \longrightarrow & P^{gp} & \longrightarrow & P^{i,gp} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{K}^i_{\bar{x}} & \longrightarrow & \overline{\mathcal{M}}^{gp}_{S,\bar{x}} & \longrightarrow & \overline{\mathcal{M}}^{gp}_{i,\bar{x}} & \longrightarrow & 0
\end{array}$$

and define

$$H = Ker(P^{gp} \to \overline{\mathcal{M}}^{gp}_{S,\bar{x}}), \quad H^i = Ker(P^{i,gp} \to \overline{\mathcal{M}}^{gp}_{i,\bar{x}}).$$

Since the map $P^{gp} \to \overline{\mathcal{M}}^{gp}_{S,\bar{x}}$ is surjective it suffices to show that the map $H \to H^i$ is surjective by the snake lemma. Now H^i is generated by differences of elements $\bar{p} \in P^i$ for which $\alpha(\bar{p}) \in O^*_{X,\bar{x}}$. But then any $p \in P$ lifting $\bar{p}$ also has this property and consequently is in H. This proves that $\tilde{K}^i \to \overline{K}^i_{\bar{x}}$ is surjective for all $\bar{x}$. Since $\tilde{K}^i$ is finitely generated this implies that F' is represented by an open subset of $\underline{X}$.

Replacing $\underline{X}$ by this open subset we can therefore assume that $\overline{K}^1 = \overline{K}^2$. It follows that if $k \in \tilde{K}^1$ then there exists a unique element $k' \in K^2$ and $u(k) \in O^*_X$ such that

$$k = k' + \lambda(u(k)).$$

The functor F is therefore represented by the closed subscheme of $\underline{X}$ represented by the conditions

$$u(k) = 1, \quad k \in \tilde{K}^1.$$

Note that since $\tilde{K}^1$ is finitely generated there are only finitely many equations for F. $\square$

To prove the general case of the theorem we make various reductions.

Reduction Step 3.1.6. *It suffices to consider the case when s_1 and s_2 are injective.*

Proof. Let $\mathcal{M}_S^1$ (resp. $\mathcal{M}_S^2$) be the image of s_1 (resp. s_2) in $\mathcal{M}_1$ (resp. $\mathcal{M}_2$). Let $\underline{X}' \subset \underline{X}$ be the locally closed subscheme obtained by applying the lemma to the diagram

$$\begin{array}{ccc} \mathcal{M}_S & \longrightarrow & \mathcal{M}_S^1 \\ \downarrow & & \\ \mathcal{M}_S^2 & & \end{array}$$

Then the morphism of functors $I \to \underline{X}$ factors through $\underline{X}'$ and hence we can replace $\underline{X}$ by $\underline{X}'$. Over $\underline{X}'$ there is a unique isomorphism $i : \mathcal{M}_S^1 \to \mathcal{M}_S^2$ and hence we may replace the diagram in 3.1.3 by the diagram

$$\begin{array}{ccc} \mathcal{M}_S^1 & \longrightarrow & \mathcal{M}_1 \\ {\scriptstyle i}\downarrow & & \\ \mathcal{M}_2 & & \end{array}$$

$\square$

Reduction Step 3.1.7. *It suffices to prove the theorem fppf-locally on $\underline{X}$.*

Proof. I comes with descent datum relative to any fppf-cover of $\underline{X}$. The result therefore follows from ([5], 6.3) which implies that descent datum for an algebraic space in the fppf-topology is effective. $\square$

Lemma 3.1.8. *Suppose $\underline{X} = Spec(A)$ where A is an artinian local ring. Then I is representable by an affine scheme.*

Proof. The property of being an affine scheme over $Spec(A)$ is local in the fppf-topology ([27], 2.23) so it suffices to prove the theorem after replacing $\underline{X}$ by an appropriate cover. If $I \neq \emptyset$ then after replacing A by an étale cover (using 1.1.7) we can assume that $\overline{\mathcal{M}}_1$ and $\overline{\mathcal{M}}_2$ are constant sheaves and that we have an isomorphism

$$\bar{i} : \overline{\mathcal{M}}_1 \simeq \overline{\mathcal{M}}_2$$

such that $\bar{s}_2 = \bar{i} \circ \bar{s}_1$. Fix a chart $Q \to \mathcal{M}_S$ such that $Q \to \overline{\mathcal{M}}_S$ is bijective (this is possible after a finite flat extension). Then after perhaps replacing A by another finite flat extension we can choose charts

$$\begin{array}{ccc} P_1 & \longrightarrow & \mathcal{M}_1 \\ \uparrow & & \uparrow \\ Q & \longrightarrow & \mathcal{M}_S \end{array}$$

and

$$\begin{array}{ccc} P_2 & \longrightarrow & \mathcal{M}_2 \\ \uparrow & & \uparrow \\ Q & \longrightarrow & \mathcal{M}_S \end{array}$$

such that $P_1 \to \overline{\mathcal{M}}_1$ and $P_2 \to \overline{\mathcal{M}}_2$ are bijective ([29], 2.13 and 2.14). Identifying P_1 and P_2 via i we can therefore assume that the log structures are induced from two morphisms $\beta_1, \beta_2 : P \to A$ where P is a fine sharp monoid, and that s_1 and s_2 are induced from an injective map $Q \to P$.

Let $Aut(P/Q)$ be the set of automorphisms of P which leave Q fixed. For $\xi \in Aut(P/Q)$, let R_ξ denote the ring

$$A[P^{gp}/Q^{gp}]/\{e(p)\beta_2(\xi(p)) = \beta_1(p)e(0)\}_{p \in P}$$

where $e(p)$ denotes the image of an element $p \in P$ in $A[P^{gp}/Q^{gp}]$. Given any map of rings $\phi : R_\xi \to B$ we get an isomorphism of log structures over B induced from the map

$$P \to B^* \oplus P, \quad p \mapsto \phi(e(p)) + \xi(p).$$

This defines a map

$$Spec(R_\xi) \to I$$

and since $P \to \overline{\mathcal{M}}_1$ and $P \to \overline{\mathcal{M}}_2$ are isomorphisms, every element $i \in I(B)$ is induced from a map $R_\xi \to B$ for a unique ξ (assuming that $Spec(B)$ is connected). It follows that I is represented by the scheme

$$\coprod_{\xi \in Aut(P/Q)} Spec(A[P^{gp}/Q^{gp}]/\{e(p)\beta_2(\xi(p)) = \beta_1(p)e(0)\}_{p \in P}).$$

$\square$

Later it will be useful to understand the construction of the above proof using more general charts:

Lemma 3.1.9. *Let A be an artinian local ring, and suppose*

$$\begin{array}{ccc} Q & \xrightarrow{h_1 \times s_1} & G_1 \oplus P_1 \\ {\scriptstyle h_2 \times s_2}\downarrow & & \downarrow{\scriptstyle \alpha_1} \\ G_2 \oplus P_2 & \xrightarrow{\alpha_2} & A \end{array}$$

is a chart for diagram 3.1 where $G_1, G_2 \subset A^*$ are subgroups, and Q, P_1, P_2 are sharp monoids. For any isomorphism

$$\sigma : P_1 \to P_2$$

such that $s_2 = \sigma \circ s_1$ let C_σ be the quotient of

$$A[P_1^{gp}]$$

by the relations

$$\alpha_2(\sigma(p))e(p) = \alpha_1(p)e(0), \quad p \in P_1$$

and

$$e(q) = \alpha_2(h_2(q))\alpha_1(h_1(q))^{-1}, \quad q \in Q.$$

Then the natural map

$$j : Spec(C_\sigma) \to I$$

defined by the isomorphism over $Spec(C_\sigma)$ induced by

$$p \mapsto e(p) + \sigma(p)$$

is an open immersion.

Proof. The statement of the lemma is fppf-local on $Spec(A)$ and hence using the argument of 3.1.8 we can assume that the log structures $\mathcal{M}_1$ and $\mathcal{M}_2$ are induced from two maps

$$\beta_1, \beta_2 : \tilde{P} \rightrightarrows A$$

and that the maps $s_j : \mathcal{M}_S \to \mathcal{M}_j$ are induced from an inclusion $\tilde{Q} \hookrightarrow \tilde{P}$. In addition we can assume that the maps

$$\tilde{P} \to \overline{\mathcal{M}}_1, \quad \tilde{P} \to \overline{\mathcal{M}}_2, \quad \tilde{Q} \to \overline{\mathcal{M}}_S$$

are bijective. From the preceding lemma we see that I is represented by

$$\coprod_{\xi \in Aut(\tilde{P}/\tilde{Q})} Spec(A[\tilde{P}^{gp}/\tilde{Q}^{gp}]/\{e(p)\beta_2(\xi(p)) = \beta_1(p)e(0)\}_{p \in \tilde{P}}).$$

α_1 and α_2 induce isomorphisms

$$P_1/F_1 \simeq \tilde{P}, \quad P_2/F_2 \simeq \tilde{P}$$

where

$$F_j = \{p \in P_j | \alpha_j(p) \in A^*\}.$$

If $Spec(C_\sigma) \neq \emptyset$ then it follows from the definition that $\sigma(F_1) = F_2$. Assume this is the case. Then σ induces an element $\xi \in Aut(\tilde{P}/\tilde{Q})$. The map

$$Spec(C_\sigma) \to I$$

maps into the component corresponding to ξ defined by a map of rings

$$A[\tilde{P}^{gp}/\tilde{Q}^{gp}]/\{e(p)\beta_2(\xi(p)) = \beta_1(p)e(0)\}_{p \in \tilde{P}} \to C_\sigma.$$

This map is defined by sending $e(\tilde{p})$ to

$$u(\tilde{p})u(\xi(\tilde{p}))^{-1}e(p)$$

where p is a lifting of $\tilde{p}$ and $u(\tilde{p})$ and $u(\xi(\tilde{p}))$ are obtained by writing

$$\beta_1(\tilde{p}) = \lambda(u(\tilde{p})) + \alpha_1(p)$$

in $\mathcal{M}_1$ and

$$\beta_2(\xi(\tilde{p})) = \lambda(\xi(\tilde{p})) + \alpha_2(\sigma(p))$$

in $\mathcal{M}_2$. It now follows from the definition of C_σ that this map is an isomorphism. $\square$

3.1.10 *(Conclusion of proof of 3.1.3).* We assume that s_1 and s_2 are injective. Replacing $\underline{X}$ by an affine cover we can assume that $\underline{X}$ is quasi-compact. We proceed by induction on

$$n = sup_{\bar{x} \to \underline{X}} \# Irr(\overline{\mathcal{M}}_{1,\bar{x}})$$

where $Irr(\overline{\mathcal{M}}_{1,\bar{x}})$ denotes the set of irreducible elements in $\overline{\mathcal{M}}_{1,\bar{x}}$. Observe that if we fix an integer n, then the set of point $x \in X$ with

$$\# Irr(\overline{\mathcal{M}}_{1,\bar{x}}) < n$$

is an open set. To see this, suppose x is such a point, and let $\bar{p}_1, \ldots, \bar{p}_r$ $(r < n)$ be the irreducible elements of $\overline{\mathcal{M}}_{1,\bar{x}}$. Let $p_1, \ldots, p_r \in \mathcal{M}_{1,\bar{x}}$ be liftings of the $\bar{p}_i$ and let $P \subset \mathcal{M}_{1,\bar{x}}$ be the sub-monoid generated by the p_i. Then $P \to \mathcal{M}_{1,\bar{x}}$ extends to a chart for $\mathcal{M}_1$ in some étale neighborhood of x. Since P is generated by r elements it follows from ([17], 2.1.2) that for each point $x' \in X$ in the image of this étale neighborhood we have

$$\#Irr(\overline{\mathcal{M}}_{1,\bar{x}'}) < n.$$

If $n = 0$ then I is represented by the open subset of $\underline{X}$ where $\mathcal{M}_2$ is isomorphic to O_X^*.

Next we prove the result for n assuming that it holds for $n-1$. Let $\underline{U} \hookrightarrow \underline{X}$ be the open set of points x such that $\#Irr(\overline{\mathcal{M}}_{1,\bar{x}}) < n$. Choose a geometric point $\bar{x} \to \underline{X} - \underline{U}$ with $I(\bar{x}) \neq \emptyset$. If no such $\bar{x}$ exists we are done so we can assume such a point exists. After perhaps replacing $\underline{X}$ by an fppf-neighborhood of $\bar{x}$ we can assume that $\mathcal{M}_1$ and $\mathcal{M}_2$ are induced from two morphisms

$$\alpha_1, \alpha_2 : P \to O_X$$

and that $s_j : \mathcal{M}_S \to \mathcal{M}_j$ are induced from an inclusion $Q \hookrightarrow P$. In addition we can assume that the maps

$$Q \to \overline{\mathcal{M}}_{S,\bar{x}}, \quad P \to \overline{\mathcal{M}}_{1,\bar{x}}, \quad P \to \overline{\mathcal{M}}_{2,\bar{x}}$$

are bijective. For each $\sigma \in Aut(P/Q)$ let

$$T_\sigma = Spec(O_X[P^{gp}/Q^{gp}]/\{e(p)\alpha_2(\sigma(p)) = \alpha_1(p)e(0)\}_{p\in P}).$$

Then there is a morphism

$$\coprod_{\sigma\in Aut(P/Q)} T_\sigma \to I$$

such that for any connected $\underline{Z}/\underline{X}$ whose image in $\underline{X}$ is not contained in $\underline{U}$ the map

$$\coprod_{\sigma\in Aut(P/Q)} T_\sigma(\underline{Z}) \to I(\underline{Z})$$

is bijective. Now for each $\sigma \in Aut(P/Q)$ we claim that the map

$$T_\sigma \times_{\underline{X}} \underline{U} \to I \times_X U$$

is an open immersion (note that by induction $I \times_X U$ is representable by an algebraic space). This is equivalent to the map being étale and injective on field valued points. In other words, we have to show that if $\underline{Z}/\underline{U}$ is a scheme isomorphic to the spectrum of an artinian local ring, then the map

$$T_\sigma(\underline{Z}) = (T_\sigma \times_X \underline{Z})(Z) \to I \times_X Z(Z) = I(Z)$$

is injective. But this follows from lemma 3.1.9.

We now conclude the proof of representability by observing that I is represented by the algebraic space obtained by gluing each of the T_σ to $I \times_{\underline{X}} \underline{U}$ via $T_\sigma \times_{\underline{X}} \underline{U}$.

It remains to verify that I is locally separated. This means that if $\underline{U} \to I$ is a morphism, then the map

$$\underline{U} \times_I \underline{U} \to \underline{U} \times_{\underline{X}} \underline{U}$$

is a locally closed immersion. That is, given two isomorphisms of log structures (we can ignore $\mathcal{M}_S$ for this) on $\underline{U} \times_{\underline{X}} \underline{U}$

$$\alpha, \beta : \mathcal{M}_1 \to \mathcal{M}_2$$

the condition that $\alpha = \beta$ is represented by a locally closed subscheme of $\underline{U} \times_{\underline{X}} \underline{U}$. But this follows from 3.1.5 applied to the diagram

$$\begin{tikzcd} \mathcal{M}_1 \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \mathcal{M}_2 \\ \mathcal{M}_2 & \end{tikzcd} .$$

□

Theorem 3.1.11. *Let Q be a fine sharp monoid and let*

$$\begin{tikzcd} Q \arrow[r, "s_1"] \arrow[d, "s_2"'] & P_1 \\ P_2 & \end{tikzcd}$$

be a diagram of fine monoids with s_1 and s_2 injective. Let

$$S = Spec(\mathbb{Z}[Q]), \quad X = Spec(\mathbb{Z}[P_1]), \quad Y = Spec(\mathbb{Z}[P_2])$$

and let $\mathcal{M}_S, \mathcal{M}_X, \mathcal{M}_Y$ be the natural log structures. Let $I/X \times_S Y$ be the functor corresponding to the diagram

$$\begin{tikzcd} \mathcal{M}_S|_{X \times_S Y} \arrow[r] \arrow[d] & p_1^*\mathcal{M}_X \\ p_2^*\mathcal{M}_Y & \end{tikzcd}$$

given by theorem 3.1.3. Then the two projections

$$I \to X, \quad I \to Y$$

are flat.

Proof. We proceed by induction on $n = \#Irr(\overline{P}_1)$. If $n = 0$, then $P_1 = P_1^*$ and

$$I \simeq Spec(\mathbb{Z}[P_1^*]) \times_{Spec(\mathbb{Z}[Q^{gp}])} Spec(\mathbb{Z}[P_2^{gp}]).$$

Now $\mathbb{Z}[P_2^{gp}]/\mathbb{Z}[Q^{gp}]$ is flat since $Q^{gp} \to P_2^{gp}$ is injective.

Next we prove the theorem for n assuming the result for $n-1$. We can assume that $P_1 \simeq P_1^* \oplus \overline{P}_1$ by replacing P_1 by $G \oplus_{P_1^*} P_1$ for a suitable group G. Fix such a splitting $P_1 \simeq P_1^* \oplus \overline{P}_1$.

We can also assume that $\#Irr(\overline{P}_2) = n$, for $I \to X \times_S Y$ maps into the union of the open sets

$$X \times_S Spec(\mathbb{Z}[P_{2,F}])$$

where $F \subset P_2$ is a face such that $\#Irr(P_2/F) \leq n$. Hence we may replace P_2 by $P_{2,F}$. In addition we may assume that $P_2 \simeq P_2^* \oplus \overline{P}_2$ by replacing Y by

$$Y' = Spec(\mathbb{Z}[G \oplus_{P_2^*} P_2])$$

for a suitable group G. Then if I' denotes the algebraic space obtained from X and Y' we have a flat surjection

$$I' \to I$$

and since the property of being flat is fppf-local on domain it suffices to consider I'. Fix such a splitting $P_2 \simeq P_2^* \oplus \overline{P}_2$.

For each isomorphism $\sigma : \overline{P}_1 \to \overline{P}_2$ define C_σ as in 3.1.9 and let

$$T_\sigma = Spec(C_\sigma).$$

Then by 3.1.9 the map $T_\sigma \to I$ is an open immersion and the union of the T_σ cover the set of points of I lying over points x for which $\#Irr(\overline{\mathcal{M}}_X) = n$. Therefore by induction it suffices to show that C_σ is flat over $\mathbb{Z}[P_1]$. This is straightforward from the definitions. Writing out the definition of C_σ we find that

$$C_\sigma \simeq \mathbb{Z}[\overline{P}_1][P_1^* \oplus_{Q^{gp}} P_2^{gp}]$$

and hence it suffices to show that

$$\mathbb{Z}[P_1^*] \to \mathbb{Z}[P_1^* \oplus_{Q^{gp}} P_2^{gp}]$$

is flat. This follows from the fact that $Q^{gp} \to P_2^{gp}$ is injective. $\square$

3.1.12 *(Conclusion of proof of 3.1.1).* Since LOG_S classifies sheaves in the fppf-topology LOG_S is a stack for this topology and 3.1.3 implies that LOG_S is relatively representable. By ([5], 6.3) fppf descent datum for an algebraic stack is effective. Therefore it suffices to prove the theorem in the case when there is a chart $Q \to \mathcal{M}_S$ where Q is a fine sharp monoid. Fix a set $\{P_j\}_{j \in J}$ of fine monoids with injective maps $i_j : Q \to P_j$ such that for any inclusion $i : Q \to P$ there exists a $j \in J$ and isomorphism $\phi : P_j \simeq P$ such that $i = \phi \circ i_j$.

Let

$$\tilde{S} = \coprod_{j \in J} X \times_{Spec(\mathbb{Z}[Q])} Spec(\mathbb{Z}[P_j])$$

with the natural log structure. By 3.1.11 the two projections

$$\tilde{S} \times_{LOG_S} \tilde{S} \rightrightarrows \tilde{S}$$

are flat, and by ([29], 2.13) LOG_S is the stack associated to this flat groupoid. By ([5], 6.3) it now follows that LOG_S is an algebraic stack. □

3.2 Deformation theory of log structures

3.2.1. It follows from Artin's theorem 2.1.11 and the fact that the stacks LOG_S of the preceding section are algebraic that a nice deformation theory of log structures exists. In this section we give a more direct approach to this deformation theory independent of the preceding sections. Throughout this section all log structures will be sheaves in the étale topology.

The following proposition records some standard facts about flatness and deformation situations.

Proposition 3.2.2. *Suppose given a deformation situation and flat modules E', E, E'' over A', A, and B respectively together with isomorphisms $E' \otimes_{A'} A \simeq E \simeq E'' \otimes_B A$.*

1. *$E' \times_E E''$ is a flat $A' \times_A B$-module.*

2. *If L is an $A' \times_A B$-module together with morphisms such that the following diagram commutes*

$$\begin{array}{ccc} L & \longrightarrow & E'' \\ \downarrow & & \downarrow \\ E' & \longrightarrow & E \end{array}$$

and $L \otimes B \simeq E''$ then $L \simeq E' \times_E E''$.

3. *Suppose $A' \to C'$, $A \to C$, and $B \to C''$ are étale algebras of finite presentation such that $C \simeq C'' \otimes_B A \simeq C' \otimes_{A'} A$. Then $A' \times_A B \to C' \times_C C''$ is étale and of finite presentation. Moreover, the set of étale $A' \times_A B$ algebras of this form are a base for the étale topology on $A' \times_A B$.*

Proof. (1) If R is any ring and $I \subset R$ a nilpotent ideal, then an R-module E is flat if and only if E/IE is flat over R/I and the natural map

$$\gamma : gr_I^0(E) \otimes_{R/I} gr_I^*(R) \to gr_I^*(E)$$

is an isomorphism ([1], V theorem 3.2). Here $gr_I^*(E)$ denotes

$$\oplus_n I^n E / I^{n+1} E.$$

Applying this criteria to E' we deduce that

$$I^i E' / I^{i+1} E' \simeq E \otimes_A (I^i / I^{i+1}).$$

The kernel of $A' \times_A B \to B$ is the ideal $I \times_A \{0\}$ which is isomorphic to I. We apply the above criteria to $E' \times_E E''$. By direct calculation we have

$$gr^i_{I \times \{0\}}(E' \times_E E'') \simeq \begin{cases} E'', & \text{if } i = 0 \\ I^i E' / (I^{i+1} E') & \text{if } i \neq 0 \end{cases}$$

and

$$gr^i_{I \times \{0\}}(A' \times_A B) \simeq \begin{cases} B, & \text{if } i = 0 \\ I^i / I^{i+1}, & \text{if } i \neq 0. \end{cases}$$

Thus checking γ is an isomorphism amounts to checking that

$$I^i E' / I^{i+1} E' \simeq E \otimes_A (I^i / I^{i+1}) \simeq (E'' \otimes_B A) \otimes_A (I^i / I^{i+1}) \simeq E'' \otimes_B (I^i / I^{i+1}).$$

(2) There is a natural map $L \to E' \times_E E''$ which is an isomorphism when tensored with B. Since the kernel of $A' \times_A B \to B$ is nilpotent and $E' \times_E E''$ is flat this implies that the map is an isomorphism. For the cokernel C of $L \to E' \times_E E''$ must be zero since it becomes zero when tensored with $A' \times_A B/(I \times \{0\})$ by Nakayama's lemma. We therefore have an exact sequence

$$0 \to K \to L \to E' \times_E E'' \to 0$$

and since $E' \times_E E''$ is flat this sequence remains exact when tensored with $A' \times_A B/(J \times \{0\})$ and hence $K = 0$ by applying Nakayama's lemma again.

(3) This follows from (1) and (2) and the fact that if $f : \underline{X} \to \underline{Y}$ is a flat morphism locally of finite type of schemes with $\underline{Y}$ locally noetherian, then f is étale at a point $x \in \underline{X}$ if and only if the map $\underline{X} \times_{\underline{Y}} k(f(x)) \to Spec(k(f(x)))$ is étale at x ([1], VI Proposition 4.8). If L is an étale $A' \times_A B$-algebra let $C' = L \otimes A'$, $C'' = L \otimes B$, and $C = L \otimes A$. Then by (1) $C' \times_C C''$ is an étale $A' \times_A B$-algebra and the natural map

$$L \to C' \times_C C''$$

is an isomorphism. □

Proposition 3.2.3. *Suppose given a deformation situation and logarithmic morphisms of monoids $\alpha_1 : M_1 \to A'$, $\alpha_2 : M_2 \to B$, $\alpha_0 : M \to A$ together with isomorphisms of monoids*

$$M_1 \oplus_{\alpha_1^{-1}(A'^*)} A^* \simeq M \simeq M_2 \oplus_{\alpha_2^{-1}(B^*)} A^*$$

compatible with the induced maps to A.

1. *The map $M_1 \times_M M_2 \to A' \times_A B$ is logarithmic.*
2. *The natural map*
$$B^* \oplus_{M_1 \times_M \alpha_2^{-1}(B^*)} (M_1 \times_M M_2) \to M_2$$
is an isomorphism.
3. *The natural map*
$$A'^* \oplus_{A'^* \times_M M_2} (M_1 \times_M M_2) \to M_1$$
is an isomorphism.

Proof. For any element m of M_2 or M_1 let $\overline{m}$ denote the image in M.

(1) This is immediate from the fact that the α_i are assumed logarithmic.

(2) The map $M_1 \to M$ is surjective, and from this it follows that the map in (2) is also surjective. As for injectivity, let $(b, m_1, m_2), (b', m_1', m_2') \in B^* \oplus M_1 \times_M M_2$ be two triples which map to the same element in M_2. Let $a \in A'^*$ be an element such that $\overline{\lambda(a)} = \overline{\lambda(b)} - \overline{\lambda(b')}$, and such that

$$m_1' = \lambda(a) + m_1.$$

This is possible since the map $A'^* \to A^*$ is surjective. Then $(a, \lambda(b) - \lambda(b')) \in A'^* \times_M M_2$ and the equations

$$\lambda(b) = \lambda(b') + \lambda(b) - \lambda(b'), \quad m_2' = \lambda(b) - \lambda(b') + m_2$$

together with the choice of a imply that the triples $(b, m_1, m_2), (b', m_1', m_2')$ define the same element of $B^* \oplus_{M_1 \times_M \alpha_2^{-1}(B^*)} (M_1 \times_M M_2)$.

(3) Since M_2 induces M on A, for every $m_1 \in M_1$ there exists $a \in A^*$ such that $\lambda(a) + \overline{m_1}$ is in the image of $M_2 \to M$. Surjectivity of the map now follows from the fact that $A'^* \to A^*$ is surjective.

For injectivity, let $(a, m_1, m_2), (a', m_1', m_2') \in A'^* \oplus M_1 \times_M M_2$ be triples which map to the same element of M_1. Let $\tilde{a} = a/a' \in A'^*$. Then we have the equation

$$m_1' = \lambda(\tilde{a}) + m_1.$$

Since $M = A^* \oplus_{\alpha_2^{-1}(A^*)} M_2$, we can choose $s, t \in \alpha_2^{-1}(A^*)$ such that

$$\overline{\lambda(\tilde{a})} + \overline{s} = \overline{t}$$

and

$$m_2' + s = m_2 + t.$$

This follows from the fact that $(\overline{\lambda(\tilde{a})}, m_2)$ and $(1, m_2')$ both map to $\overline{m_1'} \in M$. $\overline{s}$ is a unit in A so there exists $b \in A'^*$ mapping to $\overline{s}$. Then $(\tilde{a}b, t), (b, s) \in A'^* \times_M M_2$ and the equations

$$a'\tilde{a}b = ab, \quad (m_1, m_2)(\tilde{a}b, t) = (m_1', m_2')(b, s)$$

imply that the triples $(a, m_1, m_2), (a', m_1', m_2')$ define the same element of $A'^* \oplus_{A'^* \times_M M_2} (M_1 \times_M M_2)$. □

3.2.4. If $R \to A$ is a ring homomorphism, and if $\mathcal{M}_A$ is a S-log structure on $Spec(A)$, let $LOG_{S,\mathcal{M}_A}(R)$ be the category whose objects are S-log structures $\mathcal{N}_R$ on $Spec(R)$ together with an isomorphism between $\mathcal{M}_A$ and the pullback of $\mathcal{M}_R$ to $Spec(A)$. Morphisms in $LOG_{S,\mathcal{M}_A}(R)$ are morphisms of S-log structures inducing the identity on $\mathcal{M}_A$ when pulled back to $Spec(A)$.

Theorem 3.2.5. *Suppose given a deformation situation and an S-log structure $\mathcal{M}_A$ on $Spec(A)$. Then the natural functor*

$$LOG_{S,\mathcal{M}_A}(A' \times_A B) \to LOG_{S,\mathcal{M}_A}(A') \times LOG_{S,\mathcal{M}_A}(B)$$

is an equivalence of categories.

Proof. By applying the theorem to both objects and morphisms it suffices to consider the case when $\mathcal{M}_S$ is trivial. The functor is essentially surjective since the scheme

$$Spec(A' \times_A B) \simeq (|Spec(B)|, f_*O_{A'} \times_{f_*O_A} O_B)$$

and by the preceding proposition if $\mathcal{M}_{A'} \in LOG_{S,\mathcal{M}_A}(A')$ and $\mathcal{M}_B \in LOG_{S,\mathcal{M}_A}(B)$ then

$$f_*\mathcal{M}_{A'} \times_{f_*\mathcal{M}_A} \mathcal{M}_B \to f_*O_{A'} \times_{f_*O_A} O_B$$

is a log structure on $A' \times_A B$ (note here we are also using 3.2.2.3). 3.2.3 implies that this lifting of $\mathcal{M}_{A'}$ and $\mathcal{M}_B$ is unique up to unique isomorphism since any lifting $\mathcal{N}$ comes with a canonical map

$$\mathcal{N} \to f_*\mathcal{M}_{A'} \times_{f_*\mathcal{M}_A} \mathcal{M}_B$$

which must be an isomorphism since it is an isomorphism over $Spec(B)$. Any morphism

$$f_*\mathcal{M}_{A'} \times_{f_*\mathcal{M}_A} \mathcal{M}_B \to f_*\mathcal{M}'_{A'} \times_{f_*\mathcal{M}_A} \mathcal{M}'_B$$

is by the definition of fibre product uniquely determined by the map $\mathcal{M}_B \to \mathcal{M}'_B$ and $\mathcal{M}_{A'} \to \mathcal{M}'_{A'}$. From this the theorem follows. $\square$

3.2.6. It follows from 2.1.5 that if I is an A_0-module, then the set $\overline{LOG_{S,\mathcal{M}_{A_0}}(A_0[I])}$ has a natural A_0-module structure. We denote this module by

$$Ext_{A_0^\sharp}(\mathcal{A}_0, I)$$

for reasons we now explain.

Recall the definition of an extension ([17], III.2.1.4):

Definition 3.2.7. *Let $f : X \to S$ be a morphism of fine log schemes and let I be a quasi-coherent sheaf of O_X-modules on $\underline{X}$. An S-**extension of** X **by** I is a strict closed immersion*

$$i : X \hookrightarrow T$$

of fine log schemes defined by a square zero ideal together with an isomorphism between I and $ker(i^{\#})$. A morphism of extensions

$$(i_1 : X \to T_1) \to (i_2 : X \to T_2)$$

is a morphism $g : T_1 \to T_2$ such that $g \circ i_1 = i_2$ such that g induces the identity map on I. We denote the set of isomorphism classes of S-extensions of X by I by

$$Ext_S(X, I).$$

3.2.8. With notation as in 3.2.5, let X be $(Spec(A_0), \mathcal{M}_{A_0})$ and $X^\sharp$ the log scheme

$$(Spec(A_0), f^*\mathcal{M}_S).$$

Then

$$Ext_{X^\sharp}(X, I)$$

is in natural bijection with

$$\overline{LOG_{S,\mathcal{M}_{A_0}}(A_0[I])}.$$

This is immediate from the definitions.

The following theorem records the basic facts about $Ext_S(X, I)$ which we will need:

Theorem 3.2.9 ([17], III.2.3.3). *1. The set $Ext_S(X, I)$ has a natural structure of a $\Gamma(X, O_X)$-module.*

2. If $f : X \to Y$, $g : Y \to S$ are morphisms of log schemes, then there is a natural exact sequence of modules

$$0 \to Der_{X/Y}(I) \to Der_{X/S}(I) \to Der_{Y/S}(f_*I) \to Ext_Y(X, I).$$

3. If in addition f is an affine morphism, then the sequence in (2) can be extended by the sequence

$$Der_{Y/S}(f_*I) \longrightarrow Ext_Y(X, I) \longrightarrow Ext_S(X, I) \longrightarrow Ext_S(Y, f_*I).$$

Proposition 3.2.10. *Let $\underline{X} = Spec(A)$ be an affine scheme and let P be a fine monoid. Let $X[P]$ be the log scheme over $\underline{X}$ given by $Spec(P \to A[P])$. Then for all quasi-coherent I on $X[P]$ there is a natural isomorphism of A-modules*

$$Ext^1(P^{gp}, I) \simeq Ext_{\underline{X}}(X[P], I)$$

where $Ext^1(P^{gp}, I)$ denotes isomorphism classes of extensions of abelian groups.

Proof. Given an extension E of abelian groups

$$0 \longrightarrow 1 + I \xrightarrow{j} G \xrightarrow{\pi} P^{gp} \longrightarrow 0$$

define $P' = P \times_{P^{gp}} G$ and let R_E be the ring

$$A[I][P']$$

modulo the relations

$$(1+i)e(p') = e(j(1+i)+p'), \quad p' \in P', i \in I$$

and

$$ie(p') = \pi(p')i, \quad p' \in P', i \in I$$

where $\pi(p')i$ denote the action of $\pi(p')$ on the element $i \in I$. There is a natural commutative diagram

$$\begin{array}{ccc} P' & \longrightarrow & R_E \\ \big\downarrow & & \big\downarrow \\ P & \longrightarrow & A[P]. \end{array}$$

It follows from the definition of R_E that

$$I \simeq Ker(R_E \to A[P])$$

and that P' induces a lifting of $\mathcal{M}_{A[P]}$ to $Spec(R_E)$.

Conversely given a lifting $\mathcal{M}_R \to R$ of $\mathcal{M}_{A[P]} \to A[P]$ we get an extension E by pullback

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 1+I & \longrightarrow & \mathcal{M}_R^{gp} & \longrightarrow & \mathcal{M}_{A[P]}^{gp} & \longrightarrow & 0 \\ & & {\scriptstyle =}\big\uparrow & & \big\uparrow & & \big\uparrow & & \\ 0 & \xrightarrow{j} & 1+I & \longrightarrow & G & \xrightarrow[\pi]{} & P^{gp} & \longrightarrow & 0. \end{array}$$

Applying the above construction to E gives a monoid P' and a commutative diagram

$$\begin{array}{ccc} P' & \longrightarrow & \mathcal{M}_R \\ \big\downarrow & & \big\downarrow \\ P & \longrightarrow & \mathcal{M}_A. \end{array}$$

From this it follows that there is a natural isomorphism of liftings

$$(R_E, \mathcal{M}_{R_E}) \simeq (R, \mathcal{M}_R).$$

□

3.2.11. Suppose $X = Spec(A)$ and that I is a quasi-coherent A-module. Let $\mathcal{M}_{A[I]}$ be a lifting of an S-log structure $\mathcal{M}_A$ to $A[I]$. From the diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & (Spec(A[I]), \mathcal{M}_{A[I]}) \\ {\scriptstyle i}\big\downarrow & & \big\downarrow \\ (Spec(A[I]), \mathcal{M}_{A[I]}) & \longrightarrow & ((Spec(A[I]), g^*\mathcal{M}_S) \end{array}$$

we see that the set of infinitesimal automorphisms of $\mathcal{M}_{A[I]}$ is a torsor under

$$Der_{\mathcal{A}[I]/A[I]^\sharp}(i_*I)$$

which is a quasi-coherent sheaf. It follows that the functor on A-algebras

$$R \mapsto Ext_{R^\sharp}(\mathcal{R}, I \otimes R)$$

is a sheaf for the fppf-topology since

$$H^i(Spec(A), Der_{\mathcal{A}[I]/A[I]^\sharp}(i_*I)) = 0, \quad i > 0$$

because $Spec(A)$ is affine.

Theorem 3.2.12. *1. $Ext_{A^\sharp}(\mathcal{A}, I)$ is a quasi-coherent sheaf in the $fppf$-topology and is coherent if I is.*

2. Suppose A is reduced, $\mathfrak{m} \subset A$ a maximal ideal, $\hat{A}$ the completion of A along $\mathfrak{m}$. Suppose in addition that I is coherent. Then the natural map

$$Ext_{A^\sharp}(\mathcal{A}, I) \otimes \hat{A} \to \varprojlim Ext_{A^\sharp}(\mathcal{A}, I/\mathfrak{m}^n I)$$

is an isomorphism.

3. If A is reduced and I is coherent then there exists a dense set of points $p \in Spec(A)$ for which the natural map

$$Ext_{A^\sharp}(\mathcal{A}, I) \otimes k(p) \to Ext_{A^\sharp}(\mathcal{A}, I \otimes k(p))$$

is an isomorphism.

Proof. Let X (resp. $X^\sharp$) be the log scheme with underlying scheme $Spec(A)$ and log structure $\mathcal{M}_A$ (resp. $f^*\mathcal{M}_S$).

To prove the theorem it suffices to consider the case when $f^*\mathcal{M}_S = O_X^*$. For by applying 3.2.9 to the sequence of morphisms

$$X \to X^\sharp \to \underline{X}$$

we get an exact sequence

$$\begin{array}{ccccccc} 0 & \to & Der_{X/X^\sharp}(I) & \to & Der_{X/\underline{X}}(I) & \to & Der_{X^\sharp/\underline{X}}(I) & \to \\ & \to & Ext_{X^\sharp}(X, I) & \to & Ext_{\underline{X}}(X, I) & \to & Ext_{\underline{X}}(X^\sharp, I). \end{array}$$

A diagram chase combined with standard facts about the modules of derivation shows that it is enough to prove the theorem for $Ext_{\underline{X}}(X, I)$ and $Ext_{\underline{X}}(X^\sharp, I)$.

Thus we may assume that $X^\sharp = \underline{X}$ and by replacing X by an fppf-cover we can also assume that we have a chart

$$P \to \mathcal{M}_X.$$

Applying 3.2.5 to

$$\begin{array}{ccc} & & A[P] \\ & & \downarrow \\ A[I] & \longrightarrow & A \end{array}$$

and observing that $A[P] \times_A A[I] \simeq A[P][I]$ where I is viewed as an $A[P]$-module via $A[P] \to A$, we see that it is enough to consider the case when $X = X[P]$.

From the diagram

$$X[P] \to \underline{X[P]} \to \underline{X}$$

we get a sequence

$$0 \to Der_{X[P]/\underline{X[P]}}(I) \to Der_{X[P]/\underline{X}}(I) \to Der_{\underline{X[P]}/\underline{X}}(I) \to$$

$$\to Ext_{\underline{X[P]}}(X[P], I) \to Ext_{\underline{X}}(X[P], I) \to Ext_{\underline{X}}(\underline{X[P]}, I).$$

By ([15], III.1.2.3) we have an isomorphism

$$Ext_{\underline{X}}(\underline{X[P]}, I) \simeq Ext^1(L_{\underline{X}[P]/\underline{X}}, I)$$

where $L_{\underline{X}[P]/\underline{X}}$ denotes the cotangent complex of $\underline{X}[P]/\underline{X}$, and by 3.2.10 we have

$$Ext_{\underline{X}}(X[P], I) \simeq Ext^1(P^{gp}, I).$$

It follows from standard facts that the statements of the theorem hold for these modules and hence from a diagram chase it follows that the theorem holds for

$$Ext_{\underline{X[P]}}(X[P], I).$$

This concludes the proof of the theorem. $\square$

The following related result will also be used in what follows:

Theorem 3.2.13. *If $X = Spec(\mathcal{A})$ and I as a quasi-coherent sheaf on $\underline{X}$ then $Ext_S(X, I)$ is a quasi-coherent sheaf in the fppf-topology on $\underline{X}$ and is coherent if I is.*

Proof. Applying 3.2.9 to the sequence

$$X \to S \to \underline{S}$$

we see that it suffices to consider the case when $S = \underline{S}$ to prove the quasi-coherence. In addition to this assumption we can assume that we have a chart $P \to \mathcal{M}_X$. Then we get a diagram

$$X \to S[P] \to \underline{S}$$

and hence a sequence

$$Ext_{S[P]}(X, I) \to Ext_{\underline{S}}(X, I) \to Ext_{\underline{S}}(S[P], I).$$

Since the map $X \to S[P]$ is strict we can determine the cokernel of

$$Ext_{\underline{S}}(X, I) \to Ext_{\underline{S}}(S[P], I).$$

Given a an S-extension

$$S[P] \hookrightarrow T$$

of $S[P]$ the obstruction to lifting T to an S-extension of X is the same as the obstruction to lifting the extension of schemes

$$\underline{S[P]} \hookrightarrow \underline{T}$$

to $\underline{X}$. For if we have such a lifting of $\underline{T}$ we get a commutative diagram

$$\begin{tikzcd} \underline{X} \arrow[r] \arrow[d] & \underline{T'} \arrow[d] \\ \underline{S[P]} \arrow[r] & \underline{T} \end{tikzcd}$$

and if we give $\underline{T}'$ the pullback log structure from T we get a class in $Ext_{\underline{S}}(X, I)$.

By ([15], 1.2.5.3) this obstruction lies in

$$Ext^2_A(L_{A/S[P]}, I)$$

where $L_{A/S[P]}$ is the cotangent complex. We therefore have an exact sequence

$$Der_{X/S[P]}(I) \to Ext_{S[P]}(X, I) \to Ext_{\underline{S}}(X, I) \to Ext_{\underline{S}}(S[P], I) \to Ext^2_A(L_{A/S[P]}, I)$$

and from this it follows that $Ext_{\underline{S}}(X, I)$ is quasi-coherent.

To prove that $Ext_S(X, I)$ is coherent when I has this property consider the sequence

$$X \to X^\sharp \to S.$$

This gives a sequence

$$Der_{X^\sharp/S}(I) \to Ext_{X^\sharp}(X, I) \to Ext_S(X, I) \to Ext_S(X^\sharp, I)$$

and all the terms are known to be coherent except $Ext_S(X, I)$. □

3.3 A second construction of LOG_S

Using the results of the preceding section we now give a different proof of 3.1.1 using Artin's theorem 2.1.11.

3.3.1 *(Second proof of 3.1.1).* Throughout this proof we think of LOG_S as classifying log structures on the étale site.

We define an obstruction theory for LOG_S as follows. If $A \to A_0$ is as in 2.1.2 and if $\mathcal{M}_A$ is a log structure on A and I is an A_0-module of finite type we define $\mathcal{O}_a(I)$ to be

$$Coker(Ext_S(\mathcal{A}, I) \to Ext_{\underline{S}}(A, I)).$$

If $A' \to A$ is a surjection with kernel I then A' defines a class in $Ext_{\underline{S}}(A, I)$ and we define $o_a(A') \in \mathcal{O}_a(I)$ to be the image of this class. This has the right properties by 3.2.13.

In light of the results of the preceding section, the only conditions in 2.1.11 which remain to be verified are 2.1.11.2 and 2.1.11.4.

3.3.2. We claim that the functor in 2.1.11.2 is an equivalence of categories. For later use we prove a slightly more general statement:

Proposition 3.3.3. *Let $\hat{A}$ be a complete noetherian local ring with residue field of finite type over $\underline{S}$, and let R be a finite $\hat{A}$-algebra. Then the natural functor*

$$(\textit{fine log structures on } Spec(R)) \to \varprojlim(\textit{fine log structures on } Spec(R \otimes \hat{A}/\mathfrak{m}^n))$$

is an equivalence of categories (here we do not require that morphisms between log structures are isomorphisms).

Proof. Since $\hat{A}$ is complete and $R/\hat{A}$ is finite, R is a product of complete local $\hat{A}$-algebras. Because R is local and finite over $\hat{A}$, the filtrations induced by $\mathfrak{m}_{\hat{A}}$ and $\mathfrak{m}_R$ are cofinal and therefore we may assume that $R = \hat{A}$.

Let $A_n = \hat{A}/\mathfrak{m}^{n+1}$, $j_n : Spec(A_n) \to Spec(\hat{A})$ the natural map. The key observation is the following: if $\mathcal{M}$ is a log structure on $Spec(\hat{A})$, then the natural map

$$H^0(Spec(\hat{A}), \mathcal{M}) \to \varprojlim H^0(Spec(A_n), j_n^*\mathcal{M})$$

is an isomorphism. For this it suffices to show that the map

$$\pi : H^0(Spec(\hat{A}), \mathcal{M}^{gp}) \to \varprojlim H^0(Spec(A_n), j_n^*\mathcal{M}^{gp})$$

is an isomorphism. Since $\hat{A}$ is a local ring

$$H^1(Spec(\hat{A}), O^*_{\hat{A}}) = 0.$$

It follows that we have a commutative diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\hat{A}^* & \xrightarrow{\pi_1} & \varprojlim A_n^* \\
\downarrow & & \downarrow \\
H^0(Spec(\hat{A}), \mathcal{M}^{gp}) & \xrightarrow{\pi} & \varprojlim H^0(Spec(A_n), j_n^*\mathcal{M}^{gp}) \\
\downarrow & & \downarrow \\
H^0(Spec(\hat{A}), \overline{\mathcal{M}}^{gp}) & \xrightarrow{\pi_2} & \varprojlim H^0(Spec(A_n), j_n^{-1}\overline{\mathcal{M}}^{gp}) \\
\downarrow & & \downarrow \\
0 & & 0.
\end{array}$$

Here the exactness of the right column follows from the fact that the maps $A_{n+1}^* \to A_n^*$ are surjective since A_n is artinian. By ([6], VIII.8.6) π_2 is an isomorphism, and since $\hat{A}$ is complete, the map π_1 is also an isomorphism. We conclude that π is an isomorphism. Note that for each n, the map

$$H^0(Spec(A_{n+1}), j_{n+1}^*\mathcal{M}) \to H^0(Spec(A_n), j_n^*\mathcal{M})$$

is surjective. This follows from the exact sequence

$$0 \longrightarrow 1 + \mathfrak{m}^n/\mathfrak{m}^{n+1} \longrightarrow j_{n+1}^*\mathcal{M}^{gp} \longrightarrow j_n^*\mathcal{M}^{gp} \longrightarrow 0$$

and the fact that

$$H^1(Spec(A_n), \mathfrak{m}^n/\mathfrak{m}^{n+1}) = 0$$

since $Spec(A_n)$ is affine. Using the isomorphism π we conclude that the map

$$H^0(Spec(\hat{A}), \mathcal{M}) \to H^0(Spec(A_0), j_0^*\mathcal{M})$$

is surjective and identifies $H^0(Spec(A_0), j_0^*\mathcal{M})$ with the quotient of $H^0(Spec(\hat{A}), \mathcal{M})$ by the subgroup consisting of those elements in $\hat{A}^*$ which reduce to 1 modulo $\mathfrak{m}$.

This has the following implication. Suppose $\mathcal{M}$ is a fine log structure such that the map

$$H^0(Spec(A_0), j_0^*\mathcal{M}) \to j_0^*\mathcal{M}$$

is a chart. Then the map

$$H^0(Spec(\hat{A}), \mathcal{M}) \to \mathcal{M}$$

is also a chart. To see this let $\mathcal{N}$ be the log structure associated to $H^0(Spec(\hat{A}), \mathcal{M})$, K the elements in $\hat{A}$ whose reduction modulo $\mathfrak{m}$ is equal to 1. Then we have an isomorphism

$$H^0(Spec(\hat{A}), \mathcal{M})/K \simeq H^0(Spec(A_0), j_0^*\mathcal{M}).$$

By the construction of the log structure associated to a chart, it follows that the map

$$H^0(Spec(\hat{A}), \mathcal{M}) \to j_0^*\mathcal{M}$$

is a chart for $j_0^*\mathcal{M}$. Therefore, if $\bar{x} \to Spec(\hat{A})$ is a geometric point of the closed fiber, the natural map $\mathcal{N} \to \mathcal{M}$ induces an isomorphism

$$\overline{\mathcal{N}}_{\bar{x}} \simeq \overline{j_0^*\mathcal{M}}_{\bar{x}} \simeq \overline{\mathcal{M}}_{\bar{x}}.$$

By 1.1.2 the map

$$\mathcal{N}_{\bar{x}} \to \mathcal{M}_{\bar{x}}$$

is also an isomorphism.

Now since every point of $Spec(\hat{A})$ specializes to a point of the closed fiber, any map between quasi-coherent log structures which is an isomorphism at every point in the closed fiber must be an isomorphism everywhere. Therefore, $\mathcal{N} = \mathcal{M}$.

To see that the functor is fully faithful, suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are two fine log structures on $Spec(\hat{A})$ and that we are given a compatible family of isomorphisms

$$\psi_n : j_n^*\mathcal{M}_1 \to j_n^*\mathcal{M}_2.$$

We have to show that the ψ_n are induced from a unique map

$$\psi : \mathcal{M}_1 \to \mathcal{M}_2.$$

By the uniqueness, it suffices to construct ψ étale locally.

Let us show that after passing to a finite étale extension of $\hat{A}$ we can assume that the maps

$$\varprojlim H^0(Spec(A_n), j_n^*\mathcal{M}_i) \to \mathcal{M}_i$$

are charts. If we show this, then the existence and uniqueness of ψ is clear.

To prove that the map

$$\varprojlim H^0(Spec(A_n), j_n^*\mathcal{M}_i) \to \mathcal{M}_i$$

is a chart it suffices by the above discussion to show that

$$H^0(Spec(A_0), j_0^*\mathcal{M}_i) \to j_0^*\mathcal{M}_i$$

is a chart.

If B is any artin local ring, $\mathcal{M}$ a fine log structure on B, then after passing to a finite étale extension $\overline{\mathcal{M}}$ becomes constant and we have an exact sequence

$$0 \longrightarrow B^* \longrightarrow H^0(Spec(B), \mathcal{M}) \longrightarrow H^0(Spec(B), \overline{\mathcal{M}}) \longrightarrow 0.$$

Since B is local and artinian, any element $m \in H^0(Spec(B), \mathcal{M})$ which maps to a unit in some stalk $O_{B,\bar{x}}$ must be an element of B^*. Therefore, if $\mathcal{N}$ denotes the log structure associated to $H^0(Spec(B), \mathcal{M})$ we have

$$\overline{\mathcal{N}} \simeq H^0(Spec(B), \overline{\mathcal{M}})_{Spec(B)} \simeq \overline{\mathcal{M}}.$$

This implies that $\mathcal{N} \to \mathcal{M}$ is an isomorphism, and hence also the full faithfulness.

To prove essential surjectivity, let $\mathcal{M}_n$ be a compatible family of fine log structures on the $Spec(A_n)$. We wish to construct $\mathcal{M}$ on $Spec(\hat{A})$ inducing the $\mathcal{M}_n$. By the full faithfulness it suffices to construct $\mathcal{M}$ étale locally on $Spec(\hat{A})$. Replacing $Spec(\hat{A})$ by an étale cover we may therefore assume that the map

$$H^0(Spec(A_0), \mathcal{M}_0) \to \mathcal{M}_0$$

is a chart. As explained above, it follows that the map

$$H^0(Spec(A_n), \mathcal{M}_n) \to \mathcal{M}_n$$

is a chart for all n. Thus we take for $\mathcal{M}$ the log structure associated to

$$\varprojlim H^0(Spec(A_n), \mathcal{M}_n) \to \varprojlim A_n = \hat{A}.$$

This is a quasi-coherent log structure which induces the family $\mathcal{M}_n$. It remains to see that it is coherent.

We claim that any quasi-coherent log structure $\mathcal{M}$ on $Spec(\hat{A})$ for which $j_0^*\mathcal{M}$ is coherent is actually coherent. To see this, let $\bar{x} \to Spec(\hat{A})$ be a geometric point. Then

$$\overline{\mathcal{M}}_{\bar{x}} \simeq \overline{j_0^*\mathcal{M}}_{\bar{x}}$$

is a finitely generated monoid. Therefore, there exists in some étale neighborhood of $\bar{x}$ a map

$$P \to \mathcal{M}$$

from a fine monoid P which induces a chart at $\bar{x}$. After choosing a perhaps even smaller neighborhood of $\bar{x}$ we can assume that P induces a chart for $\mathcal{M}_0$ on all of $Spec(A_0)$. It follows that if $\mathcal{N}$ is the log structure associated to P, then the map

$$\mathcal{N} \to \mathcal{M}$$

is an isomorphism at every point in the closed fiber. Therefore it is an isomorphism everywhere since they are both quasi-coherent. Hence $\mathcal{M}$ is fine. □

2.1.11.4 follows from the following lemma which we state in greater generality for later use:

Lemma 3.3.4. *Let A_0 be a reduced ring, $f : \underline{X} \to Spec(A_0)$ a proper flat morphism of reduced schemes, $\mathcal{M}_X$ a log structure on $\underline{X}$, ϕ an automorphism of $\mathcal{M}_X$ which induces the identity over $Spec(k)$ for a dense set of points $A_0 \to k$. Then there exists a dense open set $\underline{U} \subset Spec(A_0)$ such that ϕ restricted to $\underline{X}_U/\underline{U}$ is the identity.*

Proof. Let $\bar{\phi} : \overline{\mathcal{M}}_X^{gp} \to \overline{\mathcal{M}}_X^{gp}$ be the induced map. $\overline{\mathcal{M}}_X^{gp}$ is a sheaf of finitely generated abelian groups which locally is a quotient of a constant sheaf, and hence the set of points where $\phi \neq id$ is a closed set $\underline{Z} \subset \underline{X}$. Since f is proper, $f(\underline{Z}) \subset Spec(A_0)$ is a closed set, and by assumption $f(\underline{Z})^c$ is dense. Thus we may replace $Spec(A_0)$ by $f(Z)^c$ and assume that $\bar{\phi}$ is the identity.

From the exact sequence

$$0 \to O_X^* \to \mathcal{M}_X^{gp} \to \overline{\mathcal{M}}_X^{gp} \to 0$$

we deduce that the automorphism ϕ must be of the form $m \mapsto \lambda(\eta(m)) + m$ where

$$\eta : \overline{\mathcal{M}}_X^{gp} \to O_X^*$$

is a group homomorphism. Consider the set of points $x \in \underline{X}$ such that $\eta_x : \overline{\mathcal{M}}_X^{gp} \to k(\bar{x})$ is the trivial map. This is a closed set since $\overline{\mathcal{M}}_X^{gp}$ is locally a quotient of a finitely generated constant sheaf. Hence the complement is an open set $\underline{V} \subset \underline{X}$. Since f is flat the image of $\underline{V}$ in $Spec(A_0)$ is open and its complement is dense by assumption. Hence $\underline{V}$ is the empty set, and η_x is the trivial map for all x. Since x is reduced it follows that η is trivial. □

This concludes the proof of the theorem. □

3.4 Basic properties of LOG_S and variants

Proposition 3.4.1. *1. If $f : X \to S$ is a morphism of log schemes then the morphism of stacks*

$$LOG_X \to LOG_S$$

induced from the map $f^*\mathcal{M}_S \to \mathcal{M}_X$ is relatively representable.

2. If $g : Y \to S$ is a second morphism of log schemes then there is a canonical isomorphism of algebraic stacks

$$LOG_X \times_{LOG_S} LOG_Y \simeq LOG_{X \times_S Y}$$

where $X \times_S Y$ denotes the fiber product in the category of integral log schemes.

Proof. (1) follows from the statement that the functor $LOG_X \to LOG_S$ is faithful. For this implies that for any $\underline{T} \to LOG_S$ the product

$$LOG_X \times_{LOG_S} \underline{T}$$

is a functor and hence an algebraic space.

(2) follows from the universal property of fiber product. □

Remark 3.4.2. The functor $LOG_X \to LOG_S$ is locally of finite presentation if and only if $\underline{X}/\underline{S}$ has this property. This can be checked on a flat cover $\underline{T}' \to LOG_X$. For any $T \to LOG_S$ the functor $\underline{T}' \times_{LOG_S} \underline{T}$ is the functor I over $\underline{X} \times_{\underline{S}} \underline{T}$ corresponding to the diagram

$$\begin{array}{ccc} \mathcal{M}_S|_{\underline{X} \times_{\underline{S}} \underline{T}} & \longrightarrow & p_1^*\mathcal{M}_X \\ \downarrow & & \\ p_2^*\mathcal{M}_T & & \end{array}$$

as in 3.1.2. From the construction of I we see that I is locally of finite presentation over $\underline{X} \times_S \underline{T}$ and hence $LOG_X \to LOG_S$ is locally of finite presentation if and only if $\underline{X}/\underline{S}$ is.

We propose the following definition for translating classical definitions into the logarithmic language:

Definition 3.4.3. *Let P be a property of morphisms $f : \underline{X} \to \underline{Y}$ between algebraic spaces over $\underline{S}$ which is stable under base change $\underline{Y}' \to \underline{Y}$ and is local for the fppf-topology on $\underline{Y}$. Then if $f : X \to S$ is a morphism of log schemes we say that f* **has property** *P if the induced relatively representable morphism of stacks*

$$LOG_X \to LOG_S$$

has property P.

Proposition 3.4.4. *Let $f : X \to S$ be a morphism of log schemes.*

1. *f is formally log smooth (resp. formally log étale) if and only if the map*

$$LOG_X \to LOG_S$$

is a formally smooth (resp. formally étale) morphism of stacks. This is also equivalent to the morphism

$$\underline{X} \to LOG_S$$

being formally smooth (resp. formally étale).

2. *There is a natural isomorphism*

$$\Omega^1_{X/S} \simeq \Omega^1_{\underline{X}/LOG_S}$$

where $\Omega^1_{X/S}$ denotes the logarithmic differentials.

3. *The morphism $LOG_X \to LOG_S$ is flat if and only if locally in the fppf topology on $\underline{X}$ and $\underline{S}$ there exists a chart*

$$\begin{array}{ccc} P & \longrightarrow & \mathcal{M}_X \\ {\scriptstyle\theta}\uparrow & & \uparrow \\ Q & \longrightarrow & \mathcal{M}_S \end{array}$$

with θ injective and the map

$$\underline{X} \to \underline{S} \times_{Spec(\mathbb{Z}[Q])} Spec(\mathbb{Z}[P])$$

flat.

Proof. That $f : X \to S$ is formally log smooth (resp. formally log étale) if and only if $LOG_X \to LOG_S$ has the corresponding property is immediate from the definitions and the infinitesimal lifting property.

The statement that X/S is formally log smooth (resp. formally log étale) if and only if $\underline{X} \to LOG_S$ is formally smooth (resp. formally étale) follows from the fact that when verifying the infinitesimal lifting property for a morphism of log schemes it suffices to consider diagrams

$$\begin{tikzcd} Z_0 \arrow[r, "z_0"] \arrow[d] & X \arrow[d] \\ Z \arrow[r] & S \end{tikzcd}$$

where z_0 is strict. For given an arbitrary such diagram let $Z_0^\sharp = (\underline{Z}_0, z_0^*\mathcal{M}_X)$ and let $Z^\sharp$ be the log scheme $\underline{Z}$ together with the sheaf of monoids

$$\mathcal{M}_Z \times_{\mathcal{M}_{Z_0}} z_0^*\mathcal{M}_X.$$

Then a lifting of z_0 is equivalent to a lifting of $z_0^\sharp$ filling in the diagram

$$\begin{tikzcd} Z_0^\sharp \arrow[r, "z_0^\sharp"] \arrow[d] & X \arrow[d] \\ Z^\sharp \arrow[r] & S. \end{tikzcd}$$

The second statement in the proposition follows from the universal property of differentials.

To prove statement (3) we may replace $\underline{S}$ by an fppf-cover and hence can assume that we have a chart $Q \to \mathcal{M}_S$. Let $\tilde{S} \to LOG_S$ be the flat cover given by 3.1.12. Suppose $LOG_X \to LOG_S$ if flat. By (1) the canonical map $\underline{X} \to LOG_X$ is étale and hence we obtain a diagram

$$\begin{tikzcd} \underline{X} & \tilde{S} \times_{LOG_S} \underline{X} \arrow[l, "g"'] \arrow[d, "h"'] \\ & \tilde{S} \end{tikzcd}$$

where g is an fppf cover and h is flat. By definition of $\tilde{S}$ this means that locally in the fppf-topology on $\underline{X}$ we have a cover as in (3).

Coversely suppose we locally have charts as in (3). The statement that $LOG_X \to LOG_S$ is flat is fppf local and hence we may assume that in fact we have a global such chart. For any injective morphism of fine monoids $P \to P'$ let $X[P']$ be as in 3.2.10. The $X[P']$ form a flat cover of LOG_X and hence it suffices to verify that each of the $X[P']$ are flat over

LOG_S. But we have a commutative diagram

$$\begin{array}{ccc} LOG_X & \longleftarrow & X[P'] \\ \downarrow & & \downarrow \\ LOG_S & \longleftarrow & S[P'] \end{array}$$

where $S[P']$ is the scheme obtained from $Q \to P \to P'$. The map $S[P'] \to LOG_S$ is flat and so is the map

$$X[P'] = X \times_{Spec(\mathbb{Z}[P])} Spec(\mathbb{Z}[P']) \to S[P'] = S \times_{Spec(\mathbb{Z}[Q])} Spec(\mathbb{Z}[P]) \times_{Spec(\mathbb{Z}[P])} Spec(\mathbb{Z}[P'])$$

and consequently their composite is also flat. □

Corollary 3.4.5. *If $f : X \to S$ is a morphism of log schemes, then the set of points $x \in \underline{X}$ where f is log smooth (resp. log étale) is open.*

Proof. The set in question is the same as the set of points where the morphism

$$\underline{X} \to LOG_S$$

is smooth. □

Next we derive some standard results on log smoothness using the stacks LOG_S.

Proposition 3.4.6. *Suppose a chart $Q \to \mathcal{M}_S$ is given and that $Q \to P$ is an injective morphism between fine monoids. Then the induced map*

$$\underline{S}[P] \to LOG_S$$

is formally smooth if and only if $\#(P^{gp}/Q^{gp})_{tor}$ is invertible on $\underline{S}$.

Proof. The infinitesimal lifting property in this case amounts to the following: Let A be an artinian local S-ring and let $\mathcal{M}_A$ be an S-log structure on A with a chart

$$\begin{array}{ccc} P & \xrightarrow{\eta} & \mathcal{M}_A \\ \uparrow & & \uparrow \\ Q & \longrightarrow & \mathcal{M}_S. \end{array}$$

Then for any surjection $A' \to A$ defined by a square zero ideal I and a lifting of $\mathcal{M}_A$ to a log structure $\mathcal{M}_{A'}$ on A' does there exist a lifting of the chart to $\mathcal{M}_{A'}$?

Let us prove the "if" direction. Write $\tilde{I}$ for the quasi-coherent sheaf on the étale site of $Spec(A)$ associate to I. We have a diagram

$$\begin{array}{ccccc} Q & \longrightarrow & \mathcal{M}_{A'} & \longrightarrow & O_{A'} \\ \downarrow & & \downarrow & & \downarrow \\ P & \xrightarrow{\eta} & \mathcal{M}_A & \longrightarrow & O_A \end{array}$$

and we wish to lift the map η to $\mathcal{M}_{A'}$. In order to fit the diagram on the page, write $\mathcal{H}om(M,N) = [M,N]$ (homomorphisms of sheaves). We have a commutative diagram

$$\begin{array}{ccccccc} [P^{gp}/Q^{gp}, \tilde{I}] & \longrightarrow & [P^{gp}/Q^{gp}, \mathcal{M}_{A'}] & \longrightarrow & [P^{gp}/Q^{gp}, \mathcal{M}_A^{gp}] & \longrightarrow & Ext^1(P^{gp}/Q^{gp}, \tilde{I}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [P^{gp}, \tilde{I}] & \longrightarrow & [P^{gp}, \mathcal{M}_{A'}^{gp}] & \longrightarrow & [P^{gp}, \mathcal{M}_A^{gp}] & \longrightarrow & Ext^1(P^{gp}, \tilde{I}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [Q^{gp}, \tilde{I}] & \longrightarrow & [Q^{gp}, \mathcal{M}_{A'}^{gp}] & \longrightarrow & [Q^{gp}, \mathcal{M}_A^{gp}] & \longrightarrow & Ext^1(Q^{gp}, \tilde{I}) \\ \downarrow & & & & & & \\ Ext^1(P^{gp}/Q^{gp}, \tilde{I}) & & & & & & . \end{array}$$

By our assumptions, $Ext^1(P^{gp}/Q^{gp}, \tilde{I}) = 0$. Thus since $\eta \in [P^{gp}, \mathcal{M}_A^{gp}]$ maps to zero in $Ext^1(Q^{gp}, \tilde{I})$ it must map to zero in $Ext^1(P^{gp}, \tilde{I})$. Thus there exists an element $\tilde{\eta} \in [Q^{gp}, \mathcal{M}_{A'}^{gp}]$ mapping to η. Since

$$[P^{gp}, \tilde{I}] \to [Q^{gp}, \tilde{I}]$$

is surjective we can choose $\tilde{\eta}$ so as to agree with the given map $Q \to \mathcal{M}_{A'}$.

Thus locally there exists a lifting of $P \to \mathcal{M}^{gp}$ to $\mathcal{M}_{A'}^{gp}$. But from the exact sequence,

$$0 \to 1 + I \to \mathcal{M}_{A'} \to \mathcal{M}_A \to 0$$

we see that in fact that image of this lifting of P lies in $\mathcal{M}_{A'}$. This gives a lifting of the chart étale locally.

From the diagram, it follows that the set of liftings of η is a torsor under

$$\mathcal{H}om(P^{gp}/Q^{gp}, \tilde{I}).$$

Since this is a quasi-coherent sheaf, and $Spec(A)$ is affine the obstruction to finding a global lifting vanishes.

To prove the other direction, it suffices to show that we can find a point $Spec(k) \to S[P]$ such that

$$I_k = Spec(k) \times_{LOG_S} S[P] \to Spec(k)$$

is not smooth. The point we will use is any point $Spec(k)$ such that $\bar{P} \to \overline{\mathcal{M}}_k$ is an isomorphism and for which the characteristic divides the order of $(P^{gp}/Q^{gp})_{tor}$. In addition by replacing k by a field extension we can choose an isomorphism

$$\mathcal{M}_k \simeq k^* \oplus P_2$$

where P_2 is a fine sharp monoid. Now in this case we can write down a scheme which represents I_k. Namely, for each isomorphism $\sigma : \overline{P} \to P_2$ let

$$T_\sigma = Spec(k[P^{gp}/Q^{gp}]).$$

Then define a map

$$T_\sigma \to I_k$$

by sending an R-valued point $\phi \in T_\sigma(R)$ corresponding to a map

$$\phi : P^{gp}/Q^{gp} \to R^*$$

to the map

$$P \to R^* \oplus P_2, \quad p \mapsto (\phi(p), \sigma(\bar{p})).$$

It is straightforward to verify that this defines an isomorphism

$$\coprod_\sigma T_\sigma \to I$$

and certainly T_σ is not smooth over k. □

3.4.7. Fix a chart $Q \to \mathcal{M}_S$. Let $LOG_S^{gt} \subset LOG_S$ be the full substack of S-log structures for which $LOG_S^{gt}(\underline{X})$ consists of those log structures for which $\mathcal{M}_{X/S}^{gp,tor} \otimes k(\bar{x}) = 0$ for all $\bar{x} \to \underline{X}$. This is an open condition so LOG_S^{gt} is an algebraic stack. In addition, we can interpret the result of ([29], 2.13) as saying that the union of the maps

$$S[P]$$

with $Q \to P$ satisfying the conditions of the proposition form a smooth cover of LOG_S^{gt}. As a consequence we get the following theorem originally due to K. Kato:

Theorem 3.4.8. *Let $f : X \to S$ be a smooth morphism of fine log schemes. Then*

1. *The morphism $\underline{X} \to LOG_S$ factors through LOG_S^{gt}.*

2. *Locally in the smooth topology on $\underline{X}$ and $\underline{S}$ there exists a chart*

$$\begin{array}{ccc} P & \longrightarrow & \mathcal{M}_X \\ {\scriptstyle\theta}\uparrow & & \uparrow \\ Q & \longrightarrow & \mathcal{M}_S \end{array}$$

with θ injective and $\#(P^{gp}/Q^{gp})^{tor}$ invertible on $\underline{X}$ such that the induced map

$$\underline{X} \to \underline{S} \times_{Spec(\mathbb{Z}[Q])} Spec(\mathbb{Z}[P])$$

is smooth.

Proof. It suffices to prove the first statement for then the second statement follows from the fact that the maps $S[P] \to LOG_S^{gt}$ of the preceding paragraph are smooth and cover LOG_S^{gt}. Thus what has to be shown is that if $\underline{X} \to LOG_S$ is smooth then $\mathcal{M}_{X/S,\bar{x}}^{gp,tor} \otimes k(\bar{x}) = 0$ for all $\bar{x} \to \underline{X}$. For this we may as well assume that $\underline{X} = Spec(R)$ where R is strictly henselian and local. Let $Spec(k) \to \underline{X}$ be a point mapping to the closed point with k algebraically closed and let $M = \Gamma(\underline{X}, \mathcal{M}_X)$, $N = \Gamma(\underline{X}, f^*\mathcal{M}_S)$. Let $J \subset R$ be the ideal generated by the images of the elements $m \in M - M^*$. It follows from the definitions that

$$I = \underline{X} \times_{LOG_S} Spec(k)$$

is isomorphic to a $Hom(M^{gp}/N^{gp}, \mathbb{G}_m)$-torsor over $R/J \otimes_{O_S} k$. In particular if $p = char(k)$ and there exists an element of order p in M^{gp}/N^{gp} then I is not reduced and hence cannot be smooth over k. $\square$

3.4.9. One can consider variants on the stack LOG_S. We denote by LOG_S^{sat} the full substack of LOG_S consisting of S-log structures which are saturated. The morphism

$$LOG_S^{sat} \to LOG_S$$

is an open immersion of algebraic stacks, because the condition of a log structure being saturated is an open condition.

Corollary 3.4.10. *Let S be a fine log scheme and let S^{sat} be the associated saturated log scheme. Then*

1. *There is a canonical equivalence of stacks*

$$LOG_S^{sat} \simeq LOG_{S^{sat}}^{sat}$$

compatible with the maps to LOG_S.

2. If $f : X \to S$ is a morphism with X saturated, then there is a canonical isomorphism

$$\Omega^1_{X/S^{sat}} \simeq \Omega^1_{X/S}.$$

Proof. Statement (1) follows from the universal property of S^{sat}. The second statement follows from the fact that

$$LOG^{sat}_{sat} \to LOG_S$$

is an open immersion. □

3.4.11. A second variant of LOG_S which is often used is the stack of idealized log structures LOG^{id}_S.

Recall that an idealized log scheme consists of a fine log scheme X and a coherent sheaf of ideals $K \subset \alpha^{-1}(0) \subset \mathcal{M}_X$.

Let S be an idealized log scheme, and let LOG^{id}_S be the fibered category over the category of $\underline{S}$-schemes for which $LOG^{id}_S(f : \underline{X} \to \underline{S})$ is the category of idealized log structures $(\mathcal{M}_X, K_X)$ together with a morphism

$$f^*(\mathcal{M}_S, K_S) \to (\mathcal{M}_X, K_X).$$

Since LOG^{id}_S classifies sheaves on the étale site LOG^{id}_S is a stack with respect to the étale topology. Given a diagram

$$\begin{array}{ccc} & & \underline{X} \\ & & \downarrow \\ \underline{Y} & \longrightarrow & LOG^{id}_S \end{array}$$

there is a natural injective morphism of functors

$$j : \underline{X} \times_{LOG^{id}_S} \underline{Y} \to \underline{X} \times_{LOG_S} \underline{Y}$$

where the map

$$LOG^{id}_S \to LOG_S$$

is obtained by forgetting the ideal. Now if X is any log scheme and K_X, K'_X are two coherent sheaves of ideals in $\mathcal{M}_X$ (not necessarily in $\alpha^{-1}(0)$) then the condition that $K_X = K'_X$ is an open condition by the coherence assumption. It follows from this that

$$\underline{X} \times_{LOG^{id}_S} \underline{Y}$$

is representable by an open subset of $\underline{X} \times_{LOG_S} \underline{Y}$.

Suppose P is a fine monoid and that $K \subset P$ is a finitely generated ideal. Then we can form the idealized log scheme

$$D(P, K) = Spec(P \to \mathbb{Z}[P]/\mathbb{Z}[K])$$

(where $\mathbb{Z}[K]$ denotes the ideal generated by K). If (P', K') is a second such pair then the projection

$$D(P, K) \times_{LOG_S^{id}} D(P', K') \to D(P, K)$$

is flat. For this product is representable by an open subset of

$$Spec(P \to \mathbb{Z}[P]/\mathbb{Z}[K]) \times_{LOG_S} Spec(P' \to \mathbb{Z}[P'])$$

and so the result follows from 3.1.11. It follows that if

$$\tilde{S} = \coprod_{(P,K)} D(P, K)$$

then

$$\tilde{S} \times_{LOG_S^{id}} \tilde{S} \rightrightarrows \tilde{S}$$

is a flat groupoid and LOG_S^{id} is represented by the associated algebraic stack.

3.5 A log version of Artin's theorem

3.5.1. In this section we apply the results of 2.2 to the case when $B = LOG_S$. Fix a fine log schemes S with $\underline{S}$ locally of finite type over an excellent Dedekind ring. Suppose $F \to FLog_S$ is a limit preserving stack over $FLog_S$ in the sense of 1.3.12. We wish to give conditions for F to be a log algebraic stack.

3.5.2. If $F^\sharp$ is a stack over the category of $\underline{S}$-schemes with a morphism of stacks $m : F^\sharp \to LOG_S$ (m is equivalent to a log structure on $F^\sharp$ if $F^\sharp$ is algebraic) then $F^\sharp$ defines in a natural way a stack F over $FLog_S$. Namely if X is an S-log schemes, the category F_X has as objects pairs (f, f^b) where $f : \underline{X} \to F$ is a 1-morphism and f^b is a morphism of S-log structures

$$f^b : m \circ f \to \mathcal{M}_X.$$

A morphism $(f', f'^b) \to (f, f^b)$ is a morphism $h : f' \to f$ in $F^\sharp_{\underline{X}}$ such that $f'^b = f^b \circ m(h)$.

If F is a log algebraic stack, then there is a tautological log structure $\mathcal{M}_F$ on $\underline{F}$, and we get a map

$$m : \underline{F} \to LOG_S.$$

This is by definition if we define log algebraic stack using 1.3.1 and by theorem 1.3.8 if we use definition 1.3.12, and the category $F/FLog_S$ is just the stack associated to $(\underline{F}, m)$.

3.5.3. This suggests that one may prove representability by a log algebraic stack for F by proceeding in two steps. First show that F is equivalent to the stack over $FLog_S$ defined by a stack in the usual sense $F^\sharp$ with a morphism of stacks

$$m : F^\sharp \to LOG_S.$$

Then apply the results of the preceding section to conclude that $F^\sharp$ is algebraic. In section 4.3 we prove a theorem useful for showing that F comes from a pair $(F^\sharp, m)$ as above. This theorem suffices for our applications, so for the remainder of this section let us assume that we are given such a pair $(F^\sharp, m)$ and work out the conditions for $F^\sharp$ to be algebraic.

Definition 3.5.4. *Suppose $F \to FLog_S$ is a stack which is obtained from a pair $(F^\sharp, m)$. If X is a log scheme and $a \in F_X$ we call a* **solid** *if the corresponding pair*

$$(f : \underline{X} \to F^\sharp, f^b : m \circ f \to \mathcal{M}_X)$$

has the property that f^b is an isomorphism.

3.5.5. The solid objects of F form a category with morphisms being isomorphisms $f' \simeq f$ (not necessarily compatible with f^b). It is immediate that the category of solid objects is naturally equivalent to $F^\sharp$. An important property of solid objects is that if $X_0 \hookrightarrow X$ is a strict closed immersion of log schemes defined by a nilpotent ideal and if $a \in F_X$ is an object then a is solid if and only if the pullback of a to F_{X_0} is solid. This follows from 1.1.2.

3.5.6. Define the category of *noetherian log rings* $\mathcal{C}$ as follows. $\mathcal{C}^{op}$ is the subcategory of $FLog$ of log schemes whose underlying schemes are isomorphic to the spectrum of a noetherian ring. The morphisms in $\mathcal{C}^{op}$ are strict morphisms of log schemes. Note that by 3.2.5, co-products exist in $\mathcal{C}^{op}$ and hence products exist in $\mathcal{C}$. We denote the objects of $\mathcal{C}$ by caligraphic letters and the underlying rings by unadorned letters (i.e. an object $\mathcal{A} \in \mathcal{C}$ has underlying scheme $Spec(A)$). If $\mathcal{A} \in \mathcal{C}$, and if I is an A-module of finite type, then we denote by $\mathcal{A}[I]$ the object of $\mathcal{C}$ with underlying ring $A[I]$ and log structure obtained from A by pullback via the retraction $A \to A[I]$. There is a canonical morphism $\mathcal{A}[I] \to \mathcal{A}$ in $\mathcal{C}$. By construction there is a contravariant functor

$$Spec : \mathcal{C} \to FLog.$$

Definition 3.5.7. *A* **logarithmic deformation situation** *is a diagram in $\mathcal{C}$*

$$\begin{array}{ccccc} & & \mathcal{B} & & \\ & & \downarrow & & \\ \mathcal{A}' & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{A}_0 \end{array}$$

such that

1. *A_0 is reduced.*

2. *$A' \to A$ and $A \to A_0$ are surjections with nilpotent kernel.*

3. *The composite $B \to A_0$ is surjective.*

4. *The kernel $I = Ker(A' \to A)$ is an A_0-module of finite type.*

Condition 2.2.8 becomes:

3.5.8 *(S1^{log}).* Suppose given a deformation situation and $a \in F_{\mathcal{A}}$. Then the canonical functor

$$F_{a,\mathcal{A}' \times_{\mathcal{A}} \mathcal{B}} \to F_{a,\mathcal{A}'} \times F_{a,\mathcal{B}}$$

is an equivalence of categories.

3.5.9. Suppose a logarithmic deformation situation is given, together with an object $a \in F_{\mathcal{A}}$. If the condition (S1^{log}) hold, then for any finite A_0-module I, the set

$$\overline{F}_{a_0,\mathcal{A}_0[I]}$$

has a natural A_0-module structure as before. Similarly, if $Aut_{a_0}(\mathcal{A}_0[I])$ denotes the set of automorphisms of a_0 in $F_{\mathcal{A}_0[I]}$ which induce the identity over $\mathcal{A}_0$ then $Aut_{a_0}(\mathcal{A}_0[I])$ has a natural A_0-module structure. Condition 2.2.10 becomes:

3.5.10 *(S2^{log}).* The modules $\overline{F}_{a_0,\mathcal{A}_0[I]}$ and $Aut_{a_0}(\mathcal{A}_0[I])$ are A_0-modules of finite type.

3.5.11 *(Obstruction Theory).* By an *obstruction theory* for F we will mean data as follows:

A. For every morphism $\mathcal{A} \to \mathcal{A}_0$ in $\mathcal{C}$, with $A \to A_0$ a surjection with nilpotent kernel and A_0 reduced, and $a \in F_{\mathcal{A}}$ a solid object a functor

$$\mathcal{O}_a : (A_0\text{-modules of finite type}) \to (A_0\text{-modules of finite type}).$$

B. For each diagram $\mathcal{A}' \to \mathcal{A} \to \mathcal{A}_0$ as in 3.5.7, an element $o_a(\mathcal{A}') \in \mathcal{O}_a(I)$ which is zero if and only if $\overline{F}_{a,\mathcal{A}'}$ is not empty.

Let $\mathcal{A} \to \mathcal{A}_0$ be a morphism in $\mathcal{C}$ such that $Ker(A \to A_0)$ is nilpotent and A_0 is reduced. Suppose $A' \to A$ is a surjection with kernel an A_0-module of finite type I and that $a \in F_{\mathcal{A}}$. Then for each lifting of $\mathcal{M}_A$ to A', we get a diagram as in (B) and hence via o_a an element of $\mathcal{O}_a(I)$. In other words, o_a defines a set map

$$(\text{liftings of } \mathcal{M}_A \text{ to } A') \to \mathcal{O}_a(I).$$

In particular, there is a natural map

$$\rho_I : Ext_{A^\sharp}(\mathcal{A}, I) \to \mathcal{O}_a(I).$$

We make the following addition requirement of the obstruction theory:

C. ρ_I is an A_0-module homomorphism.

3.5.12 *(Additional Conditions).* If F is to be algebraic, the following conditions must also hold. Let $a \in F_{\mathcal{A}}$ be a solid object with A of finite type over $\underline{S}$, and let the subscript $A_0, B_0, etc.$ denote the objects obtained by tensor product with A_{red}.

A. Compatibility with étale localization: If $\phi : \mathcal{A} \to \mathcal{B}$ is a morphism in $\mathcal{C}$ whose underlying morphism of rings is étale, and if $b \in F_B$ is the image of $a \in F_A$, then

$$\overline{F}_{b_0, \mathcal{B}_0[I \otimes B_0]} \simeq \overline{F}_{a_0, \mathcal{A}_0[I]} \otimes B_0,$$

$$Aut_{b_0}(\mathcal{B}_0[I \otimes B_0]) \simeq Aut_{a_0}(\mathcal{A}_0[I]) \otimes B_0,$$

and

$$\mathcal{O}_b(I \otimes B_0) \simeq \mathcal{O}_a(I) \otimes B_0.$$

B. Effectivity of formal objects: If $\mathfrak{m} \subset A_0$ is a maximal ideal and $\hat{A}$ the completion of A_0 along $\mathfrak{m}$, then

$$\overline{F}_{a_0, \mathcal{A}_0[I]} \otimes \hat{A} \simeq \varprojlim \overline{F}_{a_0, \mathcal{A}_0[I/\mathfrak{m}^n I]},$$

$$Aut_{a_0}(\mathcal{A}_0[I]) \otimes \hat{A} \simeq \varprojlim Aut_{a_0}(\mathcal{A}_0[I/\mathfrak{m}^n I]),$$

and

$$\mathcal{O}_{a_0}(I) \otimes \hat{A} \hookrightarrow \varprojlim O_{a_0}(I/\mathfrak{m}^n I).$$

C. Contructibility: There exists an open dense set of points $p \in Spec(A_0)$ with $k(p)$ of finite type over $\underline{S}$ such that

$$\overline{F}_{a_0, \mathcal{A}_0[I]} \otimes k(p) \simeq \overline{F}_{a_0, \mathcal{A}_0[I \otimes k(p)]},$$

$$Aut_{a_0}(\mathcal{A}_0[I]) \otimes k(p) \simeq Aut_{a_0}(\mathcal{A}_0[I \otimes k(p)]),$$

and

$$\mathcal{O}_a(I) \otimes k(p) \subset \mathcal{O}_a(I \otimes k(p)).$$

Rewriting theorem 2.2.13 we obtain the following:

Theorem 3.5.13. *Let $F \to FLog_S$ be a limit preserving stack for the strict étale topology, and suppose F is equivalent to the stack defined by a pair $(F^\sharp, m)$ as in 3.5.2. Suppose further that an obstruction theory $\mathcal{O}$ as in 3.5.11 is given and satisfies the conditions in 3.5.12. Then F is a log algebraic stack if the following conditions hold:*

1. *The conditions $(S1^{log})$, and $(S2^{log})$ hold and the associated modules satisfy the conditions of 3.5.12.*

2. *For every log ring $\hat{\mathcal{A}}$ such that $\hat{A}$ is complete and local with residue field of finite type over $\underline{S}$, the natural functor*
$$F_{\hat{\mathcal{A}}} \to \varprojlim F_{\hat{\mathcal{A}}/\mathfrak{m}^n}$$
is faithful and has dense image.

3. *Suppose given a reduced object $\mathcal{A}_0$ of $\mathcal{C}$ of finite type over $\underline{S}$, a solid object $a_0 \in F_{\mathcal{A}_0}$, and an automorphism ϕ of a_0. If ϕ induces the identity in F_k (k viewed as a log ring with the induced log structure) for a dense set of points $A_0 \to k$ with k of finite type over $\underline{S}$, then ϕ is the identity over some dense open subset of $Spec(A_0)$.*

Remark 3.5.14. As in 2.2.14 it suffices to construct local obstruction theories $\mathcal{O}_a$. For example, one can always assume that a chart is given for the log structure on $\mathcal{A}_0$ and the obstruction theory may depend on a choice of chart.

Chapter 4

Technicalities on log structures

In this chapter we gather together various facts about log structures which will be used in the chapters that follow.

4.1 A theorem about integral morphisms

Theorem 4.1.1. *Let $Y = (Spec(A), \mathcal{M}_Y)$ where A is a noetherian local ring, and let Y_0 be the reduction modulo the maximal ideal. Let $f : X \to Y$ be an integral morphism locally of finite type such that the map on schemes $\underline{X} \to \underline{Y}$ is proper and flat. Then if the reduction $f_0 : X_0 \to Y_0$ is smooth, f is also smooth.*

Proof. Let $\mathcal{U} \subset LOG_Y$ be the maximal open substack flat over $\underline{Y}$. $\mathcal{U}$ can be described as follows. If $\underline{\tilde{Y}} \to LOG_Y$ is a flat cover, let $\underline{\tilde{U}} \subset \underline{\tilde{Y}}$ be the maximal open set flat over $\underline{Y}$. Then an object of $\mathcal{U}$ over a scheme $\underline{T}$ is an object of LOG_Y over $\underline{T}$ such that the induced map

$$\underline{T} \times_{LOG_Y} \underline{\tilde{Y}} \to \underline{\tilde{Y}}$$

factors through $\underline{\tilde{U}}$.

The fact that f is integral implies that the induced map

$$\underline{X} \to LOG_Y$$

factors through $\mathcal{U}$. This can be checked locally in the flat topology on $\underline{X}$ and $\underline{Y}$. We claim that locally in the flat topology on $\underline{X}$ and $\underline{Y}$ we can find a chart

$$\begin{array}{ccc} P & \longrightarrow & \mathcal{M}_X \\ \uparrow & & \uparrow \\ Q & \longrightarrow & \mathcal{M}_Y \end{array}$$

such that $\mathbb{Z}[Q] \to \mathbb{Z}[P]$ is flat and such that $Q \to P$ is injective. In fact if $y \in \underline{Y}$ is a point then there exists a flat neighborhood $\underline{Y}' \to \underline{Y}$ and a point $y' \in \underline{Y}$ lying over y and a chart $Q \to \mathcal{M}_Y$ such that the induced map $Q \to \overline{\mathcal{M}}_{Y,\bar{y}'}$ is bijective. Replacing $\underline{X}$ by $\underline{X} \times_{\underline{Y}} \underline{Y}'$ we may therefore assume that we have such a chart. Then by ([29], 2.13 and 2.14) there exists fppf-locally on $\underline{X}$ a point $x \in X$ and a chart $P \to \mathcal{M}_X$ such that the map $P \to \overline{\mathcal{M}}_{X,\bar{x}}$ is bijective. This chart has the required properties by the integrality of f. Since

$$Y \times_{Spec(\mathbb{Z}[Q])} Spec(\mathbb{Z}[P])$$

is flat over Y and over LOG_Y by theorem 3.1.11 the map

$$Y \times_{Spec(\mathbb{Z}[Q])} Spec(\mathbb{Z}[P]) \to LOG_Y$$

and hence $\underline{X} \to LOG_Y$ factors through $\mathcal{U}$.

Now to prove that X/Y is smooth it suffices to show that the map

$$\underline{X} \to \mathcal{U}$$

is smooth. Since both $\underline{X}$ and $\mathcal{U}$ are flat over Y to check smoothness at points in the closed fiber it suffices to check it after reduction. Therefore f is smooth at all points in the closed fiber. On the other hand, the set of points where f is not smooth is closed and hence has closed image in Y. But Y has no closed sets which do not contain the closed point and hence the set must be empty. □

4.2 Log versions of the Grothendieck existence theorem.

In applications of Artin's method, the Grothendieck Existence Theorem ([10], III.5) usually plays an important role in verifying condition 2.1.11.2. In this section we prove two theorems which may be viewed as versions of the Grothendieck existence theorem for log structures.

Recall ([20], Appendix) K. Kato's definition of a DF log structure (after Deligne and Faltings):

Definition 4.2.1. *A* **DF log structure** *on a scheme $\underline{X}$ is a pair $(\mathcal{M}_X, t)$ where $\mathcal{M}_X$ is a fine log structure on $\underline{X}$, and $t : \mathbb{N}^r \to \overline{\mathcal{M}}_X$ is a homomorphism which lifts locally to a chart for $\mathcal{M}_X$.*

4.2.2. We make the collection of DF log structures into a category DF_X by defining a morphism $(\mathcal{M}_X, t) \to (\mathcal{M}'_X, t')$ to be a pair (ϕ, ψ) where $\phi : \mathcal{M}_X \to \mathcal{M}'_X$ is a morphism of log structures and $\psi : \mathbb{N}^r \to \mathbb{N}^{r'}$ is a morphism of monoids such that the diagram

$$\begin{tikzcd}
\mathbb{N}^{r'} \arrow[r, "t'"] & \overline{\mathcal{M}'}_X \\
\mathbb{N}^r \arrow[r, "t"] \arrow[u, "\psi"] & \overline{\mathcal{M}}_X \arrow[u, "\phi"']
\end{tikzcd}$$

commutes.

4.2.3. Define a second category DF'_X as follows. Objects are collection $(\mathcal{L}_i, s_i)_{i=1,\dots,r}$ where the $\mathcal{L}_i$ are invertible sheaves on $\underline{X}$, and the $s_i : \mathcal{L}_i \to O_X$ are O_X-module homomorphisms. A morphism $(\mathcal{L}_i, s_i)_{i=1,\dots,r} \to (\mathcal{L}'_j, s'_j)_{j=1,\dots,l}$ is a collection of pairs $(\phi_i, n_i)_{i=1,\dots r}$ where $n_i \in \mathbb{N}^l$ and ϕ_i is an isomorphism of O_X-modules

$$\phi_i : \mathcal{L}_i \to \otimes(\mathcal{L}'_j)^{n_{ij}}.$$

Composition of morphisms is defined in the natural way. Note that since the definition of DF'_X involves only invertible sheaves and isomorphisms between them, it makes sense to think of the $\mathcal{L}_i$ as O^*_X-torsors instead of invertible sheaves.

4.2.4. There is a functor

$$G : DF_X \to DF'_X.$$

Given $(\mathcal{M}_X, t) \in DF_X$, define $\mathcal{L}_i$ to be the O^*_X-torsor of liftings of $t(e_i)$ to $\mathcal{M}_X$, and let s_i be the map coming from $\alpha_X : \mathcal{M}_X \to O_X$. Given a morphism $\xi = (\phi, \psi) : (\mathcal{M}, t) \to (\mathcal{M}', t')$ in DF_X, define $n_i = \psi(e_i)$. Then ϕ defines a map from liftings of $t(e_i)$ to liftings of $t'(\psi(e_i))$ compatible with the maps to O_X. That is, we get a map

$$\mathcal{L}_i \to \otimes_j (\mathcal{L}'_j)^{\psi(e_i)_j}.$$

Proposition 4.2.5. *G is an equivalence of categories.*

Proof. A quasi-inverse F to G can be constructed as follows. Given $(\mathcal{L}_i, s_i)_{i=1,\dots,r}$ let $F((\mathcal{L}_i, s_i)_{i=1,\dots,r})$ be the log structure associated to the pre-log structure

$$\mathcal{M} = \{(n, a) : n \in \mathbb{N}^r, a \text{ a local generator for } \otimes_i \mathcal{L}_i^{n_{ij}}\}$$

together with the natural map $\mathbb{N}^r \to \mathcal{M}/O_X^*$. Given a morphism

$$(\phi_i, n_i)_{i=1,\ldots r} : (\mathcal{L}_i, s_i)_{i=1,\ldots,r} \to (\mathcal{L}'_j, s'_j)_{j=1,\ldots,l}$$

in $\mathcal{A}_X$, we define ψ by $e_i \mapsto n_i$ and ϕ by

$$(n, a) \mapsto (\psi(n), s_i(a)).$$

It is straightforward to check that F and G are quasi-inverses. □

Corollary 4.2.6. *Let A be a complete local ring with maximal ideal $\mathfrak{m}$ and let $\underline{X}/Spec(A)$ be a proper scheme of finite type. Denote by $X_n = X \times_A (A/\mathfrak{m}^n)$. Then the natural functor*

$$DF_X \to \varprojlim DF_{X_n}$$

is an equivalence of categories.

Proof. This follows from 4.2.5 and the usual version of the Grothendieck Existence theorem ([10], III.5). □

From this we deduce the following theorem which is originally due to K. Kato (though we have not seen his proof):

Theorem 4.2.7. *With notation as in the previous corollary, let $\mathcal{C}_n$ be the category of fine saturated log structures on $\underline{X}_n$ for which the natural map of sheaves*

$$\Gamma(\underline{X}_n, \overline{\mathcal{M}}) \to \overline{\mathcal{M}}$$

is surjective. Let $\mathcal{C}_\infty$ be the category of such log structures on $\underline{X}$. Then the natural functor

$$\mathcal{C}_\infty \to \varprojlim C_n$$

is an equivalence of categories.

Proof. Essential surjectivity. Say $\mathcal{M}_n$ is a family of log structures on the $\underline{X}_n$. Let

$$P = \Gamma(\underline{X}_1, \overline{\mathcal{M}}_1) = \Gamma(\underline{X}_n, \overline{\mathcal{M}}_n).$$

Choose an étale cover $\{U_i\}$ over $\underline{X}_1$ such that each U_i is affine and assume that we have chosen a section c_i of

$$\mathcal{M}_1 \to \overline{\mathcal{M}}_1$$

over each U_i. Since $\mathcal{M}_1$ is fine and saturated this is possible. Moreover, by the argument used in the proof of 3.4.6 we can choose a compatible collection of sections $c_{i,n}$ over U_i of the projections

$$\mathcal{M}_n \to \overline{\mathcal{M}}_n$$

(here we view $\{U_i\}$ as an étale cover of $\underline{X}_n$).

Given a map of monoids,

$$\mathbb{N}^r \to P$$

for some r we now construct a DF log structure $\mathcal{N}$ together with a map of log structures

$$\mathcal{N} \to \mathcal{M}_n$$

on $\underline{X}_n$. We first construct a pre-log structure N. On U_i define N_i to be

$$O^*_{X_n} \oplus \mathbb{N}^r.$$

N_i comes with a map $\beta_i : N_i \to \mathcal{M}_n$ coming from the composite

$$\mathbb{N}^r \longrightarrow P \longrightarrow \overline{\mathcal{M}}_n \xrightarrow{c_{i,n}} \mathcal{M}_n.$$

On $U_i \times_X U_j$, there exists a unique element $u_{ji}(e) \in O^*_X$ for each $e \in \mathbb{N}^r$ such that

$$\beta_i(e) = \lambda(u_{ji}(e)) + \beta_j(e)$$

in $\mathcal{M}_n$. Define an isomorphism

$$\phi_{ji} : N_i \to N_j, \quad (f, e) \mapsto (f u_{ji}(e), e).$$

Then $\beta_i = \phi_{ji} \circ \beta_j$ and on $U_i \times_X U_j \times_X U_k$ we have

$$\phi_{ki} = \phi_{kj} \circ \phi_{ji}.$$

Define N to be the pre-log structure obtained by gluing the N_i along the ϕ_{ji} and let $\mathcal{N}$ be the associated log structure. $\mathcal{N}$ comes equipped with a map

$$\mathcal{N} \to \mathcal{M}_n$$

and a surjection $\mathbb{N}^r \to \overline{\mathcal{N}}$ which locally lifts to a chart. In addition the induced map

$$\mathbb{N}^r \to P$$

is the map we started with. Furthermore, for each n we can apply the above construction and since the $c_{i,n}$ are compatible we get a compatible family of DF log structures $\mathcal{N}_n$ together with morphisms of log structures

$$\mathcal{N}_n \to \mathcal{M}_n.$$

If

$$\mathbb{N}^r \rightrightarrows \mathbb{N}^s \to P$$

is a presentation for P, then we can apply the above construction to both $\mathbb{N}^r$ and $\mathbb{N}^s$ to get a diagram of compatible families of log structures

$$\mathcal{N}'_n \rightrightarrows \mathcal{N}_n \to \mathcal{M}_n$$

where the morphisms

$$\mathcal{N}'_n \rightrightarrows \mathcal{N}_n$$

are morphisms of DF log structures. In fact $\mathcal{M}_n$ is the coequalizer of these two maps. To see this observe that the question is local so we may interpret the above sequence as being obtained from

$$\mathbb{N}^r \rightrightarrows \mathbb{N}^s \to P$$

viewed as a sequence of pre-log structures by taking the associated log structure. But taking the associated log structure preserves equalizers. This proves the essential surjectivity since we can realize any log structure in $\mathcal{C}_\infty$ as the coequalizer of morphisms between DF log structures.

Full faithfulness. If O is a strictly henselian local ring and if $\mathcal{M} \to O$ is a log structure, then the natural map

$$\mathcal{M} \to \widehat{\mathcal{M}} = \widehat{O}^* \oplus_O \mathcal{M}$$

is injective. This implies that the functor is faithful. To prove that it is full, let $\mathcal{M}_n$ and ${\mathcal{M}_n}'$ be two compatible collections of log structures on the $\underline{X}_n$ induced from log structures $\mathcal{M}$ and $\mathcal{M}'$ on $\underline{X}$, and say we are given a sequence of maps

$$f_n : \mathcal{M}_n \to {\mathcal{M}_n}'.$$

Let $P = \Gamma(X_n, \overline{\mathcal{M}}_n)$ and let $P' = \Gamma(X_n, \overline{\mathcal{M}}'_n)$. Choose surjections $\mathbb{N}^r \to P$ and $\mathbb{N}^{r'} \to P'$ and a map $\tilde{f} : \mathbb{N}^r \to \mathbb{N}^{r'}$ such that the diagram

$$\begin{array}{ccccc}
\mathbb{N}^r & \longrightarrow & P & \longrightarrow & \overline{\mathcal{M}}_n \\
{\scriptstyle \tilde{f}}\big\downarrow & & {\scriptstyle f}\big\downarrow & & {\scriptstyle f}\big\downarrow \\
\mathbb{N}^{r'} & \longrightarrow & P' & \longrightarrow & \overline{\mathcal{M}}'_n
\end{array}$$

commutes. Let $\mathcal{N}_n$ and $\mathcal{N}'_n$ be the associated DF log structures. We construct a compatible collection of morphisms of DF log structures $\mathcal{N}_n \to \mathcal{N}'_n$ such that the diagrams

$$\begin{array}{ccc} \mathcal{N}_n & \longrightarrow & \mathcal{M}_n \\ \downarrow & & \downarrow \\ \mathcal{N}'_n & \longrightarrow & \mathcal{M}'_n \end{array}$$

commute. Etale locally we have diagrams

$$\begin{array}{ccc} O^*_{X_n} \oplus \mathbb{N}^r & \xrightarrow{\beta} & \mathcal{M}_n \\ & & \downarrow \\ O^*_{X_n} \oplus \mathbb{N}^{r'} & \xrightarrow{\beta'} & \mathcal{M}'_n. \end{array}$$

If $e \in \mathbb{N}^r$ there exists a unique $u(e) \in O^*_{X_n}$ such that

$$f(\beta(e)) = \lambda(u(e)) + \beta'(\tilde{f}(e)).$$

Define a map

$$O^*_{X_n} \oplus \mathbb{N}^r \to O^*_{X_n} \oplus \mathbb{N}^{r'}, \quad e \mapsto (u(e), \tilde{f}(e)).$$

This defines maps of DF log structures $\mathcal{N}_n \to \mathcal{N}'_n$.

It follows that we can find a diagram in $\mathcal{C}_\infty$

$$\begin{array}{ccccc} \mathcal{N}_1 & \rightrightarrows & \mathcal{N}_2 & \to & \mathcal{M} \\ & & \downarrow & & \\ \mathcal{N}'_1 & \rightrightarrows & \mathcal{N}'_2 & \to & \mathcal{M}' \end{array}$$

such that the maps f_n are induced as the maps on co-equalizers from the reductions of the middle arrow. By the faithfullness the two arrows

$$\mathcal{N}_1 \rightrightarrows \mathcal{M}'$$

are equal and hence we get a map

$$\mathcal{M} \to \mathcal{M}'$$

inducing the f_n.

□

The second main result of this section is the following:

Theorem 4.2.8. *Suppose A is a complete noetherian local ring with perfect residue field k, and suppose A has the structure of a flat $W[[t_1, \ldots, t_s]]$-algebra where W denotes the ring of Witt vectors of k if $char(k) > 0$ and k if $char(k) = 0$. We view A and its reductions $A_n = A/\mathfrak{m}_A^n$ as having a log structure coming from the variables $t_1, \ldots t_s$. Suppose $\underline{X}/A$ is a proper scheme of finite type and that $\mathcal{M}_n$ is a compatible family of $\mathcal{M}_{A_n}$-log structures on $\underline{X}_n = \underline{X} \otimes_A A_n$. Suppose further that étale locally the log structure $\mathcal{M}_n$ is induced from a smooth map*

$$\underline{X}_n \to Spec(A_n[X_1, \ldots, X_r]/X_1 \cdots X_r - t_i)$$

for some i. Then there exists a unique log structure $\mathcal{M}$ on $\underline{X}$ inducing the $\mathcal{M}_n$

Lemma 4.2.9. *Let O be a strictly henselian local ring isomorphic to the strict henselization at some point of an $\underline{S}$-algebra of finite type, $x_1, \ldots, x_d \in O$ elements which induce an isomorphism*

$$\widehat{O} \simeq k[[x_1, \ldots, x_d]]/x_1 \cdots x_r$$

for some r. Suppose $y_1, \ldots, y_{d'} \in O$ is a second collection of elements which induce an isomorphism

$$\widehat{O} \simeq k[[y_1, \ldots, y_{d'}]]/y_1 \cdots y_s.$$

Then $d = d'$, $s = r$, and there exists a unique permuation σ of $\{1, \ldots, s\}$ such that

$$y_{\sigma(i)} = u_i x_i, \quad 1 \leq i \leq s$$

where $u_i \in O^*$.

Proof. By looking at dimensions we must have $d = d'$. View the y_i as elements of $k[[x_1, \ldots, x_d]]/x_1 \cdots x_r$ via

$$\widehat{O} \simeq k[[x_1, \ldots, x_d]]/x_1 \cdots x_r.$$

Write

$$y_i = L_i(x) + H_i(x)$$

where $L_i(x)$ consists of linear terms in the x_i and $H_i(x)$ consists of terms of degree ≥ 2. Since the y_i map to a basis for $\mathfrak{m}/\mathfrak{m}^2$ the terms $L_i(x)$ are not zero. By looking at degrees we see that

$$\prod L_i(x) = 0$$

and since none of the $L_i(x)$ are zero we must have $s \geq r$. By symmetry we therefore have $s = r$. In addition, by inspection of the ring $k[[x_1, \ldots, x_d]]/x_1 \cdots x_r$ we see that we must have $L_i(x) = a_i x_i$ with $a_i \in k^*$ after possible having to permute the y_i.

By induction on n, we can write

$$y_i = x_i u_i + H_i^{>n}(x)$$

where $u_i \in \widehat{O}^*$ consists of monomials of degree less than or equal to $n-1$ and $H_i^{>n}(x)$ consists of monomials of degree $> n$. Let $H_i^{n+1}(x)$ denote the terms of degree $n+1$. Then by looking at terms of degree $n+s$ in $y_1 \cdots y_s$ we see that

$$x_1 \cdots \hat{x}_i \cdots x_s H_i^{n+1}(x) = 0.$$

It follows that

$$H_i^{n+1}(x) = x_i(\text{element of } \widehat{O}).$$

By induction we conclude that

$$y_i = u_i x_i, \quad u_i \in \widehat{O}^*.$$

Finally to pass from $\widehat{O}$ to O apply the Artin approximation theorem ([4]). □

Lemma 4.2.10. *Let A be a an artinian local ring, $t \in \mathfrak{m}_A$, and suppose*

$$\phi : A[[x_1, \ldots, x_n]]/(x_1 \cdots x_s - t) \simeq A[[y_1, \ldots, y_n]]/(y_1 \cdots y_s - t)$$

is an isomorphism. Then after a unique permutation of the $y_1, \ldots, y_s$

$$\phi(x_i) = u_i y_i, \quad 1 \leq i \leq s$$

where u_i is a unit.

Proof. We proceed by induction on m such that $\mathfrak{m}_A^m = 0$. The case $m = 1$ is 4.2.9. Therefore assume true for m and suppose $\mathfrak{m}^{m+1} = 0$. We view

$$A[[y_1, \ldots, y_n]]/(y_1 \cdots y_s - t)$$

as a module which is the direct product of free modules with basis

$$\prod y_i^{n_i}$$

where at least one n_i, $1 \leq i \leq s$, is zero. Call these basis elements monomials. By the induction hypothesis we can write

$$\phi(x_i) = u_i y_i + H_i(y), \quad 1 \leq i \leq s$$

where $\{u_i\}$ are units and $H_i(y)$ has coefficients in $\mathfrak{m}^m$ and no monomial terms with a y_i. We show that every coefficient appearing in $H_i(y)$ can be written as

$$t(\text{something}) = y_i(y_1 \cdots \hat{y}_i \cdots y_s(\text{something})).$$

This will complete the induction step. Let k be the smallest total degree of any monomial appearing in any of the $H_i(y)$. Let $H_i^{=k}(y)$ be the monomial terms appearing in $H_i(y)$ with total degree equal to k. Then we get

$$\begin{aligned} t = \prod \phi(x_i) &= (\prod u_i)t + (\text{unit in } A)y_1 \cdots \hat{y}_i \cdots y_s H_i^{=k}(y) \\ &+ \text{ monomial terms with } y_i\text{'s and terms of degree } \geq k+s. \end{aligned}$$

Since $H_i^{=k}$ has no monomial terms with y_i's this implies that the coefficients of $H_i^{=k}$ can be written as multiples of t. Induction on k now completes the induction step for n. □

4.2.11. We can now prove theorem 4.2.8. Let $Z \hookrightarrow X$ be the closed subscheme defined by the ideal $(t_1, \ldots, t_s)$. Z defines a log structure $\mathcal{N}_X \subset O_X$. If $\underline{U} \to \underline{X}$ is a finite étale morphism, we can define

$$\tilde{\mathcal{M}}_X(\underline{U}) = \varprojlim \mathcal{M}_n(\underline{U}_n) \cap O_X(U) \subset \varprojlim O_X(\underline{U}_n).$$

We also denote by $\tilde{\mathcal{M}}_X$ the associated sheaf on the finite étale site of $\underline{X}$. Here we use the flatness of A to conclude that $\varprojlim \mathcal{M}_n(\underline{U}_n) \subset \varprojlim O_X(\underline{U}_n)$. By the previous two lemmas, for every point in the closed fiber of $\underline{X}$, the inclusion into O_X induces an isomorphism $\tilde{\mathcal{M}}_X \simeq \mathcal{N}_X$. Hence these two sheaves are isomorphic. We therefore get a map of sheaves

$$\mathcal{N}_X|_{\underline{X}_n} \to \mathcal{M}_{X_n}$$

which lemma 4.2.9 shows is an isomorphism.

4.3 Log structures on normal crossing schemes.

4.3.1. Let $f : X \to S$ be a smooth, proper, integral morphism of noetherian log schemes, and suppose that $\mathcal{M}_{X/S}$ is a sheaf of torsion free groups. Suppose that for every geometric point $\bar{s} \to \underline{S}$, the underlying scheme of $X_{\bar{s}}$ is étale locally isomorphic to

$$k(\bar{s})[X_1, \ldots, X_n]/(X_1 \cdots X_r)$$

for some natural numbers r and n. The purpose of this section is to show that under these assumptions the log structure $\mathcal{M}_X$ on $\underline{X}$ is induced by base change in a canonical manner from a smooth morphism

$$f^\sharp : X^\sharp \to S^\sharp$$

of a special form whose underlying morphism of schemes is that of the original f.

Definition 4.3.2. *A log smooth morphism $f : X \to S$ is* **essentially semi-stable** *if for each geometric point $\bar{x} \to \underline{X}$ the monoids $(f^{-1}\overline{\mathcal{M}}_S)_{\bar{x}}$ and $\overline{\mathcal{M}}_{X,\bar{x}}$ are free monoids and there exist isomorphisms $(f^{-1}\overline{\mathcal{M}}_S)_{\bar{x}} \simeq \mathbb{N}^r$ and $\overline{\mathcal{M}}_{X,\bar{x}} \simeq \mathbb{N}^{r+s}$ such that if we denote the standard generators by e_i the map*

$$(f^{-1}\overline{\mathcal{M}}_S)_{\bar{x}} \to \overline{\mathcal{M}}_{X,\bar{x}}$$

is of the form

$$e_i \mapsto \begin{cases} e_i, & \textit{if } i \neq r \\ e_r + e_{r+1} + \cdots + e_{r+s}, & \textit{if } i = r \end{cases}$$

Remark 4.3.3. If $f : X \to S$ is essentially semi-stable, then étale locally on X and S there exist charts $\mathbb{N}^r \to \mathcal{M}_S$, $\mathbb{N}^{r+s} \to \mathcal{M}_X$ such that the map $\mathbb{N}^r \to \mathbb{N}^{r+s}$ given by

$$e_i \mapsto \begin{cases} e_i, & \text{if } i \neq r \\ e_r + e_{r+1} + \cdots + e_{r+s}, & \text{if } i = r \end{cases}$$

is a chart for f and such that the map

$$O_S \otimes_{\mathbb{Z}[\mathbb{N}^r]} \mathbb{Z}[\mathbb{N}^{r+s}] \to O_X$$

is smooth. To see this observe that if $s \in \underline{S}$ is a point then the stalk $\overline{\mathcal{M}}_{S,\bar{s}}$ is a free monoid and hence in some étale neighborhood of s there exists a chart $\mathbb{N}^r \to \mathcal{M}_S$ such that the induced map $\mathbb{N}^r \to \overline{\mathcal{M}}_{S,\bar{s}}$ is bijective. If $x \in \underline{X}$ is a point lying over s, then by ([29], 2.25) there exists in some étale neighborhood of x a chart

$$\begin{array}{ccc} P & \longrightarrow & \mathcal{M}_X \\ \uparrow & & \uparrow \\ \mathbb{N}^r & \longrightarrow & \mathcal{M}_S \end{array}$$

such that the induced map

$$O_S \otimes_{\mathbb{Z}[\mathbb{N}^r]} \mathbb{Z}[P] \to O_X$$

is smooth and such that the map $P \to \overline{\mathcal{M}}_{X,\bar{x}}$ is bijective. From the bijectivity of $P \to \overline{\mathcal{M}}_{X,\bar{x}}$ we conclude that P is a free monoid and that the map $\mathbb{N}^r \to P$ has the desired form (after perhaps applying an automorphism of $\mathbb{N}^r$).

Remark 4.3.4. An essentially semi-stable morphism $f : X \to S$ is integral and vertical (this means that $\mathcal{M}_{X/S}$ is a sheaf of groups). To see the integrality we have to check that the map $\mathbb{N}^r \to \mathbb{N}^{r+s}$ described above is integral. But the diagram

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{e \mapsto e_r} & \mathbb{N}^r \\ {\scriptstyle\Delta}\downarrow & & \downarrow \\ \mathbb{N}^{s+1} & \xrightarrow{e_i \mapsto e_{i+r-s}} & \mathbb{N}^{r+s} \end{array}$$

where Δ is the diagonal map is cocartesian. Since the map Δ is integral so is the map $\mathbb{N}^r \to \mathbb{N}^{r+s}$ ([20], 4.1 (i)). To see that it is vertical, let Q be the cokernel of $\mathbb{N}^r \to \mathbb{N}^{r+s}$. By definition of cokernel, to give a map $Q \to M$ to an integral monoid M is equivalent to giving a map $\mathbb{N}^{r+s} \to M$ such that the composite $\mathbb{N}^r \to \mathbb{N}^{r+s} \to M$ is zero. A map $\mathbb{N}^{r+s} \to M$ is equivalent to giving elements

$$m_1, \dots, m_{r+s} \in M$$

and the condition that the map $\mathbb{N}^r \to M$ be zero means that

$$m_1 = m_2 = \cdots = m_{r-1} = m_r + \cdots + m_{r+s} = 0.$$

Thus giving an arrow $Q \to M$ is equivalent to giving elements $m_r, \dots, m_{r+s} \in M^*$ such that $m_r + \cdots + m_{r+s} = 0$. This in turn is equivalent to giving elements $m_r, \dots, m_{r+s-1} \in M^*$. But the functor on the category of integral monoids

$$M \mapsto \{r + s - 1\text{-tuples of elements } m \in M^*\}$$

is represented by $\mathbb{Z}^{r+s-1}$ and hence $Q \simeq \mathbb{Z}^{r+s-1}$.

4.3.5. Suppose $\underline{S} = Spec(k)$ where k is a separably closed field, and that $f : X \to S$ is essentially semi-stable. Let $x \in \underline{X}$ be a singular point. Then by remark 4.3.3 we have a chart

$$\begin{array}{ccccc} \mathbb{N}^{r+s} & \longrightarrow & \mathcal{M}_X & \longrightarrow & O_X \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{N}^r & \longrightarrow & \mathcal{M}_S & \xrightarrow{\alpha} & \Omega \end{array}$$

in an étale neighborhood of x such that

$$k \otimes_{\mathbb{Z}[\mathbb{N}^r]} \mathbb{Z}[\mathbb{N}^{r+s}] \simeq k[x_r, \dots, x_{r+s}]/(x_r \cdots x_{r+s} - \alpha(e_r)) \to O_X$$

is smooth. Since x is a singular point it follows that $\alpha(e_r) = 0$ and hence the map

$$\overline{\mathcal{M}}_S \to \overline{\mathcal{M}}_{X,\bar{x}}$$

is of the form $\mathbb{N}^{r'} \to \mathbb{N}^{r'+s}$ as in definition 4.3.2 for some $r' \leq r$. It follows that there is a unique element in the set of irreducible elements $Irr(\overline{\mathcal{M}}_S)$ of $\overline{\mathcal{M}}_S$ whose image in $\overline{\mathcal{M}}_{X,\bar{x}}$ is not irreducible. This defines a canonical map

$$s_X : \{\text{singular points of } X\} \to Irr(\overline{\mathcal{M}}_S).$$

Definition 4.3.6. *A morphism of log schemes $f : X \to S$ is* **special** *if it is essentially semistable, and if for each geometric point $\bar{s} \to \underline{S}$ the induced map*

$$s_{X_{\bar{s}}} : \{\textit{singular points of } X_{\bar{s}}\} \to Irr(\overline{\mathcal{M}}_{S,\bar{s}})$$

induces a bijection between the set of connected components of the singular locus of $X_{\bar{s}}$ and $Irr(\overline{\mathcal{M}}_{S,\bar{s}})$.

Theorem 4.3.7. *Let $f : X \to S$ be a smooth, proper, integral morphism of noetherian log schemes such that $\mathcal{M}_{X/S}$ is a sheaf of torsion free groups, and such that for every geometric point $\bar{s} \to \underline{S}$ the underlying scheme of $X_{\bar{s}}$ is étale locally isomorphic to*

$$k(\bar{s})[X_1, \dots, X_n]/(X_1 \cdots X_r)$$

for some natural numbers r and n. Then there exists a pair of log structures $(\mathcal{M}_S^\sharp, \mathcal{M}_X^\sharp)$ on $\underline{S}$ and $\underline{X}$ respectively and a morphism

$$f^\sharp : X^\sharp = (\underline{X}, \mathcal{M}_X^\sharp) \to S^\sharp = (\underline{S}, \mathcal{M}_S^\sharp)$$

which is special, together with morphisms of log structures

$$\phi : \mathcal{M}_S^\sharp \to \mathcal{M}_S, \quad \psi : \mathcal{M}_X^\sharp \to \mathcal{M}_X$$

which make the diagram

$$\begin{array}{ccc} X & \longrightarrow & X^\sharp \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^\sharp} \\ S & \longrightarrow & S^\sharp \end{array}$$

cartesian. Moreover, the datum of $(\mathcal{M}_S^\sharp, \mathcal{M}_X^\sharp, \phi, \psi)$ is unique up to unique isomorphism.

4.3.8. The proof occupies the remainder of this section. We will construct $(\mathcal{M}_S^\sharp, \mathcal{M}_X^\sharp, \phi, \psi)$ fppf-locally on $\underline{S}$ and then use the uniqueness to prove the global result.

Let us begin by considering the case when $S = Spec(O)$ where O is a strictly henselian local ring, and we have a chart $Q \to \mathcal{M}_S$ such that $Q \to \overline{\mathcal{M}}_{S,s_0}$ is bijective (here s_0 denotes the closed point). Denote by T the set of connected components of the singular locus of the closed fiber.

Lemma 4.3.9. *Let P be a fine sharp monoid, $Q \hookrightarrow P$ an integral morphism, and let $\bar{p} \in P/Q$ be an element. Denote by $\pi : P \to P/Q$ the projection map. Then there exists a unique element $p_0 \in P$ such that*

$$\pi^{-1}(\bar{p}) = \{p_0 + q | q \in Q\}.$$

Proof. Since P is fine and sharp, there exists a partial ordering on P^+ defined by $p_1 \leq p_2$ if there exists $p_3 \in P$ such that $p_1 + p_3 = p_2$. By ([30]) the set $\pi^{-1}(\bar{p})$ contains a unique finite set of minimal elements with respect to this partial ordering.

Suppose $p_1, p_2 \in \pi^{-1}(\bar{p})$ and suppose p_1 is a minimal element. By construction of the quotient, there exists $q_1, q_2 \in Q$ such that $p_1 + q_1 = p_2 + q_2$. Since the map is integral, there exists $q_3, q_4 \in Q$ and $p \in P$ such that

$$p_1 = q_3 + p, \quad p_2 = q_4 + p, \quad q_1 + q_3 = q_2 + q_4.$$

By definition of quotient, $p \in \pi^{-1}(\bar{p})$ and $p \leq p_1$. Since p_1 was minimal it follows that $p_1 = p$ and that $p_2 = p_1 + q$. □

Corollary 4.3.10. *Let t be a singular point of X, and let $n \in \mathcal{M}_{X/S,\bar{t}} \quad (\simeq \mathbb{Z}^r, \quad r \geq 0)$. Then there exists a unique element $p_n \in \overline{\mathcal{M}}_{X,\bar{t}}$ such that*

$$\pi^{-1}(n) = \{p_n + q | q \in \overline{\mathcal{M}}_{S,s_0}\}.$$

Proof. The map $\overline{\mathcal{M}}_{S,s_0} \to \overline{\mathcal{M}}_{X,\bar{t}}$ is integral by assumption. □

4.3.11. As above let s_0 denote the closed point of S and let X_{s_0} denote the fiber. By ([29], 2.13, 2.14, and 2.25) for every singular closed point t of X_{s_0} there exists a chart

$$\begin{array}{ccc} P & \xrightarrow{\beta} & \mathcal{M}_X \\ \uparrow & & \uparrow \\ Q & \xrightarrow{\beta} & \mathcal{M}_S \end{array}$$

in some étale neighborhood of t such that

1. $Q \to P$ is injective, and $P \to \overline{\mathcal{M}}_{X,\bar{t}}$ is bijective.
2. The map

   $$O \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P] \to O_X$$

   is smooth.

Observe that if $p \in P$ is an irreducible element not in Q, and if $i = \pi(p)$ (notation as in 4.3.9) then $p = p_i$. Indeed by definition we can write

$$p = p_i + q$$

for some $q \in Q$ and since p is irreducible we must have $q = 0$. Let $\{p_{i_1}, \ldots, p_{i_n}\}$ be the set of irreducible elements in P which are not in Q (we index these irreducible elements by their images in P/Q).

If $i \in P/Q$ is any element and p_i is as in 4.3.9, then p_i can be written as

$$p_i = \sum m_j p_{i_j}.$$

Indeed writing p_i as a sum of irreducible elements we have

$$p_i = \sum m_j p_{i_j} + q, \quad q \in Q$$

and by definition of p_i we must have $q = 0$. The $\{p_{i_j}\}$ define a map from a polynomial algebra

$$\gamma : O[Y_{i_1}, \ldots, Y_{i_n}] \to O \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P], \quad Y_{i_j} \mapsto 1 \otimes e(p_{i_j})$$

which is surjective. We denote by J the kernel of γ.

Proposition 4.3.12. *1. There exists $q \in Q$ such that $\sum_{j=1}^{n} p_{i_j} = q$.*

2. J is generated by the element

$$\prod_{j=1}^{n} Y_{i_j} - \alpha(q).$$

3. The natural map

$$Q \oplus_{\mathbb{N}} \mathbb{N}^n \to P, \quad (q, (\underline{n}_{i_j})) \mapsto q + \sum n_{i_j} p_{i_j}$$

is an isomorphism.

Proof. We first reduce to the case when $O = k(s_0)$. The reduction step is trivial for (1) and (3). Let us show that (2) holds if the proposition is true on the closed fiber. Since

$$\sum p_{i_j} = q, \quad q \in Q$$

we have an inclusion of ideals

$$(\prod Y_{i_j} - \alpha(q)) \subset J$$

which defines a closed immersion

$$Spec(O[Y_{i_1}, \dots, Y_{i_n}]/J) \hookrightarrow Spec(O[Y_{i_1}, \dots, Y_{i_n}]/(\prod Y_{i_j} - \alpha(q)).$$

Since both of these schemes are flat over O and the reduction is an isomorphism the closed immersion is an isomorphism also by ([1], VII.1.9). Therefore (2) holds and hence it suffices to consider the case when $O = k$ for some separably closed field k.

Lemma 4.3.13. *The ideal $J = Ker(\gamma)$ is generated by elements of the form*

$$\prod Y_{i_j}^{n_{i_j}} - \alpha(q) \prod Y_{i_j}^{m_{i_j}}$$

where

$$\sum n_{i_j} p_{i_j} = \sum m_{i_j} p_{i_j} + q, \quad q \in Q$$

in P.

Proof. We have a surjection

$$Q \oplus \mathbb{N}^n \to P, \quad (q, (\underline{n}_j)) \mapsto q + \sum_j n_j p_{i_j}$$

which identifies P with the quotient of $Q \oplus \mathbb{N}^n$ by the equivalence relation $(q, (\underline{n}_j)) = (q', (\underline{m}_j))$ if

$$\sum n_{i_j} p_{i_j} + q = \sum m_{i_j} p_{i_j} + q'.$$

It follows that the kernel of the induced map on monoid algebras

$$\mathbb{Z}[Q \oplus \mathbb{N}^n] \to \mathbb{Z}[P]$$

is the ideal generated by elements

$$e(q, (\underline{n}_j)) - e(q', (\underline{m}_j))$$

if $\sum n_{i_j} p_{i_j} + q = \sum m_{i_j} p_{i_j} + q'$ (here $e(q, (\underline{n}_j))$ denotes the element of the monoid algebra corresponding to $(q, (\underline{n}_j))$). Tensoring over $\mathbb{Z}[Q]$ with k we find that $k \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P]$ is the quotient of $k[\mathbb{N}^n]$ by the ideal generated by

$$\alpha(q) e(\underline{n}_j) - \alpha(q') e(\underline{m}_j)$$

if $\sum n_{i_j} p_{i_j} + q = \sum m_{i_j} p_{i_j} + q'$. If both q and q' are non-zero this element is zero so contributes nothing to the kernel. Therefore, writing Y_{i_j} for the irreducible element of $\mathbb{N}^n$ mapping to p_{i_j} we find the result. $\square$

Lemma 4.3.14. *1. The completion $\widehat{O}_{X,t}$ is isomorphic to*

$$k[[T_1,\dots,T_s,Y_{i_1},\dots,Y_{i_n}]]/(Y_{i_1}\cdots Y_{i_n})$$

where $\{T_1,\dots,T_s\}$ are independent variables.

2. Let $L\subset P$ be the submonoid generated by the set $\{p_i|\ i\in P/Q\}$. Then L is isomorphic to $\mathbb{N}^n$.

Proof. (1) Let Y' be the log scheme with underlying scheme $Spec(k\otimes_{\mathbb{Z}[Q]}\mathbb{Z}[P])$ and log structure induced by $P\to\mathbb{Z}[P]$. Then we have a diagram of log schemes

$$X\xrightarrow{h}Y'\longrightarrow S$$

where h is log smooth and strict. Since h is strict, the map

$$\Omega^1_{\underline{X}/\underline{S}}\to\Omega^1_{X/Y'}$$

is surjective so in some neighborhood of t we can find elements $f_1,\dots,f_s$ in the maximal ideal of t such that $df_1,\dots,df_s$ map to a basis for $\Omega^1_{X/Y'}\otimes k(t)$. This defines a map

$$h':X\to Y'\otimes_{\underline{S}}\mathbb{A}^s_{\underline{S}}$$

of log schemes which is strict and for which the map

$$h'^*\Omega^1_{Y'\otimes_{\underline{S}}\mathbb{A}^s_{\underline{S}}/Y'}\otimes k(t)\to\Omega^1_{X/Y'}\otimes k(t)$$

is an isomorphism. It follows that in some neighborhood of t the map h' is étale ([29], 2.20). Thus we can extend the map γ to an étale map

$$X\to k[T_1,\dots,T_s]\otimes_k(k[Y_{i_1},\dots,Y_{i_n}]/J)$$

sending t to the point $T_1=\cdots=T_s=Y_{i_1}=\cdots=Y_{i_n}=0$.

First we claim that the resulting map

$$k[[T_1,\dots,T_s,Y_{i_1},\dots,Y_{i_n}]]\to\widehat{O}_{X,t}$$

induces an isomorphism

$$k[[T_1,\dots,T_s,Y_{i_1},\dots,Y_{i_n}]]/(T_1,\dots,T_s,Y_{i_1},\dots,Y_{i_n})^2\simeq\widehat{O}_{X,t}/\mathfrak{m}^2.$$

To see this it suffices to show that all our defining equations for J are contained in the ideal $(Y_{i_1},\ldots,Y_{i_n})^2$. But all our equations arise from non-trivial equalities

$$\sum n_{i_j}p_{i_j} = q + \sum m_j p_{i_j}$$

in P. If $\sum n_{i_j} = 1$ then we have

$$p_{i_j} = q + \sum m_j p_{i_j}$$

which by the irreducibility of p_{i_j} implies that our equation comes from an equality

$$p_{i_j} = q \quad \text{or} \quad p_{i_j} = p_{i_{j'}}.$$

Since $p_{i_j} \notin Q$ we must have $p_{i_j} = p_{i_{j'}}$ and hence our equation is trivial.

Fix an isomorphism

$$\widehat{O}_{X,t} \simeq k[[X_1,\ldots,X_l]]/(X_1,\cdots X_h)$$

for some l and h. It follows from the above that

$$l = dim(\mathfrak{m}/\mathfrak{m}^2) = n + s.$$

For each $k \geq 1$, the dimension of $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ is equal to the number of monomials of degree k in $X_1,\ldots,X_l$ which do not contain the string $X_1\cdots X_h$. In particular, the dimension of $\mathfrak{m}^h/\mathfrak{m}^{h+1}$ is equal to one less than the number of monomials of degree h in $l = n+s$ variables. This implies that we have at least one equation involving monomials in the Y_{i_j} of degree h. Fix one such equation $f(\underline{Y}) = 0$. By lemma 4.3.13, we can choose our equation so that it is either of the form

$$f(\underline{Y}) = Y_{i_1}\cdots Y_{i_h} - Y_{i'_1}\cdots Y_{i'_m}$$

or of the form

$$f(\underline{Y}) = Y_{i_1}\cdots Y_{i_h}.$$

Let R be the ring

$$k[[T_1,\ldots,T_s,Y_{i_1},\ldots,Y_{i_n}]]/(f).$$

There is a natural surjective map

$$R \to \widehat{O}_{X,t}$$

which induces an isomorphism $\mathfrak{m}_R/\mathfrak{m}_R^2 \simeq \mathfrak{m}/\mathfrak{m}^2$. For each integer $w \geq 1$ let $R_w = R/\mathfrak{m}_R^w$ and let $M_w = \widehat{O}_{X,t}/\mathfrak{m}^w$. We view M_w as an R_w-module. Note that the map $k = R_1 \to M_w \otimes_{R_w} R_1 = k$ is an isomorphism.

For each integer k the dimension of $\mathfrak{m}_R^k/\mathfrak{m}_R^{k+1}$ is equal to the dimension of $\mathfrak{m}^k/\mathfrak{m}^{k+1}$. In fact, if we define $N(k,l)$ to be the number of monomials of degree k in l variables then the dimensions of $\mathfrak{m}_R^k/\mathfrak{m}_R^{k+1}$ and $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ are both equal to

$$N(k,l) - N(k-h,l) = N(k,n+s) - N(k-h,n+s).$$

Since we have an isomorphism $\mathfrak{m}_R/\mathfrak{m}_R^2 \simeq \mathfrak{m}/\mathfrak{m}^2$ the map

$$gr^*_{\mathfrak{m}_R}(R_w) \to gr^*_{\mathfrak{m}_R}(M_w)$$

is surjective and therefore an isomorphism by looking at dimensions. It now follows from the local criterion for flatness ([1], V.3.2) that M_w is a flat R_w-module. From this and the fact that $R_1 \to M_w \otimes_{R_w} R_1$ is an isomorphism we conclude that $R_w \simeq M_w$ (see for example the proof of proposition 3.2.2). It follows that the map

$$R = \varprojlim_w R_w \to \widehat{O}_{X_{s_0},\bar{t}} \simeq \varprojlim_w M_w$$

is also an isomorphism.

We claim that f cannot be of the form

$$f(\underline{Y}) = Y_{i_1} \cdots Y_{i_h} - Y_{i'_1} \cdots Y_{i'_m}.$$

Since P/Q is a group, for every p_{i_j} there exists $n_a \geq 0$ such that

$$Y_{i_j} \prod_{a=1}^{n} Y_{i_a}^{n_a} = 0.$$

More precisely, since P/Q is a group there exists an element p_{-i_j} such that

$$\pi(p_{i_j}) + \pi(p_{-i_j}) = 0$$

in P/Q. Writing $p_{-i_j} = \sum n_a p_{i_a}$ we conclude that we have

$$p_{i_j} + \sum_{a=1}^{n} n_a p_{i_a} \in \pi^{-1}(0).$$

By lemma 4.3.9, $\pi^{-1}(0) = Q$ and hence there exists an element $q \in Q$ such that

$$p_{i_j} + \sum_{a=1}^{n} n_a p_{i_a} = q.$$

Since p_{i_j} is not a unit, $q \neq 0$ and hence we find that

$$Y_{i_j} \prod_{a=1}^{n} Y_{i_a}^{n_a} = 0.$$

Since $\widehat{O}_{X,t}$ is reduced, if

$$\prod Y_{i_j}^{m_j} = 0$$

with the $m_j \geq 1$ then

$$\prod Y_{i_j} = 0.$$

It follows that

$$\prod Y_{i_j} = 0.$$

This is impossible if

$$f(\underline{Y}) = Y_{i_1} \cdots Y_{i_h} - Y_{i'_1} \cdots Y_{i'_m}.$$

Therefore, we have an isomorphism

$$k[[T_1, \ldots, T_s, Y_{i_1}, \ldots, Y_{i_n}]]/(Y_{i_1} \cdots Y_{i_h}) \simeq \widehat{O}_{X,t}$$

for some h.

We claim that $h = n$. To see this note that we have

$$\sum_{j=1}^{h} p_{i_j} = p + q$$

for some $p \in P$ and a nonzero $q \in Q$. If p_{i_a} is another one of our irreducible elements with $a > h$, then since P/Q is a group there exists an element p_{-i_a} as in 4.3.9 such that $p_{i_a} + p_{-i_a}$ maps to zero in P/Q. Write

$$p_{-i_a} = \sum_{j=1}^{n} b_{i_j} p_{i_j}.$$

Then at least one of the b_{i_j} with $1 \leq j \leq h$ must be zero since otherwise we can write

$$p_{-i_a} = \sum_{j=1}^{h} p_{i_j} + \sum_{j=1}^{h} (b_{i_j} - 1) p_{i_j} + \sum_{j>h} b_{i_j} p_{i_j} = p + q + \sum_{j>h} b_{i_j} p_{i_j} + \sum_{j=1}^{h} (b_{i_j} - 1) p_{i_j}$$

contradicting the minimality of p_{-i_a}. It follows that $\prod_j Y_{i_j}^{b_{i_j}} \notin J$. On the other hand $p_{i_a} + p_{-i_a} \in \pi^{-1}(0) = Q$ and since p_{i_a} is not a unit $p_{i_a} + p_{-i_a}$ is a non-zero element in Q. Therefore $Y_{i_a} \prod_j Y_{i_j}^{b_{i_j}} \in J$ and hence Y_{i_a} is a zero divisor in the ring

$$k[[T_1, \ldots, T_s, Y_{i_1}, \ldots, Y_{i_n}]]/(Y_{i_1} \cdots Y_{i_h}).$$

This is a contradiction so we must have $h = n$.

(2) Suppose we have an equality

$$\sum m_j p_{i_j} = \sum m'_j p_{i_j}$$

in P. Since P is cancellative, we can assume that at least one $m_j = 0$ and hence we get

$$\prod Y_{i_j}^{m_j} = \prod Y_{i_j}^{m'_j} \neq 0$$

in the ring

$$k(s_0)[[T_1, \dots, T_s, Y_{i_1}, \dots, Y_{i_n}]]/(\prod Y_{i_j}).$$

It follows that $m_j = m'_j$ for all j.

□

We now complete the proof of the proposition. Suppose we have an equality

$$q + \sum_{n_{i_j}} p_{i_j} = q' + \sum_{m_{i_j}} p_{i_j}$$

in P. Then by the integrality of $Q \to P$ there exists $p \in P$ and $q_3, q_4 \in Q$ such that

$$\sum_j n_{i_j} p_{i_j} = p + q_3, \quad \sum_j m_{i_j} p_{i_j} = p + q_4, \quad q + q_3 = q + q_4.$$

In addition, writing $p = \sum b_{i_j} p_{i_j} + q_5$ we see that the congruence relation on $Q \oplus \mathbb{N}^n$ defined by the surjection

$$Q \oplus \mathbb{N}^n \to P$$

is generated by equalities in P of the form

$$\sum_j n_{i_j} p_{i_j} = \sum_j b_{i_j} p_{i_j} + q.$$

In addition since P is cancellative we can assume that for each j either $n_{i_j} = 0$ or $b_{i_j} = 0$. From lemma 4.3.14 we deduce that any non-trivial equality of this form must have $n_{i_j} \geq 1$ for all j. Therefore, the congruence relation on $Q \oplus \mathbb{N}^n$ is generated by equalities

$$\sum_j n_{i_j} p_{i_j} = q$$

where all the n_{i_j} are greater than 0. In addition, from 4.3.14 we see that we must have an equality

$$\sum_{j=1}^n p_{i_j} = q$$

for some $q \in Q$. This proves (1) of the proposition. Now suppose we had an equality in P

$$\sum_j n_{i_j} p_{i_j} = q'$$

for some $q' \in Q$ and all $n_{i_j} \geq 1$. Then after perhaps reordering the p_{i_j} we can assume that n_{i_1} is the smallest of the n_{i_j}. This gives

$$q' = \sum_j n_{i_j} p_{i_j} = n_{i_1} q + \sum_j (n_{i_j} - n_{i_1}) p_{i_j}.$$

The integrality of $Q \to P$ now implies that $\sum_j (n_{i_j} - n_{i_1}) p_{i_j}$ is in Q which from above is only possible if $n_{i_j} = n_{i_1}$ for all j. This implies (2).

To see (3), note that the map

$$Q \oplus_{\mathbb{N}} \mathbb{N}^n \to P$$

surjective by the definition of the p_{i_j}. To see that it is injective, suppose $(q, (\underline{n}_j))$ and $(q', (\underline{m}_j))$ map to the same element. Without loss of generality, we may assume that at least one $n_j = 0$ and at least one $m_{j'} = 0$ (since there exists $q \in Q$ such that $(q, 0) = (0, (1, \ldots, 1))$ in $Q \oplus_{\mathbb{N}} \mathbb{N}^n$). Then we get an equality

$$q + \sum n_j p_{i_j} = q' + \sum m_j p_{i_j}$$

in P and by the integrality we get that there exists $p \in P$ and $q_3, q_4 \in Q$ such that

$$\sum n_j p_{i_j} = p + q_3, \quad \sum m_j p_{i_j} = p + q_4, \quad q_3 + q = q_4 + q'.$$

Since $\sum p_{i_j} \in Q$ we may assume that $p = \sum b_{i_j} p_{i_j}$ with at least one $b_{i_j} = 0$. If q_3 is non-zero we get that

$$\prod Y_{i_j}^{n_j} = 0$$

in $k[Y_{i_1}, \ldots, Y_{i_n}]/J$ which from above is impossible since at least one n_j is zero. Thus $q_3 = 0$ and hence

$$\sum m_j p_{i_j} = \sum n_j p_{i_j} + q_4.$$

By the same reasoning using the fact that at least one $m_{j'} = 0$ we conclude that $q_4 = 0$ and hence $q = q'$ and $(\underline{n}_j) = (\underline{m}_j)$. This completes the proof of the proposition. $\square$

4.3.15. Let $L \subset P$ be the submonoid of 4.3.14.2, and define $K \subset Q$ to be the submonoid generated by the element

$$\sum p_{i_j}.$$

Equivalently

$$K = L \cap Q \subset P.$$

If $t' \in X$ is another singular point lying over some $\zeta \in Spec(O)$, then the above shows that the submonoid $L_{t'} \subset \overline{\mathcal{M}}_{X,\bar{t}'}$ generated by the irreducible elements of $\overline{\mathcal{M}}_{X,\bar{t}'}$ not in the image of $\overline{\mathcal{M}}_{S,\bar{\zeta}}$ is isomorphic to $\mathbb{N}^r$ for some r and that the intersection $K_{t'} = L_{t'} \cap \overline{\mathcal{M}}_{S,\bar{\zeta}}$ is a free monoid of rank 1. To check this one may base change to an algebraic closure $\bar{k}$ of $k(\zeta)$ and hence can assume that we have a chart $Q \to \mathcal{M}_S$ such that $Q \to \overline{\mathcal{M}}_S$ is bijective. If t' is a closed point the result now follows from the above discussion. If t' is not a closed point, then choose a specialization t'' of t' and note that the stalk at t' is obtained by the cospecialization map from t'' ([17], II.2.3).

Lemma 4.3.16. *Suppose we have chosen a chart around t as in 4.3.11 over some étale neighborhood of t which we also denote by X. Let L_t and K_t be the submonoids of P and Q respectively given by the above construction. Then for any other singular point t' of X lying over some $\zeta \in Spec(O)$, the submonoids $L_{\bar{t}'} \subset \overline{\mathcal{M}}_{X,\bar{t}'}$ and $K_{\bar{t}} \subset \overline{\mathcal{M}}_{S,\bar{\zeta}}$ are equal to the images of L_t and K_t.*

Proof. From proposition 4.3.12, we have an isomorphism

$$Q \oplus_{K_t} L_t \simeq P,$$

and hence an isomorphism

$$\overline{\mathcal{M}}_{X,\bar{t}'} \simeq \overline{\mathcal{M}}_{S,\bar{\zeta}} \oplus_{K_t} (L_t/(p_{i_j} = 0 \text{ if } p_{i_j} \text{ maps to a unit in } O_{X,\bar{t}'}).$$

This implies the lemma.

□

4.3.17. Let D be a connected component of the singular locus of the closed fiber of X. Define log structures $\tilde{K}_D \subset \mathcal{M}_S$ and $\tilde{L}_D \subset \mathcal{M}_X$ as follows. Observe that for any point $t \in D$ we get a free monoid of rank 1 $K_t \subset \overline{\mathcal{M}}_{S,s_0}$. Proposition 4.3.16 implies that this monoid is independent of the choice of t. To see this suppose $t' \in D$ is a second singular point. Choose étale covers $\{U_i\}$ of D and singular points t_i in the image of U_i such that over each U_i we have a chart as in 4.3.16 using t_i. We can also assume that $t_1 = t$ and $t_r = t'$ for some r. Since D is connected, there exists a sequence $U_1, \ldots, U_r$ such that for each i there exists a singular point x_i of $U_i \times_X U_{i+1}$ mapping to D. It follows from 4.3.16

that the submonoid $K_{t_i} \subset \overline{\mathcal{M}}_S$ defined by t_i is equal to the submonoid defined by x_i and x_{i-1}. We conclude that $K_t = K_{t'}$.

We define $\tilde{K}_D$ to be the unique fine sub-log structure of $\mathcal{M}_S$ whose image in $\overline{\mathcal{M}}_{S,s_0}$ is equal to K_t for some $t \in D$ (and hence all). $\tilde{L}_D$ is the unique fine sub-log structure of $\mathcal{M}_X$ whose image in $\overline{\mathcal{M}}_{X,t'}$ (t' any point of $\underline{X}$) is equal to the image of $\tilde{K}_D$ if t' is not singular or if t' does not specialize to a point of D and is equal to $L_{t'}$ (notation as in 4.3.15) otherwise.

Note that $\tilde{K}_D$ and $\tilde{L}_D$ are uniquely determined if they exist since we have specified their values on stalks.

The fact that these log structures exist follows from lemma 4.3.16 and the above discussion. Indeed $\tilde{K}_D$ exists since S is just the spectrum of a strictly henselian local ring. To show that $\tilde{L}_D$ exists it suffices to construct $\tilde{L}_D$ locally by the stalk-wise description of $\tilde{L}_D$ as a subsheaf of $\mathcal{M}_X$. But locally we can choose a chart as in 4.3.11 and then lemma 4.3.16 implies the result.

Define $\mathcal{M}_S^\sharp$ to be the amalgemation over all D

$$\oplus_{O_S^*} \tilde{K}_D$$

and similarly define $\mathcal{M}_X^\sharp$ to be

$$\oplus_{O_X^*} \tilde{L}_D.$$

Then the natural diagram

$$\begin{array}{ccc} \mathcal{M}_X^\sharp & \xrightarrow{\psi} & \mathcal{M}_X \\ \uparrow & & \uparrow \\ f^*\mathcal{M}_S^\sharp & \xrightarrow{\phi} & f^*\mathcal{M}_S. \end{array}$$

is co-cartesian. Since $f^\sharp$ is smooth at every point of the closed fiber the induced morphism $f^\sharp$ is log smooth everywhere. By construction $f^\sharp$ is essentially semi-stable and the pullback to the closed fiber is special . Note also that the datum of

$$(\mathcal{M}_S^\sharp, \mathcal{M}_X^\sharp, \phi, \psi)$$

which is essentially semistable and whose pullback to the closed fiber is special is unique.

To see the uniqueness note that starting with a collection

$$(\mathcal{M}_S^\sharp, \mathcal{M}_X^\sharp, \phi, \psi)$$

we can apply the above construction to $(\mathcal{M}_S^\sharp, \mathcal{M}_X^\sharp)$. It follows from chasing through the construction that this returns the same pair and hence $\mathcal{M}_S^\sharp$ and $\mathcal{M}_X^\sharp$ have canonical decompositions

$$\mathcal{M}_S^\sharp = \oplus_{O_S^*} \tilde{K}_D, \quad \mathcal{M}_X^\sharp = \oplus_{O_S^*} \tilde{L}_D.$$

In addition the image of $\tilde{L}_D$ and $\tilde{K}_D$ in $\mathcal{M}_X$ and $\mathcal{M}_S$ are the sub-log structures obtained by applying the construction with $\mathcal{M}_X$ and $\mathcal{M}_S$. Any isomorphism between two collections of data must preserve this decomposition and since the $\tilde{K}_D$ and $\tilde{L}_D$ are sub-log structures of $\mathcal{M}_S$ and $\mathcal{M}_X$ respectively it follows that an isomorphism is unique and exists.

The following proposition 4.3.18 therefore completes the proof of theorem 4.3.7 in the case when $S = Spec(O)$ where O is a strictly henselian local ring and $Q \to \mathcal{M}_S$ is a chart such that $Q \to \overline{\mathcal{M}}_{S,s_0}$ is bijective.

Proposition 4.3.18. *Let $f : X \to S$ be as in 4.3.7 (we no longer assume that S is the spectrum of a strictly henselian local ring) and suppose in addition that f is essentially semi-stable. Then the set of points $s \in \underline{S}$ for which the fiber $X_{\bar{s}}$ is special over $Spec(k(\bar{s}))$ is open.*

Proof. Let V be the set of points $s \in \underline{S}$ for which $X_{\bar{s}}$ is special over $k(\bar{s})$. By ([1], V.4.4) it suffices to show that V is a constructible set and that if $s \in V$ then every generization of s is also in V.

Lemma 4.3.19. *V is constructible.*

Proof. The condition is local on S so we may assume that $S = Spec(A)$ for some ring A and that $\mathcal{M}_S$ is induced from some map

$$\mathbb{N}^r \to A.$$

Let $t_1, \ldots, t_r \in A$ be the images of the standard generators. For each integer n and for each choice of n distinct elements $i_1, \ldots, i_n \in \{1, \ldots, r\}$ let $A_{n,i}$ be the ring

$$A_{t_{i_1} \cdots t_{i_n}} / (t_j)_{j \notin \{i_1, \ldots, i_n\}}.$$

Let $S_{n,i} = Spec(A_{n,i})$. Then each $S_{n,i}$ is a constructible subset of S and the union of the $S_{n,i}$ is all of S. To prove the lemma it therefore suffices to consider the case of the $S_{n,i}$. This reduces us to the case when the log structure on S is given by a map

$$\mathbb{N}^r \to A$$

sending all elements to zero. In addition, by base changing to the irreducible components with the reduced structure we may assume that A is an integral domain. In this case, X is étale locally isomorphic to

$$A[X_1, \ldots, X_l]/X_1 \cdots X_s$$

for some l and s. Let X^{sing} be the singular locus of X with the reduced induced structure. From the local description of X we see that the geometric fibers of X^{sing} are all reduced and that X^{sing} is flat over A. It follows that the function

$$s \mapsto \text{number of connected components of } X^{sing}_{\bar{s}}$$

can be identified with the function

$$s \mapsto h^0(X^{sing} \times_S Spec(k(s)), O_{X^{sing} \times_S Spec(k(s))}).$$

By the semi-continuity theorem ([14], III.12.8) we conclude that the function

$$s \mapsto \text{number of connected components of } X^{sing}_{\bar{s}}$$

is upper semi-continous. It follows that the set of points s where this number is equal to r is constructible. On the other hand, the set of points s where the map in 4.3.5 is surjective is open by the description of the charts in remark 4.3.3. Therefore the set of points where the map in 4.3.5 is bijective is constructible. $\square$

Lemma 4.3.20. *V is stable under generization.*

Proof. We can assume that $S = Spec(A)$, where A is a local ring and that we want to prove that if the closed fiber is special then the fiber over some prime $\mathfrak{p} \subset A$ is also. Replacing A by $A/\mathfrak{p}$ we may assume that A is a local integral domain and that we wish to prove that the generic fiber is special if the closed fiber is special. In addition by replacing A by an étale cover we may assume that we have a global chart

$$\mathbb{N}^T \to \mathcal{M}_S$$

where T is the set of connected components of the singular locus of the closed fiber. Let $n_t \in \mathbb{N}^T$ be an irreducible element such that $\alpha(n_t) = 0$. Let Z denote the set of singular points of X which map to n_t. Z is a closed set and we view Z as a subscheme with the reduced structure. What has to be shown is that the generic fiber of Z is non-empty and geometrically connected.

Let t_0 be a point of the closed fiber whose corresponding element of $\mathbb{N}^T$ is n_t. In an étale neighborhood of t_0 we have an étale morphism

$$U \to Spec(A[X_1, \ldots, X_n]/X_1 \cdots X_s)$$

for some n and s. From this it follows that the generic fiber of Z is not empty. From this local description it also follows that for each point $s \in \underline{S}$ the fiber Z_s is reduced and that Z is flat over A. Therefore, the generic fiber of Z is geometrically connected if and only if

$$h^0(Z \otimes_A Frac(A), O_{Z \otimes_A Frac(A)}) = 1$$

where $Frac(A)$ denotes the field of fractions of A. This number is at least 1 since $Z \otimes_A Frac(A)$ is not empty. On the other hand, the semi-continuity theorem ([14], III.12.8) combined with the fact that the closed fiber of Z is geometrically connected implies that the generic fiber of Z is geometrically connected. This implies the lemma. $\square$

Proposition 4.3.18 now follows from the two lemmas. $\square$

We can now complete the proof of 4.3.7. By the uniqueness we may assume that S is affine. Note first that if s is a point, and $S_s = Spec(O_{S,\bar{s}})$ then by 1.3.9 there is an equivalence of categories

$$\varinjlim_{U \to S} (\text{fine log structures on } X \times_S U) \to (\text{fine log structures on } X \times_S S_s)$$

where the limit is taken over étale neighborhood $U \to S$ of $\bar{s}$. If X is affine then this is exactly the statement of 1.3.9. In general, we can cover X by finitely many affines $\{V_i\}$ and then the categories above can be described as collections of log structures on the V_i together with isomorphisms on the $V_i \cap V_j$ satisfying the cocycle condition. Since X is separated (it is proper over an affine scheme) the intersections are again affine and hence the statement follows from this case.

It follows that if we have two collections of data

$$(\mathcal{M}^\sharp_{S,1}, \mathcal{M}^\sharp_{X,1}, \phi_1, \psi_1), \quad (\mathcal{M}^\sharp_{S,2}, \mathcal{M}^\sharp_{X,2}, \phi_2, \psi_2)$$

then there exists exactly one isomorphism between them. Indeed if $s \in S$ is a point then after replacing S by an fppf-cover we can assume that we have a chart $Q \to \mathcal{M}_S$ such that the map $Q \to \overline{\mathcal{M}}_{S,\bar{s}}$ is bijective. Over the strict henselization of the local ring of s there is a unique isomorphism, and this isomorphism can by the above discussion be extended to some étale neighborhood of s. It follows that fppf-locally there exists a unique isomorphism and hence by descent theory there exists a global isomorphism.

Finally to prove existence, it suffices to prove existence in some fppf-neighborhood of a point $s \in S$. Thus we may assume that we have a chart $Q \to \mathcal{M}_S$ such that $Q \to \overline{\mathcal{M}}_{S,\bar{s}}$ is bijective. We have shown that there exists data $(\mathcal{M}_S^\sharp, \mathcal{M}_X^\sharp, \phi, \psi)$ over the strict henselization of the local ring at s. From above we see that we can extend the data $(\mathcal{M}_S^\sharp, \mathcal{M}_X^\sharp, \phi, \psi)$ over the henselization to some étale neighborhood of s. After perhaps shrinking S we can assume that the resulting morphism is still log smooth. We claim that after perhaps shrinking S some more the morphism $f^\sharp$ will be essentially semi-stable. We may assume that we have a chart $\mathbb{N}^T \to \mathcal{M}_S$ where T denotes the set of connected component of the singular locus of $X_{\bar{s}}$. Consider the set $\underline{V}$ of points $t \in X$ for which the map

$$(f^{-1}\overline{\mathcal{M}}_S^\sharp)_{\bar{t}} \to \overline{\mathcal{M}}_{X,\bar{t}}^\sharp$$

is as in the definition of essentially semi-stable. By the discussion in 4.3.3 the set $\underline{V}$ is open and by construction $\underline{V}$ contains every point of the fiber over s. Since f is proper the image of $\underline{V}^c$ in $\underline{S}$ is a closed set not containing s. Therefore by replacing S by the complement of $f(\underline{V}^c)$ we may assume that $f^\sharp$ is essentially semi-stable everyhwere.

It was also shown above that the set of points where the morphism $f^\sharp$ is special is open on the base and hence after shrinking S some more we have constructed datum $(\mathcal{M}_S^\sharp, \mathcal{M}_X^\sharp, \phi, \psi)$ satisfying the conditions of 4.3.7 in an fppf-neighborhood of s. This concludes the proof of 4.3.7.

Chapter 5

Moduli of log curves

Throughout this chapter $S = Spec(\mathbb{Z})$ with the trivial log structure. $FLog$ is the category of fine log schemes with the strict étale topology.

5.1 Statement of Theorem

Recall the definition of a stable curve:

Definition 5.1.1 ([8], [9]). *Let $g \geq 2$ (resp. $g = 1$) be an integer. A* **stable curve of genus** *g over a scheme $\underline{X}$ is a proper, flat morphism $f : \underline{C} \to \underline{X}$ such that for every geometric point $\bar{x} \in \underline{X}$, the fiber $\underline{C}_{\bar{x}}$ is a reduced, connected 1-dimensional scheme satisfying:*

1. *$\underline{C}_{\bar{x}}$ has only ordinary double points.*
2. *If $\underline{E}$ is a non-singular rational component of $\underline{C}_{\bar{x}}$ then $\underline{E}$ meets the other components of $\underline{C}_{\bar{x}}$ in more than 2 points (resp. $\underline{C}_{\bar{x}}$ is a Néron polygon [9]).*
3. *$\underline{C}_{\bar{x}}$ has arithmetic genus g.*

5.1.2. If $g \geq 2$, let $\underline{\mathcal{M}}_g$ be the stack of stable curves of genus g, and if $g = 1$ let $\underline{\mathcal{M}}_g$ be the stack of irreducible pointed stable curves of genus 1. It is an algebraic stack, and the complement of the open substack $\underline{\mathcal{M}}_g^0 \subset \underline{\mathcal{M}}_g$ classifying smooth objects is a divisor $\underline{Z}$ with normal crossings ([8], [24]). Hence there is a natural log structure on $\underline{\mathcal{M}}_g$. By 1.3.8, $\underline{\mathcal{M}}_g$ determines a log algebraic stack $\mathcal{M}_g \to FLog$.

The purpose of this chapter is to determine what objects this log algebraic stack classifies.

5.1.3. Let $\underline{\mathcal{C}} \to \underline{\mathcal{M}}_g$ be the universal curve. Then the inverse image of the divisor $\underline{Z}$ in $\underline{C}$ is again a divisor and hence determines a log structure on $\underline{C}$. The resulting morphism of log stacks $\mathcal{C} \to \mathcal{M}_g$ satisfies the conditions in the following definition:

Definition 5.1.4. *Let $g \geq 2$ be an integer, and let $X \in FLog$. Then a* **stable log curve of genus** g *over X is a morphism $f : C \to X$ such that:*

1. *f is log smooth, proper, and integral.*
2. *For every geometric point $\bar{x} \in \underline{X}$, the fiber $\underline{C}_{\bar{x}}$ is a reduced connected curve of arithmetic genus g.*
3. *$\mathcal{M}_{C/X}$ is a sheaf of saturated and torsion free monoids.*
4. *For every geometric point $\bar{x} \to X$*

$$H^0(C_{\bar{x}}, Der(C_{\bar{x}}/\bar{x})) = 0.$$

Here $Der(C_{\bar{x}}/\bar{x})$ means the log derivations.

Definition 5.1.5. *A stable log curve $f : C \to X$ is* **vertical** *if $\mathcal{M}_{C/X}$ is a sheaf of groups.*

5.1.6. If $g \geq 2$, let $\mathcal{M}_g^{log}$ denote the stack over $FLog$ of vertical stable log curves of genus g, and if $g = 1$ we denote by $\mathcal{M}_g^{log}$ the stack of vertical morphisms $f : C \to X$ satisfying 5.1.4.1-5.1.4.3 with a strict section $e : X \to C$ such that for every geometric point $\bar{x} \to X$

$$H^0(C_{\bar{x}}, Der(C_{\bar{x}}/\bar{x})(-e)) = 0.$$

There is a natural functor

$$\mathcal{M}_g \to \mathcal{M}_g^{log}. \tag{5.1}$$

The purpose of this chapter is to prove the following:

Theorem 5.1.7. *Let $g \geq 1$ be an integer.*

1. *$\mathcal{M}_g^{log}$ is a log algebraic stack.*
2. *The functor 5.1 is an equivalence of log stacks.*

5.2 Basic properties of log curves.

5.2.1. The results of this section are essentially contained in ([19]). Let $f : C \to X$ be a stable log curve, and let $c \in \underline{C}$ be a point with $x = f(c)$. Define

$$P = \overline{\mathcal{M}}_{C,\bar{c}}, \quad Q = \overline{\mathcal{M}}_{X,\bar{x}}.$$

After perhaps replacing X by an fppf-cover we can find an étale neighborhood of $\bar{c}$ and a chart for f

$$h : C \to X \times_{Spec(\mathbb{Z}[Q])} Spec(\mathbb{Z}[P])$$

with h smooth ([29], 2.14).

Lemma 5.2.2. *Suppose $\underline{X} = Spec(k)$ with k separably closed and that $\mathcal{M}_{C/X,\bar{c}} \neq 0$. Then h is étale and*

$$\mathcal{M}^{gp}_{C/X,\bar{c}} \simeq \mathbb{Z}.$$

Proof. Let $G = \mathcal{M}^{gp}_{C/X,\bar{c}}$. G is a free abelian group of positive rank. Since f is integral, the map

$$Spec(\mathbb{Z}[P]) \to Spec(\mathbb{Z}[Q])$$

is flat, and consequently we have

$$1 = dim(C/X) \geq dim(\mathbb{Z}[P]/\mathbb{Z}[Q]) = rank(G) > 0.$$

It follows that the rank of G is 1 and that the inequality is an equality. □

Corollary 5.2.3. *If $f : C \to X$ is a stable log curve and if $\bar{c} \in C$ is a geometric point in the support of $\mathcal{M}_{C/X}$ then*

$$\mathcal{M}_{C/X,\bar{c}} \simeq \mathbb{N} \text{ or } \mathbb{Z}.$$

Proof. Any fine saturated submonoid of $\mathbb{Z}$ is $\mathbb{N}$ or $\mathbb{Z}$. □

Lemma 5.2.4. *With notation as above, if $c \in \underline{C}$ is a point with $\mathcal{M}_{C/X,\bar{c}} \simeq \mathbb{N}$ then c is a smooth point.*

Proof. Since $\underline{C}/\underline{X}$ is flat we may assume that $\underline{X}$ is the spectrum of a separably closed field k. Choose a chart as in 5.2.1. Then P/Q is isomorphic (canonically) to $\mathbb{N}$. Choose an element $p_1 \in P$ mapping to 1 as in 4.3.10. Then with the notation of 4.3.10 we have

$$np_1 = p_n, \quad \text{for all } n \in \mathbb{N}.$$

If not, we have $np_1 = p_n + q$ for some non-zero $q \in Q$ and hence $e(np_1) \otimes 1 = (e(p_1) \otimes 1)^n$ is zero in the ring

$$\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} k.$$

Therefore, $e(p_1) \otimes 1$ is nilpotent, which contradicts the assumption that the fibers are reduced. Thus we get a canonical isomorphism

$$\mathbb{N} \oplus Q \simeq P.$$

Hence in an étale neighborhood of c,

$$\underline{C} \simeq Spec(k[t]).$$

□

Lemma 5.2.5. *If $\mathcal{M}_{C/X,\bar{c}} \simeq \mathbb{Z}$ and if $\underline{X} = Spec(k)$ with k separably closed then $\bar{c}$ is an ordinary double point.*

Proof. Continuing with the notation of 4.3.10, choose an isomorphism $\mathcal{M}_{C/X,\bar{c}} \simeq \mathbb{Z}$ and let p_n be the primitive element of P mapping to n. We claim that if $n > 0$ then $p_n = np_1$ and if $n < 0$ then $p_n = (-n)p_{-1}$. If not, we can write

$$np_1 = p_n + q, \quad q \in Q$$

if $n > 0$ or

$$(-n)p_{-1} = p_n + q, \quad q \in Q$$

if $n < 0$. This implies that the image of p_1 (or p_{-1}) in $k \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P]$ is nilpotent which is a contradiction. We therefore get a co-cartesian diagram

$$\begin{array}{ccc} \mathbb{N}^2 & \longrightarrow & P \\ \uparrow & & \uparrow \\ \mathbb{N} & \xrightarrow{1 \mapsto p_1 + p_{-1}} & Q \end{array}$$

and so

$$X \times_{Spec(\mathbb{Z}[Q])} Spec(\mathbb{Z}[P]) \simeq Spec(k[u,v]/uv).$$

□

Corollary 5.2.6. *If $(f : C \to X) \in \mathcal{M}_g^{log}$ then there exists a cartesian diagram unique up to unique isomorphism*

$$\begin{array}{ccc} C & \xrightarrow{\tilde{\beta}} & C^\dagger \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle f^\dagger}\big\downarrow \\ X & \xrightarrow{\beta} & X^\dagger \end{array}$$

where $C^\dagger/X^\dagger$ is a special log curve in the sense of 4.3.6, and the morphisms $\underline{C} \to \underline{C}^\dagger$ and $\underline{X} \to \underline{X}^\dagger$ are isomorphisms.

Proof. The previous lemma implies that the hypotheses of 4.3.7 are satisfied. □

Corollary 5.2.7. *$\mathcal{M}_g^{log}$ is naturally equivalent to the log stack defined by the stack of special log curves with its natural morphism to $LOG_{Spec(\mathbb{Z})}$ as in 3.5.2.*

In light of definition 3.5.4 we might also refer to the special log curves as *solid* log curves.

Corollary 5.2.8. *If C/X is an object of $\mathcal{M}_g^{log}$ then $\underline{C}/\underline{X}$ is an object of $\underline{\mathcal{M}}_g$.*

Proof. It suffices to consider the case when $\underline{X}$ is isomorphic to the spectrum of a separably closed field. In this case it follows from ([19], 1.2) that $\Omega^1_{C/X}$ is isomorphic to the dualizing sheaf of $\underline{C}/\underline{X}$. The corollary now follows from the definition in ([24]). □

Remark 5.2.9. This last corollary (5.2.8) is the only result in this section which used the assumption 5.1.4.4.

5.3 Cohomological calculations.

5.3.1. The contents of this section are not needed to prove that $\mathcal{M}_g^{log}$ is a log algebraic stack but are necessary for showing that the functor 5.1 is an equivalence of log stacks.

Let $f : C \to X$ be a special curve of genus $g \geq 2$ where $\underline{X} = Spec(A)$ for an artinian local ring A. Suppose I is an A-module of finite type killed by the maximal ideal of A (and hence a vector space over $A_0 = A/\mathfrak{m}$). Assume for this section that Schlessinger's conditions $(S1^{log})$ and $(S2^{log})$ (3.5.8 and 3.5.10) hold. Then the 8-term exact sequence 2.1 in the proof of 2.2.13 gives in this case a short exact sequence

$$0 \to E_a(I) \to \overline{\underline{\mathcal{M}}}^{log}_{g,A[I]} \to Ext_{A^\sharp}(\mathcal{A}, I) \to 0$$

where the module $E_a(I)$ is the cokernel of

$$i : Hom(\overline{\mathcal{M}}_A^{gp}, I) \to H^1(C, Der_X(C, I \otimes O_C)).$$

The map i is defined by identifying $Hom(\overline{\mathcal{M}}_A^{gp}, I)$ with the infinitesimal automorphisms of $\mathcal{M}_{A[I]}$ and sending such an automorphism $\sigma : Spec(\mathcal{A}[I]) \to Spec(\mathcal{A}[I])$ to the class of

$$C \times_X Spec(\mathcal{A}[I]) \longrightarrow Spec(\mathcal{A}[I]) \xrightarrow{\sigma} Spec(\mathcal{A}[I]).$$

Lemma 5.3.2. *i is injective.*

Proof. We can replace A by an étale cover and hence can assume that the sheaf $\overline{\mathcal{M}}_A$ is a constant sheaf. Let $X' = Spec(\mathcal{A}[I])$ and let C'/X' be a lifting of C, and suppose $\sigma : X' \to X'$ is an automorphism coming from an infinitesimal automorphism of the log structure on X'. To prove that i is injective it suffices to show that $C'_\sigma = C' \times_{X',\sigma} X'$ is not isomorphic over X' to C'/X'. Suppose

$$h : C'_\sigma \to C'$$

were such an isomorphisms. The underlying schemes of C'_σ and C' are the same and hence h induces an infinitesimal automorphism of $\underline{C}'$. On the other hand such infinitesimal automorphisms are classified by ([34], 4.4)

$$Ext^0(\Omega_{\underline{C}/\underline{X}}, O_{\underline{C}})$$

which is zero by ([8], 1.4). Thus the underlying morphism of schemes of h is the identity. On the other hand, if T denotes the singular points of the closed fiber of C, then σ is given by a map

$$\partial : \mathbb{N}^T \to I$$

where we identify $\mathbb{N}^T$ and $\overline{\mathcal{M}}_A$ via the canonical isomorphism given by the map in 4.3.5. Suppose $t \in T$ is such that $\partial(e_t)$ is not zero. The log structure on the completion $\hat{O}_{C',t}$ is given by the pushout via

$$\mathbb{N}^t \hookrightarrow \mathbb{N}^T$$

from

$$\begin{array}{ccc} \mathbb{N}^2 & \longrightarrow & A[I][[x,y]]/(xy-a) \\ {\scriptstyle\Delta}\uparrow & & \uparrow \\ \mathbb{N} & \longrightarrow & A[I] \end{array}$$

where the map $\mathbb{N}^2 \to A[I][[x,y]]/(xy-a)$ is the standard one. Thus h induces an automorphism h^b of $\hat{O}^*_{C,t} \oplus \mathbb{N}^2$ compatible with the maps to $\hat{O}_{C',t}$ and such that $h^b \circ \Delta = \Delta \circ \sigma$. It follows that h^b is of the form

$$(1,0) \mapsto \lambda(1+yf) + (1,0), \quad (0,1) \mapsto \lambda(1+xg) + (0,1)$$

where f and g are elements of $A[I][[x,y]]/(xy-a)$ with coefficients in I. Their product must be in the image of $\mathcal{M}_{A[I]}$ so f and g are both zero which means that h and hence σ is the identity. □

5.3.3. Forgetting the log structure defines a morphism of sheaves

$$Der_X(C, I \otimes O_C) \to Der_{\underline{X}}(\underline{C}, I \otimes O_C)$$

which can be shown to be injective by reasoning similar to the proof of the lemma. Let Q be the cokernel. From the exact sequence

$$0 \to Der_X(C, I \otimes O_C) \to Der_{\underline{X}}(\underline{C}, I \otimes O_C) \to Q \to 0$$

we obtain a map

$$H^1(C, Der_X(C, I \otimes O_C)) \to H^1(C, Der_{\underline{X}}(\underline{C}, I \otimes O_C)).$$

This map is surjective for Q is a sheaf with support at the singular points of $\underline{C}$ and hence $H^1(C,Q) = 0$.

Proposition 5.3.4. *The map*

$$H^1(C, Der_X(C, I \otimes O_C)) \to H^1(C, Der_{\underline{X}}(\underline{C}, I \otimes O_C)).$$

induces an isomorphism

$$E_a(I) \simeq H^1(C, Der_{\underline{X}}(\underline{C}, I \otimes O_C)).$$

Proof. There is an injection

$$H^1(C, Der_{\underline{X}}(\underline{C}, I \otimes O_C)) \to Ext^1(\Omega_{\underline{C}/\underline{X}}, I \otimes O_C)$$

coming from the spectral sequence of a composite of functors, and the composite

$$H^1(C, Der_X(C, I \otimes O_C)) \to Ext^1(\Omega_{\underline{C}/\underline{X}}, I \otimes O_C)$$

is the natural one coming from $\Omega_{\underline{C}/\underline{X}} \to \Omega_{C/X}$. On the other hand, the module

$$Ext^1(\Omega_{\underline{C}/\underline{X}}, I \otimes O_C)$$

is in bijection with flat liftings of the scheme $\underline{C}/\underline{X}$ to $\underline{X}'$ by ([34], 4.4), and the map

$$H^1(C, Der_X(C, I \otimes O_C)) \to Ext^1(\Omega_{\underline{C}/\underline{X}}, I \otimes O_C)$$

can also be described as the map obtained by forgetting the log structure. This last statement can be seen as follows. Let $\{U_i\}$ be an affine cover of C and let $r : X' \to X$ be the natural retraction. Then a class in $H^1(C, Der_X(C, I \otimes O_C))$ can be described by a collection of sections $D_{ij} \in Der_X(C, I \otimes O_C)(U_{ij})$ over the $U_{ij} = U_i \cap U_j$ such that

$$D_{ij}|_{U_{ijk}} - D_{ik}|_{U_{ijk}} + D_{jk}|_{U_{ijk}} = 0.$$

The corresponding log smooth lifting of C to X' is obtained by gluing $U_i \times_{X,r} X'$ and $U_j \times_{X,r} X'$ over U_{ij} via D_{ij} which can be thought of as an isomorphism between $U_{ij} \times_{X,r} X' \to U_{ij} \times_{X,r} X'$ ([20], 3.14). That is, we glue as in the following diagram

$$\begin{tikzcd} U_{ij} \times_{X,r} X' \arrow[r, "D_{ij}"] \arrow[d] & U_{ij} \times_{X,r} X' \arrow[d] \\ U_i \times_{X,r} X' & U_j \times_{X,r} X'. \end{tikzcd}$$

If $\underline{D}_{ij}$ denotes the image of D_{ij} in $Der_{\underline{X}}(\underline{C}, I \otimes O_{\underline{C}})(\underline{U}_{ij})$ then the class

$$[D_{ij}] \in H^1(C, Der_X(C, I \otimes O_C))$$

maps to the class

$$[\underline{D}_{ij}] \in H^1(C, Der_{\underline{X}}(\underline{C}, I \otimes O_{\underline{C}})).$$

From the construction given in ([34], section 4) we see that the image of this class in

$$Ext^1(\Omega_{\underline{C}/\underline{X}}, I \otimes O_C)$$

corresponds to the deformation of $\underline{C}$ obtained by gluing using the $\{\underline{D}_{ij}\}$, which is just the deformation of $\underline{C}$ obtained by forgetting the log structures.

From this it follows that the map

$$H^1(C, Der_X(C, I \otimes O_C)) \to H^1(C, Der_{\underline{X}}(\underline{C}, I \otimes O_C)).$$

factors through $E_a(I)$. $E_a(I)$ and $H^1(C, Der_{\underline{X}}(\underline{C}, I \otimes O_C))$ are both finite dimensional vector spaces over A_0. Since the map between them is surjective to prove that it is an isomorphism it suffices to prove that they have the same dimension. Let d be the dimension of I as an A_0-vector space, and let $|T|$ be the number of singular points of $C \otimes A_0$. The dimension of

$$H^1(C, Der_X(C, I \otimes O_C))$$

is by ([13], 3.3.9) $d(3g-3)$ and the dimension of

$$Hom(\overline{\mathcal{M}}_X^{gp}, I))$$

is $d|T|$. Thus the dimension of $E_a(I)$ is

$$d(3g - 3 - |T|).$$

To compute the dimension of $H^1(C, Der_{\underline{X}}(\underline{C}, I \otimes O_C))$ we may as well replace C by $C \otimes A_0$. In fact we may even assume that A is an algebraically closed field. An element of $H^1(C, Der_{\underline{X}}(\underline{C}, I \otimes O_C))$ corresponds to a vector field on the normalization $\tilde{\underline{C}}$ which vanishes at all points lying over singular points of $\underline{C}$. Let $\underline{C}_i$ be an irreducible component of $\underline{C}$ and $\tilde{\underline{C}}_i$ its normalization. Let $q_1^i, \ldots, q_{r_i}^i$ be the points of $\tilde{\underline{C}}_i$ mapping to singular points of $\underline{C}$. Then the dimension of $H^1(C, Der_{\underline{X}}(\underline{C}, I \otimes O_C))$ is equal to

$$\sum_i h^1(\tilde{\underline{C}}_i, Der_{\underline{X}}(\tilde{\underline{C}}_i, I \otimes O)(-q_1^i - \cdots - q_{r_i}^i))).$$

By Serre duality,

$$h^1(\tilde{\underline{C}}_i, Der_{\underline{X}}(\tilde{\underline{C}}_i, I \otimes O)(-q_1^i - \cdots - q_{r_i}^i))) = h^0(\tilde{\underline{C}}_i, \Omega^{\otimes 2}(\tilde{\underline{C}}_i, I \otimes O)(q_1^i + \cdots + q_{r_i}^i))).$$

which by the Riemann-Roch theorem is equal to

$$d(3g_i - 3 + r_i).$$

On the other hand, the genus g of C is given by the formula

$$g = (\sum g_i) + |T| - \nu + 1$$

where ν is the number of irreducible components. It follows that the dimension of

$$H^1(C, Der_{\underline{X}}(\underline{C}, I \otimes O_C))$$

is also equal to

$$d(3g - 3 - |T|).$$

$\square$

5.3.5. The cokernel of the map

$$H^1(C, Der_{\underline{X}}(\underline{C}, I \otimes O)) \to \mathrm{Ext}^1(\Omega_{\underline{C}/\underline{X}}, I \otimes O)$$

is equal to

$$H^0(C, \mathcal{E}xt^1(\Omega_{\underline{C}/\underline{X}}, I \otimes O)).$$

This group also has a natural logarithmic interpretation. We state the following proposition slightly more generally because we found the more general statement interesting:

Proposition 5.3.6. *Instead of assuming that I is killed by the maximal ideal of A assume just that $I^2 = 0$. Then there is a natural map*

$$\beta : Der_A(A[\overline{\mathcal{M}}_A^{gp}], I) \to H^0(\underline{C}, \mathcal{E}xt^1(\Omega^1_{\underline{C}/A}, f^*I))$$

which induces an isomorphism

$$Coker(Hom(\overline{\mathcal{M}}_A^{gp}, I) \to Der_A(A[\overline{\mathcal{M}}_A^{gp}], I)) \simeq H^0(\underline{C}, \mathcal{E}xt^1(\Omega^1_{\underline{C}/A}, f^*I)).$$

Proof. First construct a map of sheaves

$$\beta : f^*Der_A(A[\overline{\mathcal{M}}_A^{gp}], I) \to \mathcal{E}xt^1(\Omega^1_{\underline{C}/A}, f^*I)$$

as follows. We have maps

$$C \xrightarrow{\ f\ } X \longrightarrow \underline{X}$$

which since f is smooth induces an exact sequence

$$0 \to f^*\Omega^1_{X/\underline{X}} \to \Omega^1_{C/\underline{X}} \to \Omega^1_{C/X} \to 0$$

and hence a class in

$$\mathcal{E}xt^1(\Omega^1_{C/X}, f^*\Omega_{X/\underline{X}}).$$

Note that there is an isomorphism

$$\Omega^1_{X/\underline{X}} \simeq O_X \otimes \overline{\mathcal{M}}_A^{gp}.$$

From the natural map $\Omega^1_{\underline{C}/\underline{X}} \to \Omega^1_{C/X}$ and functoriality the above class induces an element

$$\xi_C \in \mathcal{E}xt^1(\Omega^1_{\underline{C}/\underline{X}}, O_C \otimes \overline{\mathcal{M}}_A^{gp}).$$

Note that in our case, $\overline{\mathcal{M}}_A^{gp}$ is isomorphic to the free monoid on the elements of T, and hence an element of $Der_A(A[\overline{\mathcal{M}}_A^{gp}], I)$ can be thought of as a T-tuple of elements $\{\alpha_t\} \in I$. A section of $f^*Der_A(A[\overline{\mathcal{M}}_A^{gp}], I)$ thus induces a map

$$O_C \otimes \overline{\mathcal{M}}_A^{gp} \to f^*I$$

by $e_t \mapsto \alpha_t$. By looking at the image of ξ_C a section of $f^*Der_A(A[\overline{\mathcal{M}}_A^{gp}], I)$ defines a class in

$$\mathcal{E}xt^1(\Omega^1_{\underline{C}/A}, f^*I).$$

This defines the map β.

The sheaf $\mathcal{E}xt^1(\Omega^1_{\underline{C}/A}, f^*I)$ is a skyscraper sheaf with support at the singular points T. Choose a chart $\mathbb{N}^T \to A$. If $t \in T$, then we can choose local coordinates

$$A[x, y]/(xy - \alpha_t)$$

where α_t is the image in A of the element $e_t \in \mathbb{N}^T$. The sheaf $\mathcal{E}xt^1(\Omega^1_{\underline{C}/A}, f^*I)$ can be described explicitly in terms of these coordinates as the cokernel of ([13] p. 100-101)

$$Hom(O_{C,t} < dx, dy >, I \otimes O_{C,t}) \to Hom(O_{C,t} < xy - \alpha_t >, I \otimes O_{C,t}).$$

Here $O_{C,t} < dx, dy >$ denotes the free module on generators dx and dy and $O_{C,t} < xy - \alpha_t >$ denotes the free module with generator $xy - \alpha_t$. If $dx \mapsto \phi_x$ and $dy \mapsto \phi_y$ is an element of $Hom(O_{C,t} < dx, dy >, I \otimes O_{C,t})$ then its image is the map which sends $xy - \alpha_t$ to $\phi_y \otimes x + \phi_x \otimes y$. It follows from direct calculation that the cokernel is isomorphic to $I/\alpha_t I$ and hence

$$\mathcal{E}xt^1(\Omega^1_{\underline{C}/A}, f^*I) \simeq \oplus_t (I/\alpha_t I)_t.$$

It is straightforward to check that if $\partial \in Der_A(A[\mathbb{N}^T], I)$ then the associated element of $\mathcal{E}xt^1(\Omega^1_{\underline{C}/A}, f^*I)$ is the one coming from the map

$$xy - \alpha_t \mapsto \partial(e_t).$$

From this the proposition follows. $\square$

Now

$$Coker(Hom(\overline{\mathcal{M}}_A^{gp}, I) \to Der_A(A[\overline{\mathcal{M}}_A^{gp}], I))$$

is naturally identified with $Ext_{A^\sharp}(\mathcal{A}, I)$. For we have a diagram

$$A \to A[\overline{\mathcal{M}}_A^{gp}] \to \mathcal{A}$$

and hence and exact sequence

$$Der_{A[\overline{\mathcal{M}}_A^{gp}]/A}(I) \to Ext_{A[\overline{\mathcal{M}}_A^{gp}]}(\mathcal{A}, I) \to Ext_{A^\sharp}(\mathcal{A}, I) \to Ext_A(A[\overline{\mathcal{M}}_A^{gp}], I).$$

By 3.2.10

$$Ext_A(A[\overline{\mathcal{M}}_A^{gp}], I) = 0$$

and we can identify

$$Hom(\overline{\mathcal{M}}^{gp}_A, I) \simeq Der_{A[\overline{\mathcal{M}}^{gp}_A]/A}(I).$$

Finally to give an element in

$$Ext_{A[\overline{\mathcal{M}}^{gp}_A]}(\mathcal{A}, I)$$

is equivalent to giving a map

$$A[\overline{\mathcal{M}}^{gp}_A] \to A[I]$$

which induces the canonical map to A. This in turn is equivalent to an element of

$$Der_A(A[\overline{\mathcal{M}}^{gp}_A], I).$$

Combining the propositions we get the following theorem

Theorem 5.3.7. *Suppose I is killed by the maximal ideal of A. Then there is a commutative diagram obtained by forgetting the log structures*

$$\begin{array}{ccccc} E_a(I) & \to & \underline{\overline{\mathcal{M}}}^{log}_{g,a,\mathcal{A}[I]} & \to & Ext_{A^\sharp}(\mathcal{A}, I) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(C, Der_{\underline{X}}(\underline{C}, I \otimes O)) & \to & Ext^1(\Omega_{\underline{C}/\underline{X}}, I \otimes O) & \to & H^0(C, \mathcal{E}xt^1(\Omega_{\underline{C}/\underline{X}}, I \otimes O)) \end{array}$$

in which all vertical arrows are isomorphisms.

5.3.8. The case when $g = 1$ is similar. Let $(C, e)/X$ be a special curve. By the same reasoning as in 5.3.2, the map

$$Hom(\overline{\mathcal{M}}^{gp}_A, I) \to \mathrm{Ext}^1(\Omega_{C/X}(e), I \otimes O)$$

is injective. Moreover,

$$\mathrm{Ext}^1(\Omega_{C/X}(e), I \otimes O) \simeq H^1(C, I \otimes O(-e)) \simeq I$$

since $\Omega_{C/X}$ is trivial. Thus since the map is injective it is an isomorphism. Consequently,

$$\underline{\overline{\mathcal{M}}}^{log}_{g,a,\mathcal{A}[I]} \simeq Ext_{A^\sharp}(\mathcal{A}, I).$$

Similarly, deformations of $(\underline{C}, e)/\underline{X}$ are classified by

$$\mathrm{Ext}^1(\Omega_{\underline{C}/\underline{X}}(e), I \otimes O).$$

We have

$$h^1(C, Der_{\underline{X}}(\underline{C}, I \otimes O(-e))) = h^1(\tilde{C}, Der_{\underline{X}}(\underline{\tilde{C}}, I \otimes O(-e-(o)-(\infty))) = 0$$

and therefore

$$\mathrm{Ext}^1(\Omega_{\underline{C}/\underline{X}}(e), I\otimes O) \simeq H^0(C, \mathcal{E}xt^1(\Omega_{\underline{C}/\underline{X}}(e), I\otimes O)) \simeq H^0(C, \mathcal{E}xt^1(\Omega_{\underline{C}/\underline{X}}, I\otimes O))$$

(note that $\mathcal{E}xt^1(\Omega_{\underline{C}/\underline{X}}(e), I\otimes O)$ is supported away from e). Combining this with the argument in 5.3.6 we obtain:

Theorem 5.3.9. *Forgetting the log structures induces a commutative diagram of isomorphisms*

$$\begin{array}{ccc} \underline{\overline{\mathcal{M}}}^{log}_{1,a,\mathcal{A}[I]} & \longrightarrow & Ext_{A^\sharp}(\mathcal{A}, I) \\ \downarrow & & \downarrow \\ Ext^1(\Omega_{\underline{C}/\underline{X}}(e), I\otimes O) & \longrightarrow & H^0(C, \mathcal{E}xt^1(\Omega_{\underline{C}/\underline{X}}, I\otimes O)). \end{array}$$

5.4 Verification of 3.5.13

5.4.1. Using the results of the preceding sections, we now verify the conditions of 3.5.13 for $\mathcal{M}^{log}_g$, and thereby prove that $\mathcal{M}^{log}_g$ is a log algebraic stack.

5.4.2 *(Obstruction Theory).* Let $\mathcal{A} \to \mathcal{A}_0$ be a morphism of log rings as in 3.5.11 with $I = Ker(A \to A_0)$ a square zero ideal. Suppose $C_0/Spec(\mathcal{A}_0)$ is a log stable curve. It is immediate from the definition of log stable curve that any log smooth lifting of C_0 is again a log stable curve. It follows from ([20] 3.14.4) that the obstruction to finding a log smooth lifting of C_0 lies in

$$H^2(C_0, \mathcal{H}om(\Omega^1_{C_0/X_0}, IO_{C_0}))$$

which is zero since C_0/X_0 is a curve ([14], III.11.2). In the case of $g = 1$, it follows from ([20], 3.11) that the section also lifts. Thus we take for our obstruction theory $\mathcal{O}$ the zero modules.

5.4.3 *(Schlessinger's Criteria).* (S1^{log}). This follows from 3.2.2 and 3.2.5 and the fact that we know that the functor $\mathcal{M}^{log}_g$ is unobstructed (one could also argue directly from the constructions in 3.2.2 and 3.2.5 that the liftings obtained from these theorems are log smooth).

(S2^{log}). By ([20], 3.14.3), the set of liftings of a log stable curve $f : C_0 \to Spec(\mathcal{A}_0)$ to $\mathcal{A}_0[I]$ are classified by

$$H^1(C_0, \mathcal{H}om_{C_0}(\Omega^1_{C_0/\mathcal{A}_0}, f^*I)).$$

In the case when C_0 is pointed, say $e : Spec(\mathcal{A}_0) \to C_0$ it follows as in the classical case that the set of liftings of the pair (C_0, e) are classified by

$$H^1(C_0, \mathcal{H}om_{C_0}(\Omega^1_{C_0/\mathcal{A}_0}, f^*I \otimes O_{C_0}(-e))).$$

In both cases they are A_0-modules of finite type.

Again by ([20], 3.14.2), the infinitesimal automorphisms of a lifting of $C_0/\mathcal{A}_0$ to $\mathcal{A}_0[I]$ are classified by

$$H^0(C_0, \mathcal{H}om_{C_0}(\Omega^1_{C_0/\mathcal{A}_0}, f^*I)).$$

It follows that the module $Aut_{a_0}(\mathcal{A}_0[I])$ is equal to this cohomology group when $g \geq 2$ and when $g = 1$ is equal to the kernel of

$$H^0(C_0, \mathcal{H}om_{C_0}(\Omega^1_{C_0/\mathcal{A}_0}, f^*I)) \to H^0(Spec(A_0), \mathcal{H}om_{A_0}(e^*\Omega^1_{C_0/\mathcal{A}_0}, I)). \tag{5.2}$$

The following lemma shows that $(S2)^{log}$ holds.

Lemma 5.4.4. $Aut_{a_0}(\mathcal{A}_0[I]) = 0$.

Proof. We prove that when $g \geq 2$, the module $H^0(C_0, \mathcal{H}om_{C_0}(\Omega^1_{C_0/\mathcal{A}_0}, f^*I)) = 0$ and in the case $g = 1$ that the map in 5.2 is an isomorphism.

We first reduce to the case when $I \simeq A_0$. I is finitely generated so there exists a resolution

$$0 = I_0 \subset I_1 \subset \cdots \subset I_n = I$$

such that $I_i/I_{i-1} \simeq A_0/\mathfrak{p}_i$ for some prime $\mathfrak{p}_i \subset A_0$. Since $\Omega^1_{C_0/\mathcal{A}_0}$ is locally free, and f is a flat morphism, the sequence

$$0 \to \mathcal{H}om_{C_0}(\Omega^1_{C_0/\mathcal{A}_0}, f^*I_{i-1}) \to \mathcal{H}om_{C_0}(\Omega^1_{C_0/\mathcal{A}_0}, f^*I_i) \to \mathcal{H}om_{C_0}(\Omega^1_{C_0/\mathcal{A}_0}, f^*(A_0/\mathfrak{p}_i)) \to 0$$

is exact. Let $C_i = C_0 \times_{A_0} (A_0/\mathfrak{p}_i)$ and let $j : C_i \hookrightarrow C$ be the closed immersion. Then

$$\mathcal{H}om_{C_0}(\Omega^1_{C_0/\mathcal{A}_0}, f^*(A_0/\mathfrak{p}_i)) \simeq j_*\mathcal{H}om_{C_i}(\Omega^1_{C_i}, O_{C_i}).$$

Applying the global section functor and induction now gives the reduction step for $g \geq 2$. To handle the case when $g = 1$, note that since $\Omega^1_{C_0}$ is the dualizing sheaf, it follows that $\Omega^1_{C_0}$ is trivial in this case. Thus the above exact sequence can be rewritten as

$$0 \to f^*I_{i-1} \to f^*I_i \to j_*O_{C_i} \to 0$$

and hence taking cohomology gives a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_{i-1} & \longrightarrow & I_i & \longrightarrow & A_i/\mathfrak{p}_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0(C_0, f^*I_{i-1}) & \longrightarrow & H^0(C_0, f^*I_i) & \longrightarrow & H^0(C_i, O_{C_i}). & & \end{array}$$

Since C_0/A_0 is proper, the right vertical arrow is an isomorphism and by induction the left arrow is also. The five lemma now implies that the middle arrow is an isomorphism.

Thus we may assume that $I = A$. The case $g = 1$ now follows from the above argument since $\Omega^1_{C_0/A_0}$ is trivial. For $g \geq 2$, the case when A is a field is given in ([8], 1.4) and the general case follows from this by standard base change results. $\square$

5.4.5 *(Additional Conditions 3.5.12).* These conditions follows from the above descriptions of the modules together with standard results on cohomology.

5.4.6 *(Condition 3.5.13.2).* In this case the functor is an equivalence of categories. The full faithfulness follows from the following lemma and the usual Grothendieck existence theorem applied to the underlying schemes:

Lemma 5.4.7. *Fix a complete noetherian local ring A, and let $\mathcal{M}_A$ be a log structure on A. Suppose $\underline{C}/A$ is a stable curve and that $\mathcal{M}^1_C$ and $\mathcal{M}^2_C$ are two $\mathcal{M}_A$-log structures on $\underline{C}/A$ such that the resulting log schemes $C^1 = (\underline{C}, \mathcal{M}^1_C)$, $C^2 = (\underline{C}, \mathcal{M}^2_C)$ over $(Spec(A), \mathcal{M}_A)$ are special log curves. Then given a compatible collection of isomorphisms*

$$\phi_n : \mathcal{M}^1_{C_n} \to \mathcal{M}^2_{C_n}$$

over C_n, there exists a unique isomorphism $\phi : \mathcal{M}^1_C \simeq \mathcal{M}^2_C$ inducing the ϕ_n.

Proof. It suffices to construct ϕ after a finite étale extension of A by the uniqueness. Therefore, we may assume that we have a chart $\mathbb{N}^T \to \mathcal{M}_A$ where T denotes the set of singular points of the closed fiber. Note that the canonical isomorphism $\mathbb{N}^T \to \overline{\mathcal{M}}_{A,s_0}$ (where s_0 denotes the closed point) defined by the map in 4.3.5 depends only on the closed fiber of C, and hence C^1 and C^2 define the same isomorphism $\mathbb{N}^T \to \overline{\mathcal{M}}_{A,s_0}$ which we lift to a chart.

Let $t \in T$ be a singular point. In some étale neighborhood U of t we have by remark 4.3.3 a map

$$\beta^i : \mathbb{N}^2 \to \mathcal{M}^i_C$$

such that the diagram

$$\begin{tikzcd} \mathbb{N}^2 \arrow[r, "\beta^i"] & \mathcal{M}^i_C \\ \mathbb{N}^t \arrow[u, "\Delta"] \arrow[r] & \mathbb{N}^T \arrow[u] \end{tikzcd}$$

commutes (here Δ denotes the diagonal), and such that the resulting map

$$\mathbb{N}^2 \oplus_{\mathbb{N}} \mathbb{N}^T \to \mathcal{M}^i_C$$

is a chart for $\mathcal{M}^i_C$. At the stalk at t, we have an isomorphism

$$\mathbb{N}^2 \oplus_{\mathbb{N}} \mathbb{N}^T \xrightarrow{\simeq} \overline{\mathcal{M}}^1_{C,\bar{t}} \xrightarrow{\phi_n} \overline{\mathcal{M}}^2_{C,\bar{t}} \xrightarrow{\simeq} \mathbb{N}^2 \oplus_{\mathbb{N}} \mathbb{N}^T$$

and after perhaps composing β^1 by the automorphism of $\mathbb{N}^2$ that switches the generators we may assume that the ϕ_n induce the identity map

$$\mathbb{N}^2 \oplus_{\mathbb{N}} \mathbb{N}^T \to \mathbb{N}^2 \oplus_{\mathbb{N}} \mathbb{N}^T.$$

Let (x, y) be the pair $(\alpha_{\mathcal{M}^1_C}(\beta^1(1,0)), \alpha_{\mathcal{M}^1_C}(\beta^1(0,1)))$ of elements of O_C and let (u, v) be the pair $(\alpha_{\mathcal{M}^2_C}(\beta^2(1,0)), \alpha_{\mathcal{M}^2_C}(\beta^2(0,1)))$. To give an isomorphism ϕ between $\mathcal{M}^1_C$ and $\mathcal{M}^2_C$ over U such that the map

$$\phi \circ \beta^1 : \mathbb{N}^2 \to \overline{\mathcal{M}}^1_C \to \overline{\mathcal{M}}^2_C$$

equals

$$\beta^2 : \mathbb{N}^2 \to \overline{\mathcal{M}}^2_C$$

is equivalent to giving units $a, b \in O^*_C$ such that $ab = 1$ and

$$x = au, \quad y = bv.$$

We claim that a solution to these equations is unique if it exists. To see this suppose we have two solutions (a, b) and (a', b'). The completion of O_C at t is isomorphic to

$$A[[u, v]]/(uv - \alpha(e_t))$$

where $\alpha(e_t) \in \mathfrak{m}_A$ is the image if $1 \in \mathbb{N}^t$. Therefore, if $x = au = a'u$ we must have

$$(a - a')u = 0$$

and hence $a - a' \in (v)$. Similarly, if $y = bv = b'v$ we must have $b - b' \in (u)$. Thus we can write $a' = a + vf$ and $b' = b + ug$ for some f and g. We show that vf and ug are zero modulo $\mathfrak{m}^n_A$ for all n. If $n = 1$, then $\alpha(t) = 0$ so we have

$$1 = a'b' = (a + vf)(b + ug) = ab + bvf + aug = 1 + bvf + aug$$

which implies that $vf = 0$ and $ug = 0$. Thus assume true for $n-1$. Then $(vf)(ug)$ is zero modulo $\mathfrak{m}^n$ and hence modulo $\mathfrak{m}^n$ we again have the equation

$$1 = a'b' = (a+vf)(b+ug) = ab + bvf + aug = 1 + bvf + aug.$$

This implies the uniqueness of the pair (a,b).

Since we are given ϕ_n over C_n it follows from the Artin approximation theorem ([4]) that étale locally we have elements a, b as above. By the uniqueness of the solutions to the equations

$$x = Xu, \quad y = Yv, \quad XY = 1$$

it follows that the isomorphisms constructed locally glue, and that the resulting global isomorphism $\mathcal{M}_C^1 \simeq \mathcal{M}_C^2$ induces the ϕ_n. □

The essential surjectivity follows from:

Lemma 5.4.8. *Let A be a complete local ring with residue field k of finite type and let $\underline{C}/A$ be a stable curve. Suppuse further that a log structure $\mathcal{M}_A$ on A is given together with a compatible collection of $\mathcal{M}_A$-log structures $\mathcal{M}_{C_n}$ on $\underline{C} \otimes (A/\mathfrak{m}^n)$ making $C_n/\mathcal{A}_n$ a special log curve. Then there exists a unique log structure on $\underline{C}$ inducing those on the C_n making C/A a special log curve.*

Proof. By the uniqueness we may assume that the log structure on A is induced by a chart

$$\mathbb{N}^s \to A.$$

Let W denote the ring of Witt vectors of k if $char(k) > 0$ and k if $char(k) = 0$. Consider the functor F on artinian local $W[[t_1, \ldots, t_s]]$-algebras Q with residue field k for which $F(Q)$ consists of isomorphism classes of log smooth liftings of C_1/k to Q where Q is given the log structure associated to the images of the t_i. We claim that the functor F is pro-representable.

By ([32], 2.11) it suffices to show that given a deformation situation (see 2.1.2)

$$\begin{array}{ccc} & & B \\ & & \downarrow \\ A' & \longrightarrow & A \end{array}$$

of $W[[t_1, \ldots, t_s]]$-algebras with residue field k, the natural map

$$F(A' \times_A B) \to F(A') \times_{F(A)} F(B)$$

is bijective, and that the tangent space $F(k[\epsilon])$ is finite dimensional (these are conditions (S1) and (S2) of 2.1.4 and 2.1.6 for F). This follows from $(S1^{log})$ and $(S2^{log})$ for $\mathcal{M}_g^{log}$ and the fact that we know that a special log curve admits no infinitesimal automorphisms (5.4.4). Indeed given curves C_1/A' and C_2/B defining a class

$$([C_1],[C_2]) \in F(A') \times_{F(A)} F(B).$$

We can view C_1 in a unique way as a lifting of $C_{2,A}$ to A' since there are no infinitesimal automorphisms. Thus $(S1^{log})$ implies that there exists a curve $C/A' \times_A B$ lifting C_1 and C_2. If $\tilde{C}/A' \times_A B$ is a second such curve, then there are unique isomorphisms $\tilde{C}_{A'} \simeq \tilde{C}_{A'}$ and $\tilde{C}_B \simeq C_B$ and hence $(S1^{log})$ again implies that $\tilde{C} \simeq C$. That $(S2)$ follows from $(S2^{log})$ is clear, and hence F is pro-representable.

By 5.4.2 F is formally smooth, and hence F is pro-represented by a formally smooth $W[[t_1,\ldots,t_s]]$-algebra R. Thus over the reductions R_n of R we have a compatible family $(\underline{C}_n^u, \mathcal{M}_{C_n^u})$ which is the universal data. By the Grothendieck existence theorem ([10], III.5) and 4.2.8 there exists a log curve $(\underline{C}^u, \mathcal{M}_{C^u})$ over R inducing the $(\underline{C}_n^u, \mathcal{M}_{C_n^u})$. Pulling back $(\underline{C}^u, \mathcal{M}_{C^u})$ via the map $R \to A$ induced by the C_n we get a log curve $\tilde{C}$ over A inducing the $(\underline{C}_n, \mathcal{M}_{C_n})$. Applying the Grothendieck existence theorem again we conclude that there is a unique isomorphism $\underline{\tilde{C}} \simeq \underline{C}$ compatible with the isomorphism on reductions. This proves existence of $\mathcal{M}_C$ on $\underline{C}$ and uniqueness follows from 5.4.7. □

5.4.9 *(Separation Condition).* Condition 3.5.13.3 follows from the following lemma which shows that there is a dense open set where the underlying automorphism of schemes is trivial and lemma 3.3.4.

Lemma 5.4.10. *Let A_0 be a reduced ring, and let $\pi : \underline{X} \to Spec(A_0)$ be a separated flat morphism of reduced schemes. Then if $\phi : \underline{X} \to \underline{X}$ is an automorphism of $\underline{X}$ over $Spec(A_0)$ such that for a dense set of points $s \in Spec(A_0)$, $\phi \otimes k(s) = id$, then $\phi = id$.*

Proof. Let $\underline{U} = \underline{X} - \Gamma_\phi^{-1}(\Delta_X)$. Then $\underline{U}$ is an open subscheme of $\underline{X}$. Since π is flat $\pi(\underline{U})$ is open in $Spec(A_0)$. Moreover, $\pi(\underline{U})^c$ contains a dense set of points. Hence $\pi(\underline{U})$ equals the empty set. □

5.5 Comparison with [8], [9]

Proposition 5.5.1. *Let F_1 and F_2 be log algebraic stacks, and suppose*

$$G : F_1 \to F_2$$

is a functor such that

1. *G is fully faithful.*

2. *If $X \in FLog$ has $\underline{X}$ isomorphic to the spectrum of an artinian local ring, then G induces a bijection on isomorphism classes*

$$\overline{F}_{1,X} \to \overline{F}_{2,X}.$$

Then G is an equivalence of log algebraic stacks.

Proof. If $x : X \to F_1$, $y : Y \to F_1$ are 1-morphisms, then we abuse notation and write $X \times_{F_2} Y$ for the product $G(X) \times_{F_2} G(Y)$. Note that the full faithfulness implies that in this situation there is a natural isomorphism

$$X \times_{F_1} Y \simeq X \times_{F_2} Y.$$

Choose $X \to F_1$ which is smooth, relatively strict and surjective. Then F_1 is equivalent to the stack associated to the groupoid $X \times_{F_1} X \rightrightarrows X$. Since G is fully faithful, to prove that G is an equivalence it suffices that the induced 1-morphism $X \to F_2$ is also smooth, strict, and surjective.

Suppose $Y \to F_2$ is a 1-morphism. To prove that $X \times_{F_2} Y \to Y$ is strict and surjective it suffices to consider the case when $\underline{Y}$ is the spectrum of a field. But in that case, by assumption 2, the 1-morphism $Y \to F_2$ arises from a 1-morphism $Y \to F_1$.

To prove that $X \times_{F_2} Y \to Y$ is smooth, consider a diagram

$$\begin{array}{ccc} X \times_{F_2} Y & \longleftarrow & Z_0 \\ \downarrow & & \downarrow \\ Y & \longleftarrow & Z \end{array}$$

where $\underline{Z}$ is isomorphic to the spectrum of an artinian local ring, and $Z_0 \to Z$ is a closed immersion defined by a nilpotent ideal. From assumption 2, the composite $Z \to Y \to F_2$ is induced from a 1-morphism $Z \to F_1$ and from this it follows that there exists a lifting filling in the diagram. □

5.5.2. In the case of the functor (5.1.6)

$$G : \mathcal{M}_g \to \mathcal{M}_g^{log}$$

the full faithfulness follows from 4.3.7. To verify the second condition proceed as follows. Suppose that $\mathcal{A}' \to \mathcal{A}$ is a surjective morphism between artinian local log rings with kernel I, an $A/\mathfrak{m}$-module of finite type, and that $a \in \mathcal{M}_{g,\mathcal{A}}$ is obtained from a strict morphism to $\mathcal{M}_g$. Then we claim that the map

$$\overline{\mathcal{M}}_{g,a,\mathcal{A}'} \to \overline{\mathcal{M}}^{log}_{g,G(a),\mathcal{A}'} \tag{5.3}$$

is bijective. From the full faithfulness the map is injective. Thus suppose $C/\mathcal{A}'$ is an object of $\overline{\mathcal{M}}^{log}_{g,G(a),\mathcal{A}'}$. Certainly there exists a log structure $\mathcal{N}$ on A' lifting $\mathcal{M}_{\mathcal{A}}$ and a morphism

$$(Spec(A'), \mathcal{N}) \to \mathcal{M}_g$$

lifting a, such that the induced log curve $C'/(Spec(A'), \mathcal{N})$ has underlying scheme isomorphic to $\underline{C}$ (just pull back the log structure on $\mathcal{M}_g$ via the map $Spec(A') \to \underline{\mathcal{M}}_g$ defined by $\underline{C}$). Now the set of liftings of $\mathcal{M}_{\mathcal{A}}$ to A' are a torsor under

$$Ext_{A^\sharp}(\mathcal{A}, I).$$

Therefore $\mathcal{M}_{\mathcal{A}'} - \mathcal{N}$ defines a class in this group which from 5.3.7 and 5.3.9 is the image of the class of $\underline{C} - \underline{C}'$ under the map given in those theorems. It follows that the class is zero and that $\mathcal{N}$ and $\mathcal{M}_{\mathcal{A}'}$ are isomorphic as liftings of $\mathcal{M}_{\mathcal{A}}$. Fix such an isomorphism and view C' as a lifting of a to $\mathcal{A}'$. The isomorphism classes of log smooth liftings are a torsor under

$$H^1(C \otimes A/\mathfrak{m}, Der_{\mathcal{A}/\mathfrak{m}}(C \otimes A/\mathfrak{m}, I \otimes O_{C \otimes A/\mathfrak{m}}))$$

and the class of $C - C'$ maps to zero in

$$\underline{\overline{\mathcal{M}}}^{log}_{g,\mathcal{A}[I]}$$

It follows from this and 5.3.1 that there exists an infinitesimal automorphism σ of $\mathcal{A}'$ such that

$$C' \longrightarrow Spec(\mathcal{A}') \xrightarrow{\sigma} Spec(\mathcal{A}')$$

is isomorphic to $C/Spec(\mathcal{A}')$. This proves the bijectivity of the map (5.3).

It follows that it is enough to verify that G induces a bijection on isomorphism classes over separably closed fields. Thus the following lemma concludes the proof of 5.1.7:

Lemma 5.5.3. *Let k be a separable closed field, $\mathcal{M}_k$ a log structure on $Spec(k)$, $X = (Spec(k), \mathcal{M}_k)$, and suppose $\underline{C}/\underline{X}$ is a curve. If $\mathcal{M}^1_C$ and $\mathcal{M}^2_C$ are two $\mathcal{M}_k$-log structures on $\underline{C}$ which make C/X a special log curve then there exists a unique automorphism $\sigma : \mathcal{M}_k \to \mathcal{M}_k$ and an isomorphism*

$$(\underline{C}, \mathcal{M}^1_C) \simeq (\underline{C}, \mathcal{M}^2_C) \times_{X,\sigma} X.$$

Proof. Let T be the set of singular points of $\underline{C}$ and choose a chart

$$\mathbb{N}^T \to \mathcal{M}_k.$$

If $t \in T$ let $\mathbb{N}^2 \to \mathcal{M}^1_C$ and $\mathbb{N}^2 \to \mathcal{M}^2_C$ be charts inducing isomorphisms on the strict henselizations

$$(k[X,Y]/XY)^{sh} \simeq O_{C,\bar{t}} \simeq (k[U,V]/UV)^{sh}.$$

By 4.2.9 there exists units $\zeta_1, \zeta_2 \in O^*_{C,\bar{t}}$ such that

$$X = \zeta_1 U,\ Y = \zeta_2 V, \quad \text{or} \quad X = \zeta_1 V,\ Y = \zeta_2 U.$$

Moreover by the proof of 4.2.9 we see that we can choose ζ_1, ζ_2 such that $\zeta_1\zeta_2 = a_t \in k$ and that the condition of having their product in k determines the ζ_i uniquely. Define σ on $\mathcal{M}_k$ by sending the generator of $\mathbb{N}^T$ corresponding to t to $\lambda(a_t) + e_t$.

Then we see that for each geometric point $\bar{x} \to \underline{C}$ there exists a unique morphism $\tilde{\sigma}_{\bar{x}}$ making the diagram

$$\begin{array}{ccc} \mathcal{M}^1_{C,\bar{x}} & \xrightarrow{\tilde{\sigma}_{\bar{x}}} & \mathcal{M}^2_{C,\bar{x}} \\ \uparrow & & \uparrow \\ \mathcal{M}_k & \xrightarrow{\sigma} & \mathcal{M}_k \end{array}$$

commute. Since our log structures are fine, we can extend the $\tilde{\sigma}_{\bar{x}}$ to étale neighborhoods and therefore obtain maps $\tilde{\sigma}_i$ on some étale cover $\{U_i\}$ of C. By the uniqueness the induced maps $p_1^*\tilde{\sigma}_i$ and $p_2^*\tilde{\sigma}_i$ agree on $U_i \times_C U_j$ and hence glue to give a global isomorphism. $\square$

Chapter 6

Modular Curves: First Construction

Throughout this chapter N will denote a fixed integer greater than 0, and S denotes $Spec(\mathbb{Z}[1/N])$ with the trivial log structure.

6.1 Full level N structure.

6.1.1. Let $\mathcal{M}_N^{log}$ be the fibered category over $FLog$ such that for $X \in FLog$ the category $\mathcal{M}_{N,X}^{log}$ consists of data $(f : C \to X, e, i)$ such that

A. f is a vertical morphism satisfying conditions 1-3 of 5.1.4.

B. $e \in C(X)$ is a strict section.

C. i is an inclusion of groups

$$i : (\mathbb{Z}/N\mathbb{Z})^2 \hookrightarrow Aut_{\underline{X}}(\underline{C}).$$

D. For every geometric point $\bar{x} \in X$, the fiber $C_{\bar{x}}$ is either a smooth curve or isomorphic to the Néron N-gon ([9]) and $p_a(\underline{C}_{\bar{x}}) = 1$.

E. If $\underline{C}_{\bar{x}}$ is smooth then the action of $(\mathbb{Z}/N\mathbb{Z})^2$ commutes with the canonical group law on $(\underline{C}_{\bar{x}}, e)$.

F. If $\underline{C}_{\bar{x}}$ is singular then $(\mathbb{Z}/N\mathbb{Z})^2$ acts trivially on $Pic^0(\underline{C}_{\bar{x}})$.

G. Etale locally on X there exists $\sigma \in (\mathbb{Z}/N\mathbb{Z})^2$ which extends to an isomorphism of log schemes $C \to C$ over X such that for each geometric point $\bar{x} \to \underline{X}$, the restriction $\sigma_{\bar{x}}$ of σ to the fiber $C_{\bar{x}}$ acts transitively on the irreducible components of $\underline{C}_{\bar{x}}$.

We hope that the ambiguity of the notation $\mathcal{M}_N^{log}$ as compared to our earlier notation of $\mathcal{M}_g^{log}$ for the moduli stack of stable curves does not lead to much confusion. Both unfortunately seem to be standard notations in their respective contexts.

The main result of this chapter is the following theorem:

Theorem 6.1.2. *$\mathcal{M}_N^{log}$ is a log algebraic stack, and is naturally equivalent to the stack $\mathcal{M}_N$ of ([9], III.2.1) endowed with the log structure corresponding to the divisor at infinity.*

The proof of this theorem will occupy the following 4 sections.

Remark 6.1.3. We will show (6.3.1) that the data of (C) above can be replaced with an inclusion of groups

$$i : (\mathbb{Z}/N\mathbb{Z})^2 \hookrightarrow Aut_X(C)$$

(automorphisms of C/X as a log scheme).

6.2 Solid objects.

Definition 6.2.1. *An object $(f : C \to X, e, i)$ is* **solid** *if there exists locally on X and C a chart*

$$\begin{array}{ccc} \mathbb{N}^2 & \longrightarrow & \Gamma(C, \mathcal{M}_C) \\ {\scriptstyle\Delta}\uparrow & & \uparrow \\ \mathbb{N} & \longrightarrow & \Gamma(X, \mathcal{M}_X) \end{array}$$

for f such that the induced map

$$\underline{C} \to \underline{X} \times_{\mathbb{Z}[\mathbb{N}]} \mathbb{Z}[\mathbb{N}^2]$$

is étale.

Remark 6.2.2. In 3.5.4 we had a notion of a solid object of a stack over the category of log schemes. Our abuse of language is justified by corollary 6.2.4 below.

Theorem 6.2.3. *Given $(f : C \to X, e, i) \in \mathcal{M}_N^{log}$ there exists a solid object $(f^\sharp : C^\sharp \to X^\sharp, e^\sharp, i^\sharp)$ and a morphism whose underlying morphism of schemes is an isomorphism*

$$X \to X^\sharp$$

such that $(f : C \to X, e, i)$ is obtained by base change. This data is unique in the sense of 4.3.7.

Proof. Uniqueness. Let $\bar{x} \in X$ be a geometric point, $t \in C_{\bar{x}}$ a singular point. Then $\overline{\mathcal{M}}^{\sharp}_{C,\bar{t}}$ is the submonoid of $\overline{\mathcal{M}}_{C,\bar{t}}$ generated by the irreducible elements which are not in the image of $\overline{\mathcal{M}}_{X,\bar{x}}$, and $\overline{\mathcal{M}}^{\sharp}_{X,\bar{x}}$ is the inverse image of $\overline{\mathcal{M}}^{\sharp}_{C,\bar{t}}$ in $\overline{\mathcal{M}}_{X,\bar{x}}$. From this the uniqueness follows.

Existence. By remark 5.2.9, theorem 4.3.7 applies. Let

$$\phi : \mathcal{M}'_X \to \mathcal{M}_X, \quad \psi : \mathcal{M}'_C \to \mathcal{M}_C$$

be the log structures provided by that theorem (we write $\mathcal{M}'$ instead of $\mathcal{M}^{\sharp}$ so as not to confuse the notation). Define $\mathcal{M}^{\sharp}_X$ to be the image of ϕ and $\mathcal{M}^{\sharp}_C$ to be the pushout

$$\begin{array}{ccc} f^*\mathcal{M}'_X & \longrightarrow & f^*\mathcal{M}^{\sharp}_X \\ \downarrow & & \downarrow \\ \mathcal{M}'_C & \longrightarrow & \mathcal{M}^{\sharp}_C. \end{array} \tag{6.1}$$

We claim that the pair $(\mathcal{M}^{\sharp}_X, \mathcal{M}^{\sharp}_C)$ has the desired properties. To see this we may assume that $\underline{X}$ is isomorphic to the spectrum of a strictly henselian local ring. Let T be the set of singular points of the closed fiber. Then we have decompositions (see 4.3.17)

$$\mathcal{M}'_X \simeq \oplus_{t \in T, O^*_X} \tilde{K}_t, \quad \mathcal{M}'_C \simeq \oplus_{t \in T, O^*_C} \tilde{L}_t.$$

For each $t \in T$, the map $\tilde{K}_t \to \mathcal{M}_X$ is injective and induced from a map $\mathbb{N} \to \mathcal{M}_X$.

Recall that this decomposition of $\mathcal{M}'_X$ arose from a canonical isomorphism

$$\mathbb{N}^T \simeq \overline{\mathcal{M}}'_{X,x_0}$$

where x_0 denotes the closed point. $\tilde{K}_t$ is simply the inverse image in $\mathcal{M}'_X$ of the subsheaf of $\overline{\mathcal{M}}'_X$ defined by the image of $\mathbb{N}^t \hookrightarrow \mathbb{N}^T$. Therefore, if σ is an automorphism of C/X then it follows from the construction of $\tilde{K}_t$ that

$$\tilde{K}_t = \tilde{K}_{\sigma(t)}.$$

Combining this with 6.1.1.G we see that all the $\tilde{K}_t$ are equal when viewed as sub-log structures of $\mathcal{M}_X$. It follows from 4.3.6 that the diagram (6.1) locally admits a chart

$$\begin{array}{ccc} \mathbb{N}^2 \oplus \mathbb{N}^{|T|-1} & \longrightarrow & \mathbb{N}^2 \\ {\scriptstyle \Delta\times 1}\uparrow & & \uparrow{\scriptstyle \Delta} \\ \mathbb{N} \oplus \mathbb{N}^{|T|-1} & \longrightarrow & \mathbb{N}. \end{array}$$

This implies that $(\mathcal{M}^{\sharp}_X, \mathcal{M}^{\sharp}_C)$ has the desired form. ∎

Corollary 6.2.4. *$\mathcal{M}_N^{log}$ is naturally equivalent to the log stack defined by the stack of solid objects with its natural morphism to $LOG_{Spec(\mathbb{Z}[1/N])}$ as in 3.5.2.*

6.3 The action of $(\mathbb{Z}/N\mathbb{Z})^2$.

Theorem 6.3.1. *Let $(f : C \to X, e, i) \in \mathcal{M}_N^{log}$ be a solid object. Then for every $\sigma \in (\mathbb{Z}/N\mathbb{Z})^2$ there exist a unique endomorphism $\sigma_C : C \to C$ of C as a log scheme over X whose underlying morphism of schemes is $i(\sigma)$.*

Lemma 6.3.2. *Let $\underline{X}$ be a scheme of finite type over $\underline{S}$, and let*

$$F : (\underline{X} - schemes) \to Set$$

be a relatively representable (1.2.8.2), limit preserving functor such that

1. *F is a sheaf for the étale topology.*
2. *The natural morphism of functors $F \to \underline{X}$ is formally étale. That is, given any diagram*

$$\begin{array}{ccc} Spec(A_0) & \xrightarrow{z_0} & F \\ \downarrow & & \downarrow \\ Spec(A) & \longrightarrow & \underline{X} \end{array}$$

   where $A \to A_0$ is a surjective morphism between artinian local rings, there exists a unique lifting $z : Spec(A) \to F$ of z_0.
3. *For every complete local ring $\hat{A}$ with residue field of finite type over $\underline{X}$, the natural map*

$$F(\hat{A}) \to \varprojlim F(\hat{A}/\mathfrak{m}^n)$$

   is bijective.
4. *If k is a field, then $F(k)$ consists of one element.*

Then the natural morphism of functors $F \to \underline{X}$ is an isomorphism.

Proof. We verify the conditions of ([2], 3.4).

Conditions (2) and (3) imply that the functor F is effectively pro-representable.

Condition (2) implies that for any

$$y_1 : \underline{Y}_1 \to F, \quad y_2 : \underline{Y}_2 \to F$$

with $\underline{Y}_1$ and $\underline{Y}_2$ locally of finite type over $\underline{X}$, the map

$$\underline{Y}_1 \times_F \underline{Y}_2 \to \underline{Y}_1 \times_{\underline{X}} \underline{Y}_2$$

is an open immersion.

Now since the map $F \to \underline{X}$ is formally étale, a map $\underline{Y} \to F$ is formally étale at a point $p \in \underline{Y}$ if and only if the composite $\underline{Y} \to \underline{X}$ is formally étale at p. Therefore the set of points where $\underline{Y} \to F$ is formally étale is open. This implies the lemma. $\square$

6.3.3. With notation as in the theorem, consider the functor F on $\underline{X}$-schemes which to any $\underline{T}/\underline{X}$ associates the set of automorphisms σ_C as in 6.3.1 for $C \times_X T/T$. Since F classifies morphism between étale sheaves, F is a sheaf with respect to this topology. By 1.3.9 F is also limit preserving.

If

$$\sigma_C : \underline{Z}_1 \to F, \quad \sigma'_C : \underline{Z}_2 \to F$$

are two elements of F, then the product $\underline{Z}_1 \times_F \underline{Z}_2$ is the functor over $\underline{Z}_1 \times_{\underline{X}} \underline{Z}_2$ expressing the condition that the two maps

$$p_1^* \sigma_C^b, p_2^* \sigma_C^{',b} : \sigma^* \mathcal{M}_C \to \mathcal{M}_C$$

are equal. By 3.1.5 applied to the diagram

$$\begin{array}{ccc} \sigma^* \mathcal{M}_C & \xrightarrow{p_1^* \sigma_C^b} & \mathcal{M}_C \\ {\scriptstyle p_2^* \sigma_C^{',b}} \downarrow & & \\ \mathcal{M}_C & & \end{array}$$

we see that $\underline{Z}_1 \times_F \underline{Z}_2$ is represented by a locally closed subscheme of $\underline{Z}_1 \times_{\underline{X}} \underline{Z}_2$.

6.3.4. In order to understand the deformation theory of objects of $\mathcal{M}_N^{log}$ we need to make sense of quotients by finite group actions. Suppose X is a log scheme and that G is a finite group acting on X such that the orbit of any point is contained in an affine open set of $\underline{X}$. Then the scheme-theoretic quotient $\underline{X}/G$ exists. Moreover, if the action of G is free, then the map

$$\pi : \underline{X} \to \underline{X}/G$$

is étale. Suppose we are in this situation. Then the G-action defines descent datum on $\mathcal{M}_X$ relative to the morphism π and hence we get a fine log structure $\mathcal{M}_{X/G}$ on $\underline{X}/G$. Explicitly, the log structure is given by

$$\underline{U} \mapsto (\mathcal{M}_X(\underline{U} \times_{\underline{X}/G} \underline{X}))^G.$$

In what follows we denote by X/G the quotient $\underline{X}/G$ with this log structure, and $\pi : X \to X/G$ is the natural morphism of log schemes.

Lemma 6.3.5. *Let C/X be a solid object of $\mathcal{M}_N^{log}$ and let $\sigma \in (\mathbb{Z}/N\mathbb{Z})^2$ be an element satisfying 6.1.1.G. Choose an extension of σ to an automorphism of C/X. Then the log scheme $C/ < \sigma >$ is a special object of $\mathcal{M}_1^{log}$ (see 5.2.6 for the definition of special object).*

Proof. The action of σ on $\underline{C}$ is free and hence the map

$$\underline{C} \to \underline{C}/ < \sigma >$$

is finite and étale. The extension of σ to C/X defines descent datum for the X-log structure on C to $\underline{C}/ < \sigma >$. The question of whether the induced object is special is étale local and hence follows from the fact that C/X is solid. □

Lemma 6.3.6. *The map $F \to \underline{X}$ is formally étale.*

Proof. The question is étale local on $\underline{X}$ so we may assume that we have an element $\sigma \in (\mathbb{Z}/N\mathbb{Z})^2$ as in 6.1.1.G. Choose an extension of σ to an automorphism $\sigma : C \to C$ over X. Suppose now that $A \to A_0$ is a surjective morphism of artinian local rings with kernel I killed by the maximal ideal of A. Denote by $(f_0 : C_0 \to \mathcal{A}_0, e_0, i_0)$ the induced object over A_0 and by $(f : C \to \mathcal{A}, e, i)$ the induced object over A. We suppose that $F(A_0)$ is not empty.

Fix an element $g \in (\mathbb{Z}/N\mathbb{Z})^2$. Since g commutes with σ it induces a map which we denote by the same letter

$$g : \underline{C}/ < \sigma > \to \underline{C}/ < \sigma > .$$

By assumption we have a map between objects of $\mathcal{M}_1^{log}$

$$g_{C_0} : (C_0/ < \sigma >, e_0) \to (C_0/ < \sigma >, g(e)_0).$$

Using g_{C_0} we can view $(C/ < \sigma >, e)$ as a log smooth lifting of $(C_0/ < \sigma >, g(e)_0)$.

It follows from the fact that $\mathcal{M}_1^{log}$ is algebraic that there exists a unique infinitesimal automorphism ϵ of $\mathcal{M}_A$ and an isomorphism h such that the diagram

$$\begin{array}{ccc} (C/<\sigma>, e) & \xrightarrow{h} & (C/<\sigma>, g(e)) \\ f\downarrow & & f\downarrow \\ Spec(\mathcal{A}) & \xrightarrow{\epsilon} & Spec(\mathcal{A}) \end{array}$$

commutes and h reduces to g_{C_0} modulo the maximal ideal. Since an irreducible pointed nodal curve of genus 1 admits no infinitesimal automorphisms, the underlying map of schemes associated to h must be equal to g. It follows that for all $i \in \mathbb{Z}$, h^i defines a map

$$\begin{array}{ccc} (C/<\sigma>, e) & \xrightarrow{h^i} & (C/<\sigma>, (ig)(e)) \\ f\downarrow & & f\downarrow \\ Spec(\mathcal{A}) & \xrightarrow{\epsilon^i} & Spec(\mathcal{A}). \end{array}$$

In particular, for $i = N$ we have a commutative diagram

$$\begin{array}{ccc} (C/<\sigma>, e) & \xrightarrow{h^N} & (C/<\sigma>, e) \\ f\downarrow & & f\downarrow \\ Spec(\mathcal{A}) & \xrightarrow{\epsilon^N} & Spec(\mathcal{A}). \end{array}$$

From 5.3.2 we conclude that ϵ^N is the identity. On the other hand, the set of infinitesimal automorphisms of $\mathcal{M}_{A_0}$ is an A_0 module and hence multiplication by N is injective. It follows that ϵ is the identity.

We now have a diagram

$$\begin{array}{ccc} (C, e) & & (C, g(e)) \\ \downarrow & & \downarrow \\ (C/<\sigma>, e) & \xrightarrow{h} & (C/<\sigma>, g(e)). \end{array}$$

The vertical maps are étale and strict, and hence to find a map

$$C \to C$$

filling in the diagram, it suffices to construct a map of schemes. On the other hand, the étale sites of C and C_0 are equivalent so such a map is obtained from the one given over $\mathcal{A}_0$.

This proves that $F \to \underline{X}$ is formally étale. $\square$

6.3.7. Now let $\hat{A}$ be a complete local ring and consider the map

$$F(\hat{A}) \to \varprojlim F(\hat{A}/\mathfrak{m}^n)$$

Suppose $g \in (\mathbb{Z}/N\mathbb{Z})^2$, and we are given a compatible system of isomorphisms

$$g_n : C_n \to C_n$$

extending the action of g on $\underline{C}$. It follows from 5.4.7 that there exists an isomorphism $g^*\mathcal{M}_C \simeq \mathcal{M}_C$ on $\underline{C}$.

It remains to verify that $F(k)$ consists of a single element for every algebraically closed k.

Thus suppose k is an algebraically closed field. By ([9], II.3.2) there exists on $\underline{C}$ a unique structure of a generalized elliptic curve such that the action of $(\mathbb{Z}/N\mathbb{Z})^2$ is given by translation by N-torsion points. Moreover, the arguments of the proof of 5.5.3 show that the log structure on C is isomorphic to the natural one. In other words, if t is a singular point of C then the completion of the local ring at t is isomorphic to

$$k[[x,y]]/(xy)$$

with the standard log structure. Identifying the local rings via the translation under which the log structure is invariant, we see that the action on $k[[x,y]]/(xy)$ of an element in $(\mathbb{Z}/N\mathbb{Z})^2$ is given by $x \mapsto \zeta x$, $y \mapsto \zeta^{-1}y$ for an N-th rooth of unity ζ. From this it follows that there is a unique extension to a morphism of log schemes over $Spec(k)$.

This concludes the proof of theorem 6.3.1.

6.4 Proof that $\mathcal{M}_N^{log}$ is algebraic.

We verify the conditions of 3.5.13.

Proposition 6.4.1. *Let $\mathcal{A}' \to \mathcal{A}$ be a strict surjection of log rings with nilpotent kernel I such that $I^2 = 0$, $(C', e')/\mathcal{A}' \in \mathcal{M}_1^{log}$ a special object, and $(C, e)/\mathcal{A}$ the reduction of C'. Then for any automorphism (which need not fix e)*

$$g : C \to C$$

such that g^N is the identity, there exists a unique lifting

$$g' : C' \to C'$$

of g for which $g'^N = id$.

Proof. The set of liftings of g are a torsor under the infinitesimal automorphisms of C'/C which are given by

$$H^0(C, Der_{\mathcal{A}}(C, I)).$$

If g' and h' are two liftings of g, then for all $i \in \mathbb{Z}$, we get a class

$${g'}^i - {h'}^i \in H^0(C, Der_{\mathcal{A}}(C, I)).$$

The formula

$${g'}^i {h'}^{-i} = {g'}^{(i-1)}(g'{h'}^{-1}){g'}^{(1-i)}{g'}^{(i-2)}(g'{h'}^{-1}){g'}^{(2-i)} \cdots g'{h'}^{-1}$$

implies that this class is equal to

$$\sum_{j=0}^{i-1} Lie(g^j)(g' - h').$$

Now in our case, the action of $Lie(g)$ is trivial by ([9], II.3.2 and the description of the Tate curve VII.1.12). Therefore, we get the formula

$${g'}^i - {h'}^i = i(g' - h').$$

This implies the uniqueness of the lifting. For if g' and h' both satisfy the conditions of the proposition, then

$$N(g' - h') = {g'}^N - {h'}^N = 0$$

which is only possible if $g' - h' = 0$ since N is invertible.

As for existence, we observe that the obstruction to lifting g to any map is a class

$$c \in H^1(C, Der_{\mathcal{A}}(C, I)).$$

For if $\alpha : C' \otimes_{\mathcal{A}'} \mathcal{A} \to C$ is the map making C' a lifting of C, then the existence of a lifting h' is equivalent to the statement that the two log smooth liftings of C

$$(C', \alpha) \text{ and } (C', g \circ \alpha)$$

are isomorphic. The class c is defined by

$$c = (C', g \circ \alpha) - (C', \alpha) \in H^1(C, Der_{\mathcal{A}}(C, I)).$$

It follows from the construction of c that for any $i \in \mathbb{Z}$ the class

$$(C', g^{i+1} \circ \alpha) - (C', g^i \circ \alpha) \in H^1(C, Der_{\mathcal{A}}(C, I))$$

is also equal to c. Therefore, for any $i \in \mathbb{Z}$ we have an equality

$$ic = (C', g^i \circ \alpha) - (C', \alpha).$$

Since $g^N \circ \alpha$ we must have $Nc = 0$ and hence $c = 0$ since N is invertible.

This shows that we can lift g to some h'. Let

$$\partial = id - {h'}^N \in H^0(C, Der_{\mathcal{A}}(C, I))$$

and let $g' = h' + \frac{1}{N}\partial$. Then

$$id - {g'}^N = (id - {h'}^N) - \partial = 0$$

and hence ${g'}^N = 0$. $\square$

Corollary 6.4.2. *With notation as above, if $C \in \mathcal{M}^{log}_{N,\mathcal{A}}$ is a solid object, σ as in 6.1.1.G, and $C/ < \sigma > \in \mathcal{M}^{log}_{1,\mathcal{A}}$ the corresponding log elliptic curve, then the natural functor*

$$\mathcal{M}^{log}_{N,\mathcal{A}',C} \rightarrow \mathcal{M}^{log}_{1,\mathcal{A}',C/<\sigma>}$$

is an equivalence of categories.

Proof. Since the map

$$C \rightarrow C/ < \sigma >$$

is strict and étale, the functor

$$(\text{liftings of } C \text{ to } \mathcal{A}') \rightarrow (\text{liftings of } C/ < \sigma > \text{ with } (\mathbb{Z}/N\mathbb{Z})^2/ < \sigma > \text{-action})$$

is an equivalence of categories. The result therefore follows from the proposition. $\square$

6.4.3. Let $\mathcal{A}' \rightarrow \mathcal{A}$ be a strict surjection of log rings, with nilpotent kernel, and $C/\mathcal{A} \in \mathcal{M}^{log}_N$ a formally solid object. If $C'/\mathcal{A}'$ is a lifting of $C/\mathcal{A}$, then it follows from the corollary that the map

$$Aut_{\mathcal{M}^{log}_{N,\mathcal{A}'}}(C') \rightarrow Aut_{\mathcal{M}^{log}_{N,\mathcal{A}}}(C)$$

is bijective. Indeed both sides are sheaves on the étale site of A' and hence the assertion can be verified étale locally. Thus we may assume that we have a σ as in 6.1.1.G and in this case the automorphisms of C and C' are in bijection with the automorphisms of $C/ < \sigma >$ and $C'/ < \sigma >$ which respect the $(\mathbb{Z}/N\mathbb{Z})^2/ < \sigma >$-action.

The abscence of infinitesimal automorphisms implies that the functor on étale A-algebras

$$L : B \mapsto \{\text{isomorphism classes of liftings of } C/\mathcal{A} \text{ to } B \times_{\mathcal{A}} \mathcal{A}'\}$$

is a sheaf (here $B \times_{\mathcal{A}} \mathcal{A}'$ has the log structure induced from $\mathcal{A}'$). As in 5.3.1, there is a natural action of

$$Der_A(\mathcal{A}, I)$$

on this sheaf which is locally simply transitive by the corresponding result for $\mathcal{M}_1^{log}$. Since the sheaf $Der_A(\mathcal{A}, I)$ is coherent the functor L is smooth.

In addition, this implies that given a deformation situation (3.5.7) and $C \in \mathcal{M}_{N,\mathcal{A}}^{log}$ the natural map

$$\overline{\mathcal{M}}_{N,C,\mathcal{A}' \times_{\mathcal{A}} \mathcal{B}} \to \overline{\mathcal{M}}_{N,C,\mathcal{A}'} \times \overline{\mathcal{M}}_{N,C,\mathcal{B}}$$

is bijective. It is injective because any automorphism of C lifts. To prove surjectivity, let

$$(C_{\mathcal{A}'}, C_{\mathcal{B}}) \in \overline{\mathcal{M}}_{N,C,\mathcal{A}'} \times \overline{\mathcal{M}}_{N,C,\mathcal{B}}$$

and choose an element $\tilde{C} \in \overline{\mathcal{M}}_{N,C,\mathcal{A}' \times_{\mathcal{A}} \mathcal{B}}$ whose image in $\overline{\mathcal{M}}_{N,C,\mathcal{B}}$ is equal to $C_{\mathcal{B}}$. Let $\tilde{C}_{\mathcal{A}'}$ be the image of $\tilde{C}$ in $\overline{\mathcal{M}}_{N,C,\mathcal{A}'}$. Then the difference

$$\partial = \tilde{C}_{\mathcal{A}'} - C_{\mathcal{A}'} \in Der_A(\mathcal{A}, I) \simeq Der_B(\mathcal{B}, I).$$

Therefore, $\tilde{C} + \partial$ maps to the pair $(C_{\mathcal{A}'}, C_{\mathcal{B}})$.

The bijectivity of

$$\overline{\mathcal{M}}_{N,C,\mathcal{A}' \times_{\mathcal{A}} \mathcal{B}} \to \overline{\mathcal{M}}_{N,C,\mathcal{A}'} \times \overline{\mathcal{M}}_{N,C,\mathcal{B}}$$

combined with the fact that all automorphisms lift implies that the functor

$$\mathcal{M}_{N,C,\mathcal{A}' \times_{\mathcal{A}} \mathcal{B}} \to \mathcal{M}_{N,C,\mathcal{A}'} \times \mathcal{M}_{N,C,\mathcal{B}}$$

is an equivalence of categories. This verifies 3.5.13.1.

6.4.4. Condition 3.5.13.2 follows from the same argument as in 5.4.7-5.4.9. Suppose $\{C_n\}$ is a compatible family of solid objects of $\mathcal{M}_N^{log}$ over the reductions $\hat{\mathcal{A}}/\mathfrak{m}^n$ of a complete noetherian local log ring $\hat{\mathcal{A}}$. We show that there exists a unique object in $\mathcal{M}_{N,\hat{\mathcal{A}}}^{log}$ inducing the C_n. The uniqueness follows from 5.4.7 and the usual Grothendieck existence theorem. To prove essential surjectivity we may by descent theory and the uniqueness replace $\hat{A}$ by an étale cover and therefore can assume that we have $\sigma \in (\mathbb{Z}/N\mathbb{Z})^2$ as in 6.1.1.G for C_0. In addition we may assume that we have a chart

$$\mathbb{N} \to \hat{A}$$

for the log structure on $\hat{\mathcal{A}}$. Let W be the ring of Witt vectors of $k = \hat{A}/\mathfrak{m}$ if $char(k) > 0$ and k otherwise, and define F to be the functor on artin local $W[[\mathbb{N}]]$-algebras Q with residue field k for which $F(Q)$ is equal to

$$\overline{\mathcal{M}}_{N,C_0,Spec(\mathbb{N}\to Q)}$$

(notation as in 2.3.3). Here the map $\mathbb{N} \to Q$ is the map induced from $W[[\mathbb{N}]] \to Q$. As shown above, the functor F is isomorphic to the deformation functor of $C_0/ < \sigma >$ and hence is pro-representable by a formally smooth $W[[\mathbb{N}]]$-algebra. 3.5.13.2 follows by the usual Grothendieck existence theorem and 4.2.8.

Finally condition 3.5.13.3 follows from 3.3.3 and 5.4.10. This concludes the proof of the statement that $\mathcal{M}_N^{log}$ is a log algebraic stack.

6.5 Comparison with ([9], III.2.1).

The following theorem of Deligne-Rapoport shows that an object of $\mathcal{M}_N^{log}$ admits a canonical structure of a generalized elliptic curve:

Theorem 6.5.1 ([9], II.3.2). *Let C/X be a solid object of $\mathcal{M}_N^{log}$. Then there exists a unique structure of a generalized elliptic curve on $(\underline{C}/\underline{X}, e)$ such that the action of $(\mathbb{Z}/N\mathbb{Z})^2$ is given by translation by N-torsion points on $\underline{C}^{reg}$.*

6.5.2. Let $\mathcal{M}_N$ be the classifying stack of generalized elliptic curves with full level N-structure ([9], IV.2.4), and give $\mathcal{M}_N$ the log structure corresponding to its divisor at infinity. Denote by $\mathcal{E}$ the tautological family of $\mathcal{M}_N$ endowed with the log structure coming from the inverse image of the divisor at infinity on $\mathcal{M}_N$. A concrete description of these log structures is given by the Tate curve ([9], VII.2.4). In particular, for any morphism

$$\underline{T} \to \underline{\mathcal{M}}_N$$

we obtain a solid object of $\mathcal{M}_N^{log}$ by pulling back the log structures and the tautological family $\mathcal{E}$. This defines a functor

$$G : \mathcal{M}_N \to \mathcal{M}_N^{log}.$$

It follows from 6.5.1 that this functor is fully faithful. From the argument in 5.5.3, if $\underline{X}$ is isomorphic to the spectrum of a field then

$$G : \overline{\mathcal{M}}_{N,X} \to \overline{\mathcal{M}}_{N,X}^{log}$$

is bijective. By the discussion in 6.4.3, if $\mathcal{A} \to \mathcal{A}_0$ is a surjection of artinian local log rings with kernel I (an A_0-module of finite type), then for any $a \in \mathcal{M}^{log}_{N,\mathcal{A}_0}$ the set

$$\overline{\mathcal{M}}^{log}_{N,a,\mathcal{A}}$$

is a torsor under the set of infinitesimal automorphisms of $\mathcal{M}_A$ over $\mathcal{M}_{A_0}$. It follows that the map

$$\overline{\mathcal{M}}_{N,a,\mathcal{A}} \to \overline{\mathcal{M}}^{log}_{N,a,\mathcal{A}}$$

is surjective and hence bijective by the full faithfulness. This proves that 5.5.1.2 holds, and hence completes the proof of 6.1.2.

Chapter 7

Modular Curves: Second Construction

As in the preceding chapter, S denotes the scheme $Spec(\mathbb{Z}[1/N])$ with the trivial log structure, where N is a fixed integer greater than 0.

7.1 Valuative Log Spaces.

Throughout this section all log schemes will be schemes with log structure on the *Zariski* site. The results of this section are due to K. Kato (unpublished, outlined in [16]).

Definition 7.1.1. *A* **log space** *is a pair* $Z = (\underline{Z}, \mathcal{M}_Z)$ *consisting of a locally ringed space* $\underline{Z}$ *together with an integral log structure* $\mathcal{M}_Z$ *on* $\underline{Z}$. Z *is* **valuative** *if for every point* $z \in \underline{Z}$, $\mathcal{M}_{Z,z}$ *has the property that for each* $a \in \mathcal{M}_{Z,z}^{gp}$ *either* a *or* $-a$ *is in* $\mathcal{M}_{Z,z}$.

If M is an abstract integral monoid, we also say that M is valuative if for each $a \in M^{gp}$ either a or $-a$ is in M.

Lemma 7.1.2. *Let* M *be an integral monoid.*

1. *M is valuative if and only if every finitely generated ideal is principal.*

2. *If M is valuative then M is saturated. M is valuative if and only of M/M^* is valuative.*

3. *Suppose M is valuative and $t, a \in M^{gp}$ are elements such that there exists integers $i, j \in \mathbb{Z}$ such that*

$$-it + a \in M \quad \text{and} \quad jt - a \in M.$$

Then there exists an integer $n \in \mathbb{Z}$ such that we can take $i = n$, $j = n + 1$. If m is a second integer with this property then either $t \in M^*$ or $m = n \pm 1$ and not both $n + 1$ and $n - 1$ work.

Proof. (1) Suppose M is an integral monoid such that every ideal is principal. To prove that M is valuative it suffices to show that the set

$$\{\pm a | a \in M\} \subset M^{gp}$$

is a group. For this it suffices to show that for any $a, b \in M$ either $a - b$ or $b - a$ is in M. Let K be the ideal generated by a and b. By assumption K is principal, say generated by $k \in M$. It follows that there exists natural numbers $n_a, n_b \in \mathbb{Z}$ not both zero and $m \in M$ such that

$$k = m + n_a a + n_b b.$$

Without loss of generality assume $n_a \neq 0$. Since $a \in K$, there exists $m_2 \in M$ such that

$$a = m_2 + k.$$

Let $m_3 = m + (n_a - 1)a + n_b b$. Then

$$k = m_2 + m_3 + k$$

and consequently $m_2, m_3 \in M^*$. It follows that K is generated by a which implies that $b = a + m$ for some $m \in M$, and hence $b - a \in M$. The case when $n_b \neq 0$ gives $a - b \in M$.

Suppose now that M is valuative, and let K be a finitely generated ideal; say generated by $a_1, \ldots, a_r \in M$. For each i, either $a_i - a_{i-1}$ or $a_{i-1} - a_i$ is in M by the valuative condition, and consequently one of the sets $a_1, \ldots, \hat{a}_i, \ldots, a_r$ or $a_1, \ldots, \hat{a}_{i-1}, \ldots, a_r$ generate K. By induction on the number of generators we conclude that K is principal.

(2) If $a \in M^{gp}$, then either a or $-a$ is in M. If $na \in M$ for $n > 0$, and $-a \in M$ then $-na \in M$ and hence $na \in M^*$. It follows that $a \in M^*$. This proves that M is saturated if M is valuative. The second statement is immediate.

(3) If $it + a \in M$ for all $i \in \mathbb{Z}$ and for some $j \in \mathbb{Z}$, $jt - a \in M$. Then $-jt + a$ and $jt - a$ are both in M, hence are units. This implies that t and a are both units. Assume this is not the case.

Then there exists $i \in \mathbb{Z}$ such that $-it + a \in M$ but $(-i-1)t + a \notin M$. Therefore $(i+1)t - a \in M$, and we can take $n = i$. Suppose n and m both have the desired property and that $|n - m| > 1$. Then

$$(-nt + a) + ((m+1)t - a), \quad (-a + (n+1)t) + (a - mt)$$

are both in M. This implies that $(n - m + 1)t, (m - n + 1)t \in M$. It follows that $t \in M^*$ which we assumed was not the case. □

7.1.3. Let X be a fine log scheme, and let $K \subset \mathcal{M}_X$ be a coherent sheaf of ideals. Consider maps

$$f : Z \to X$$

where Z is a log space with the property that for every $z \in Z$, the ideal generated by K in $\mathcal{M}_{Z,z}$ is principal.

Lemma 7.1.4. *There exists a universal such map*

$$b : B_K(X) \to X.$$

Moreover, $B_K(X)$ is a fine log scheme.

Proof. The question is local on X so we may assume that X is given as $Spec(\mathbb{Z}[P])$ where P is a fine monoid, and that the sheaf of ideals is induced from some ideal $K \subset P$.

Denote by $P^\cdot$ the graded monoid

$$P^\cdot = \coprod_{n=0}^{\infty} K^n,$$

and let $S^\cdot$ be the graded ring $\mathbb{Z}[P^\cdot]$. For each $k \in K$ let $P^\cdot_k[0]$ be the monoid consisting of the degree zero elements of the monoid $P^\cdot_k$ obtained by inverting $k \in K^1$.

Let $\underline{B}_K(X)$ be the scheme

$$Proj(S^\cdot).$$

There is a natural cover $\{\underline{U}_k\}_{k \in K}$ where

$$\underline{U}_k = Spec(\mathbb{Z}[P^\cdot_k[0]]).$$

Each $\underline{U}_k$ has a natural log structure, and there are natural isomorphisms between these log structures on the $\underline{U}_k \cap \underline{U}_{k'}$. Thus the log structures glue to make $B_K(X)$ a log scheme. The natural map $P \to P^\cdot$ induces a morphism of log schemes

$$b : B_K(X) \to X.$$

It follows from the construction that this is the desired map. □

7.1.5. Now suppose X is a log space. Associated to X are two functors X^{val} and $B(X)$ defined as follows. X^{val} is the functor on valuative log spaces

$$Z \mapsto Hom(Z, X).$$

$B(X)$ is the functor on the category of log spaces which to any Z associates the set of morphisms $f : Z \to X$ such that for all $z \in Z$ and finitely generated ideal, $K \subset \overline{\mathcal{M}}_{X,f(z)}$, the ideal generated by K in $\overline{\mathcal{M}}_{Z,z}$ is principal.

Lemma 7.1.6. *Suppose X is locally isomorphic to $Spec(P \to \mathbb{Z}[P])$ where P is a fine monoid. Then $B(X)$ is representable by a log space locally of the form $\varprojlim X_i$ where each X_i is a log scheme locally isomorphic to $Spec(P_i \to \mathbb{Z}[P_i])$ where P_i is a fine monoid.*

Proof. The question is local on X so without loss of generality we may assume that $X = Spec(P \to \mathbb{Z}[P])$. For every ideal $K \subset P$ let $B_K(X)$ be the log blowup defined in 7.1.4. The $B_K(X)$ form a projective system, and we let

$$\tilde{X} = \varprojlim B_K(X).$$

If $f : Z \to X$ defines an object of $B(X)(Z)$, then for every ideal $K \subset P$ and point $z \in Z$, the ideal generated by K in $\mathcal{M}_{Z,z}$ is principal. It follows that there exists a unique factorization of the map f through $\tilde{X}$.

On the other hand, let $\tilde{x} \in \tilde{X}$ be a point. Then

$$\overline{\mathcal{M}}_{\tilde{X},\tilde{x}} = \varinjlim \overline{\mathcal{M}}_{B_K(X),x_K}.$$

If $K_x \subset \overline{\mathcal{M}}_{X,x}$ is an ideal, then the inverse image of K in P is an ideal whose image generates K. It follows that the image of K in $\overline{\mathcal{M}}_{\tilde{X},\tilde{x}}$ generates a principal ideal. From this it follows that $\tilde{X}$ represents $B(X)$. $\square$

Theorem 7.1.7. *Suppose X is a fine log scheme. Then the functor X^{val} is representable.*

Proof. By the universal property, it follows that the question is local on X. Moreover, choosing a chart locally we may assume that

$$X = Spec(P \to \mathbb{Z}[P]).$$

We now construct log spaces $B^r(X)$ for $r \in \mathbb{Z}$ inductively together with morphisms $B^r(X) \to B^{r-1}(X)$ which represent $B(B^{r-1}(X)) \to B^{r-1}(X)$. Moreover, each $B^r(X)$ will be of the form $\varprojlim B^{r-1}(X_i)$ where X_i is a log scheme locally isomorphic to $Spec(Q \to \mathbb{Z}[Q])$ for a fine monoid Q.

Let $B^0(X) = X$, $B^1(X) = B(X)$. To construct $B^r(X)$ from $B^{r-1}(X)$, write $B^{r-1}(X) = \varprojlim B^{r-2}(X_i)$ where X_i is of the above form. The collection

$$\{B(B^{r-2}(X_i))\}_i$$

form a projective system by functoriality and we let $B^r(X) = \varprojlim B(B^{r-2}(X_i))$. There is a natural map $B^r(X) \to B^{r-1}(X)$ by the universal property of inverse limit. It is straightforward to verify that this map has the correct properties.

From this construction we obtain a projective system $\{B^r(X)\}$ and we let

$$B^{\cdot}(X) = \varprojlim B^r(X).$$

This space together with the natural map $B^{\cdot}(X) \to X$ represents X^{val}. To prove this recall the characterization of valuative monoids 7.1.2.1. Suppose $\tilde{x} \in B^{\cdot}(X)$ is a point. Then

$$\overline{\mathcal{M}}_{B^{\cdot}(X),\tilde{x}} = \varinjlim_{r} \overline{\mathcal{M}}_{B^r(X),\tilde{x}_r}.$$

If $K \subset \overline{\mathcal{M}}_{B^{\cdot}(X),\tilde{x}}$ is a finitely generated ideal then K is induced from some ideal $K_r \subset \overline{\mathcal{M}}_{B^r(X),\tilde{x}_r}$ for some r. But K_r induces a principal ideal in $\overline{\mathcal{M}}_{B^{r+1}(X),\tilde{x}_{r+1}}$ and hence K is principal. It follows that $B^{\cdot}(X)$ is valuative.

Suppose $f : Z \to X$ is a morphism where Z is valuative. Then by the universal property and 7.1.2.1 there exists a unique lifting of f to $B(X)$. Again by the universal property this map lifts to $B^2(X)$. By induction we obtain a compatible system of maps $Z \to B^r(X)$ lifting f and hence a unique map

$$Z \to B^{\cdot}(X).$$

It follows that $B^{\cdot}(X)$ represents X^{val}. □

From this point on we do not distinguish between the functor X^{val} and the space which represents it.

7.1.8. Suppose X, Y, and Z are fine log schemes. Then it follows that the valuative log space

$$(X \times_Z Y)^{val}$$

represents the functor on valuative log spaces

$$X^{val} \times_{Z^{val}} Y^{val}.$$

Therefore, the full subcategory of the category of valuative log spaces consisting of spaces which are locally of the form X^{val} for some fine log schemes X has fiber products. We refer to this subcategory as the category of algebraic valuative log spaces. It follows that a group object in the category of algebraic valuative log spaces is equivalent to a valuative log space X together with morphisms

$$+ : X \times X \to X$$

and

$$- : X \to X$$

such that the usual diagrams commute.

7.2 The Kato-Tate Curve.

7.2.1. Let $\underline{T}$ be a scheme, and let $t \in \Gamma(\underline{T}, O_T)$ be a global section. Let T denote the log scheme obtained by giving $\underline{T}$ the log structure coming from $\mathbb{N} \to O_T$, $1 \mapsto t$. Construct a log scheme $\mathbb{G}_m^t$ over T as follows. $\mathbb{G}_m^t$ is covered by open sets $\{U_{i+\frac{1}{2}}\}_{i\in\mathbb{Z}}$ given by

$$U_{i+\frac{1}{2}} = \underline{T}[x_i, y_{i+1}]/(x_i y_{i+1} - t)$$

with the log structure coming from $\mathbb{N}^2 \to O_{U_{i+\frac{1}{2}}}$, $(1,0) \mapsto x_i$, $(0,1) \mapsto y_{i+1}$. $U_{i+\frac{1}{2}}$ and $U_{i-\frac{1}{2}}$ are glued along

$$U_{i+\frac{1}{2}}[1/x_i] \simeq \underline{T}[x_i^{\pm 1}] \simeq \underline{T}[y_i^{\pm 1}] \simeq U_{i-\frac{1}{2}}[1/y_i]$$

via $x_i = 1/y_i$. This isomorphism extends to an isomorphism of log schemes, and $\mathbb{G}_m^t$ is by definition the log scheme obtained by gluing.

On $U_{i+\frac{1}{2}}$ let $\alpha_i \in \Gamma(U_{i+\frac{1}{2}}, \mathcal{M}^{gp}_{\mathbb{G}_m^t})$ be the section corresponding to $(-i-1, -i) \in \mathbb{Z}^2$ (i.e. $t^{-1-i}y_{i+1}$). From the definitions it follows that α_i and α_{i-1} define the same section over

$$U_{i+\frac{1}{2}} \cap U_{i-\frac{1}{2}}.$$

Let $\alpha \in \Gamma(\mathbb{G}_m^t, \mathcal{M}_{\mathbb{G}_m^t})$ be the resulting global section.

Proposition 7.2.2. *If Z is a valuative log space, then there is a natural bijection between $\mathbb{G}_m^t(Z)$ and the set of $a \in \Gamma(Z, \mathcal{M}_Z^{gp})$ such that for each point $z \in Z$, there exists integers $i \leq j$ such that $-a - it$ and $jt + a$ are both in $\mathcal{M}_{Z,z}$. This bijection is given by*

$$(f : Z \to \mathbb{G}_m^t) \mapsto \textit{image of } \alpha \textit{ in } \Gamma(Z, \mathcal{M}_Z^{gp}).$$

Proof. To give a map $Z \to U_{i+\frac{1}{2}}$ is equivalent to giving a pair $(g_1, g_2) \in \Gamma(Z, \mathcal{M}_Z)$, where g_1 is the image of x_i and g_2 is the image of y_{i+1}. For such a map the global section α maps to $(-i-1)t + g_2 \in \Gamma(Z, \mathcal{M}_Z^{gp})$. Note that $-it - \alpha = t - g_2 = g_1$ and hence this defines a morphism of functors from $U_{i+\frac{1}{2}}$ to the functor which to any Z associates the set of elements $a \in \Gamma(Z, \mathcal{M}_Z^{gp})$ such that $(i+1)t + a$ and $-it - a$ are both in $\Gamma(Z, \mathcal{M}_Z)$. On the other hand, given such an $a \in \Gamma(Z, \mathcal{M}_Z^{gp})$ we can recover the pair (g_1, g_2) as $(-it - a, (i+1)t + a)$, and so $U_{i+\frac{1}{2}}$ represents this functor. Note that $a \in \Gamma(Z, \mathcal{M}_Z^{gp})$ maps to

$$U_{i+\frac{1}{2}} \cap U_{i-\frac{1}{2}}$$

if and only if both i and $i+1$ have the property of 7.1.2.3. The gluing of the $U_{i+\frac{1}{2}}$ is done exactly so that the intersection represents the functor of a's such that both i and $i+1$ work.

Suppose now that $a \in \Gamma(Z, \mathcal{M}_Z^{gp})$ and that for each point $z \in Z$ there exists integers $i \leq j$ such that $-it + a$ and $jt - a$ are both in $\mathcal{M}_{Z,z}$. By 7.1.2.3 we may assume that $j = i+1$. Moreover, since $\mathcal{M}_Z$ is a sheaf we can cover Z by open sets V such that over V the same integer i works for each $z \in V$. From the above there exists a unique map

$$V \to U_{(-i-1)+\frac{1}{2}}$$

such that $a|V$ is the image of α. By the uniqueness these maps glue to give a map

$$V \to \mathbb{G}_m^t.$$

This proves the proposition. $\square$

7.2.3. If Z is a valuative log space, then there is a natural group structure on the set of elements $a \in \Gamma(Z, \mathcal{M}_Z^{gp})$ which have the property described in proposition 7.2.2. This gives $\mathbb{G}_m^t$ the structure of a group functor. Denote by

$$(\mathbb{G}_m^t)^{val} \to \mathbb{G}_m^t$$

the universal valuative log space associated to the $\mathbb{G}_m^t$ by 7.1.7. By the universal property it is a group object in the category of algebraic valuative log spaces.

If Z is a valuative log space, then there is an action of $\mathbb{Z}$ on $\mathbb{G}_m^t(Z)$ given by $a \mapsto a + t$. This action can be described on the log scheme by the map

$$U_{i+\frac{3}{2}} \to U_{i+\frac{1}{2}}, \quad x_i \mapsto x_{i+1}, \quad y_{i+1} \mapsto y_{i+2}.$$

Also the automorphism $a \mapsto -a$ on the functor $\mathbb{G}_m^t$ can be described on the log scheme by

$$U_{i+\frac{1}{2}} \to U_{-i-\frac{1}{2}}, \quad x_{-i-1} \mapsto y_{i+1}, \quad y_{-i} \mapsto x_i.$$

7.2.4. There are certain functors associated to $(\mathbb{G}_m^t)^{val}$ which will be used. To define these functors, we need to be more careful about the distinction between the element of $t \in \Gamma(T, O_T)$ and the section of $\mathcal{M}_T$ which we have also been denoting by t. Therefore, for the purposes of this paragraph, let us denote by $t^\dagger$ the section of $\mathcal{M}_T$ and reserve t for the element in $\Gamma(T, O_T)$. If Z is a valuative log space, define

$$(\mathbb{G}_m^t)^{val}[N](Z) = \{a \in \Gamma(Z, \mathcal{M}_Z^{gp}) | \text{ for all } z \in Z \ \exists\ j \in \mathbb{Z} \text{ s.t. } Na = jt^\dagger \text{ in } \mathcal{M}_{Z,z}^{gp}\},$$

$$(E^t)^{val}[N](Z) = (\mathbb{G}_m^t)^{val}[N](Z)/\mathbb{Z}$$

where the action of $\mathbb{Z}$ is given by $a \mapsto a + t^\dagger$. If j is an integer, we also define

$$[N, j](Z) = \{a \in \Gamma(Z, \mathcal{M}_Z^{gp}) | \ Na = jt^\dagger\}.$$

Suppose now that our chosen $t \in \Gamma(T, O_T)$ is not a unit in $O_{T,t}$ for any $t \in T$. That is, t is locally nilpotent. In this case, if $a \in (\mathbb{G}_m^t)^{val}[N](Z)$ for some connected valuative log space Z, then there exists a unique integer j such that

$$Na = jt^\dagger$$

globally. The point is that if $V_1, V_2 \subset Z$ are open sets such that $Na = j_1 t^\dagger$ on V_1 and $Na = j_2 t^\dagger$ on V_2, then on $V_1 \cap V_2$, $(j_1 - j_2)t^\dagger \in \mathcal{M}_Z^*$ and hence $j_1 = j_2$. It follows that under this hypothesis there is an isomorphism of functors

$$(\mathbb{G}_m^t)^{val}[N] \simeq \coprod_{j \in \mathbb{Z}} [N, j].$$

Also if $a \in (\mathbb{G}_m^t)^{val}[N](Z)$ then there exists a unique integer

$$j \in \{0, 1, \ldots, N-1\}$$

and $s \in \mathbb{Z}$ such that $N(a + st) = jt^\dagger$. It follows that there is an isomorphism of functors

$$(E^t)^{val}[N] \simeq \coprod_{j=0}^{N-1} [N, j].$$

The following explicit description of $[N, j]$ will be important in what follows:

Proposition 7.2.5. *Let j be an integer, and let $r = gcd(N, j)$. Choose integers α and β such that $r = \alpha N + \beta j$. The functor $[N, j]$ is represented by the fine, saturated, valuative log scheme*

$$\underline{T}[w_1, w_2]/(w_1^r = 1, w_2^{N/r} w_1^\beta = t)$$

with log structure induced from $\mathbb{N}$ via $1 \mapsto w_2$. The composite

$$[N, j] \to (\mathbb{G}_m^t)^{val} \to \mathbb{G}_m^t$$

maps $[N, j]$ into the intersection of those $U_{i+\frac{1}{2}}$ with

$$-iN \geq j \geq -Ni - N$$

and for those i the map

$$\underline{T}[x_i, y_{i+1}]/(x_i y_{i+1} - t) \to \underline{T}[w_1, w_2]/(w_1^r = 1, w_2^{N/r} w_1^\beta = t)$$

is given by

$$x_i \mapsto w_1^{\alpha - i\beta} w_2^{(-j-iN)/r}, \quad y_{i+1} \mapsto w_1^{-\alpha+\beta(i+1)} w_2^{(j+iN+N)/r}.$$

Proof. Suppose $Z \to U_{i+\frac{1}{2}}$ is a morphism such that $\alpha \mapsto a \in \Gamma(Z, \mathcal{M}_Z^{gp})$ satisfies

$$Na = jt$$

(here we have returned to our earlier abuse of notation and think of t as a section of $\mathcal{M}_T$). By 7.2.2 a is equal to the image of y_{i+1} minus $(i + 1)t$. It follows that $(j + Ni + N)t \in \Gamma(Z, \mathcal{M}_Z)$ and hence

$$j + Ni + N \geq 0.$$

Similarly, the image of x_i is $-it - a$ and hence $(-iN - j)t \in \Gamma(Z, \mathcal{M}_Z)$ from which it follows that

$$-iN - j \geq 0.$$

From this we obtain the inequalities

$$-iN \geq j \geq -iN - N.$$

Let $P_{i+\frac{1}{2}}$ be the free monoid on generators x_i and y_{i+1}, $t \in P_{i+\frac{1}{2}}$ the element $x_i + y_{i+1}$. $U_{i+\frac{1}{2}}$ is obtained as the fiber product

$$\begin{array}{ccc} & & Spec(P_{i+\frac{1}{2}} \to \mathbb{Z}[P_{i+\frac{1}{2}}]) \\ & & \downarrow \\ T & \longrightarrow & Spec(\mathbb{N} \to \mathbb{Z}[\mathbb{N}]). \end{array}$$

The condition that $Na = jt$ is represented by the subscheme corresponding to the monoid

$$M = P_{i+\frac{1}{2}}/Ny_{i+1} = (j + N(i+1))t.$$

To calculate the saturation of M, observe that there is an isomorphism

$$M^{gp} \to \mathbb{Z}/r\mathbb{Z} \oplus \mathbb{Z}$$

given by

$$x_i \mapsto (\alpha - i\beta, (j + iN)/r), \quad y_{i+1} \mapsto (-\alpha + \beta(i+1), (-j - (i+1)N)/r).$$

It follows that

$$M^{sat} \simeq \mathbb{Z}/r\mathbb{Z} \oplus -\mathbb{N} \subset \mathbb{Z}/r\mathbb{Z} \oplus \mathbb{Z}.$$

Since

$$T \times_{\mathbb{Z}[\mathbb{N}]} \mathbb{Z}[M^{sat}]$$

is valuative it represents $[N, j]$. □

7.2.6. From this description, it follows that the product in the category of fine saturated log schemes

$$(E^t)^{val}[N] \times_T (E^t)^{val}[N]$$

is valuative and hence is also equal to the product in the category of algebraic valuative log schemes. In particular, $(E^t)^{val}[N]$ is a group object in the category of fine saturated log schemes. It is not however a group object in the category of log schemes. If it were, then the underlying scheme would be a finite flat group scheme of rank N^2. If N is invertible in $\underline{T}$ this would imply that the underlying scheme is étale which it is not.

Lemma 7.2.7. *There are no automorphisms of $[N, j]$ over $U_{i+\frac{1}{2}}$.*

Proof. Denote by $\tilde{w}_2$ the element of $\mathcal{M}_{[N,j]}$ mapping to w_2. $\overline{\mathcal{M}}_{[N,j]}$ is generated by the image of $\tilde{w}_2$ and hence an automorphism must be of the form

$$\tilde{w}_2 \mapsto \lambda(\eta) + \tilde{w}_2, \quad w_1 \mapsto \lambda(\zeta) + w_1$$

where $\eta, \zeta \in O^*_{[N,j]}$. By the explicit description of the map

$$[N, j] \to U_{i+\frac{1}{2}}$$

it follows that η and ζ are subject to the relations

$$\zeta^r = 1, \quad \eta^{N/r}\zeta^{\beta} = 1, \quad \zeta^{\alpha - i\beta}\eta^{(-j-iN)/r} = 1, \quad \zeta^{-\alpha+\beta(i+1)}\eta^{(j+iN+N)/r} = 1$$

where r, α, β are as in 7.2.5. Combining the second and third equations gives

$$\eta^{N/r} = \zeta^{-\beta}, \quad \eta^{j/r} = \zeta^{\alpha}.$$

This in turn gives

$$1 = (\zeta^{-\beta})^{\alpha}(\zeta^{\alpha})^{\beta} = \eta^{(\alpha N/r)+(\beta j/r)} = \eta.$$

From this we get the equations

$$\zeta^{\beta} = 1, \quad \zeta^{\alpha} = 1$$

and hence

$$1 = (\zeta^{\beta})^{j/r}(\zeta^{\alpha})^{N/r} = \zeta.$$

$\square$

7.2.8. Under the hypothesis that t is locally nilpotent, the quotient of $\mathbb{G}_m^t$ by $\mathbb{Z}$ exists ([9], VII.1.6) and we denote it by E^t. By 7.2.3 this action of $\mathbb{Z}$ is compatible with the action of $\mathbb{Z}$ on the functor $(\mathbb{G}_m^t)^{val}$ and hence there is an induced map

$$(E^t)^{val}[N] \to E^t.$$

E^t admits one nontrivial automorphism

$$- : E^t \to E^t$$

which on the underlying pointed curve with its canonical structure of a generalized elliptic curve is given by negation ([9], VII.1.3). If $- : (E^t)^{val}[N] \to (E^t)^{val}[N]$ denotes negation as a group functor then it follows from the above explicit description that the induced diagram of morphisms of log schemes

$$\begin{tikzcd} (E^t)^{val}[N] \arrow[r] \arrow[d, "-"'] & E^t \arrow[d, "-"] \\ (E^t)^{val}[N] \arrow[r] & E^t \end{tikzcd}$$

commutes. In terms of the description

$$(E^t)^{val}[N] \simeq \coprod_{j=0}^{N-1} [N, j]$$

the negation map is given by the maps

$$[N,0] \to [N,0], \quad [N,j] \to [N, N-j]$$

$$a \mapsto -a, \quad a \mapsto -a+t.$$

Lemma 7.2.9. *1. For any integer j, there are no non-trivial automorphisms of $[N,j]$ over E^t.*

2. Suppose $j, j' \in \{0, 1, \ldots, N-1\}$ are distinct integers, and $x_1 \to [N,j]$, $x_2 \to [N,j']$ are points endowed with the pullback log structure. Then there are no isomorphisms of log points

$$x_1 \to x_2$$

over E^t.

3. There are no non-trivial automorphisms of $(E^t)^{val}[N]$ over E^t.

Proof. (1). If $j = 0$, then $[N,0]$ mapsto $U_{\frac{1}{2}} \cap U_{\frac{3}{2}}$. For any integer i, the map

$$U_{i+\frac{1}{2}} \to E^t$$

is étale ([9], proof of 1.6). On the other hand, the intersection of $U_{\frac{1}{2}} \cap U_{\frac{3}{2}}$ with the orbit of any point under the action of $\mathbb{Z}$ is either empty or consists of one point. It follows that the map

$$U_{\frac{1}{2}} \cap U_{\frac{3}{2}} \to E^t$$

is an open immersion into the regular locus of E^t.

For $j \neq 0$, the images of x_i and y_{i+1} under the map

$$[N,j] \to U_{i+\frac{1}{2}}$$

are both nilpotent. Hence the image of $[N,j]$ in E^t has support in the singular locus. Let M be an integer sufficiently large such that $[N,j] \to U_{i+\frac{1}{2}}$ factors through the closed subscheme $Z_{i+\frac{1}{2}}$ defined by the M-th power of the ideal of the singular locus in $U_{i+\frac{1}{2}}$. If J denotes the ideal of the singular locus in E^t, then the following diagram is cartesian

$$\begin{array}{ccc} Z_{i+\frac{1}{2}} & \longrightarrow & U_{i+\frac{1}{2}} \\ \uparrow & & \uparrow \\ Z(J^M) & \longrightarrow & E^t. \end{array}$$

Since the orbit of any point intersects $Z_{i+\frac{1}{2}}$ in at most on point, it follows that

$$Z_{i+\frac{1}{2}} \to Z(J^M)$$

is an isomorphism. It follows from this that any automorphism of $[N,j]$ over E^t induces an automorphism of $[N,j]$ over $U_{i+\frac{1}{2}}$. From 7.2.6 there are no non-trivial such automorphisms.

(2) By the above discussion, it suffices to prove that there are no isomorphisms of log points $x_1 \to x_2$ over $U_{\frac{-1}{2}}$. If such an isomorphism existed, we would get an isomorphism

$$\overline{\mathcal{M}}_{[N,j],x_1} \simeq \overline{\mathcal{M}}_{[N,j'],x_2}.$$

Looking at the image of t it follows that $gcd(N,j) = gcd(N,j')$ and by looking at the image of y_0 we get

$$j + (i+1)N = j' + (i+1)N.$$

From this the assertion follows.

(3) By (2), any automorphism of $(E^t)^{val}[N]$ must preserve the decomposition

$$(E^t)^{val}[N] \simeq \coprod_{j=0}^{N-1} [N,j]$$

and hence induces an automorphism of $[N,j]$. By (1) there are no non-trivial such automorphisms. $\square$

7.2.10. Let X be a log scheme, and let E/X be a special object of $\mathcal{M}_1^{log}$ (4.3.6). Consider the functor

$$F : (\underline{X} - \text{schemes}) \to Set$$

which to any $\underline{Y} \to \underline{X}$ associates the isomorphism classes of pairs (E_N, i) having the following properties (let Y denote the scheme $\underline{Y}$ with the pullback log structure):

1. E_N is a group object in the category of fine saturated log schemes over Y such that $\underline{E}_N$ is a finite flat scheme of rank N^2.

2. $i : E_N \to E \times_X Y$ is a morphism of log schemes.

3. If $\underline{Y}_0$ is isomorphic to the spectrum of an artinian local ring and $\underline{Y}_0 \to \underline{Y}$ is a morphism, then either $E \times_X Y_0$ is smooth and E_N are the usual N-torsion points, or there exists étale locally on $\underline{Y}_0$ a global section $t \in \Gamma(\underline{Y}_0, \mathcal{M}_{Y_0})$ which generates $\overline{\mathcal{M}}_{Y_0}$ and a commutative diagram
$$\begin{array}{ccc} E_N \times_Y Y_0 & \xrightarrow{\simeq} & (E^t)^{val}[N] \\ {\scriptstyle i}\downarrow & & \downarrow \\ E \times_X Y_0 & \xrightarrow{\simeq} & E^t. \end{array}$$

4. Two pairs (E_N, i) and (E'_N, i') are isomorphic if there exists an isomorphisms $E_N \simeq E'_N$ of log schemes over $E \times_X Y$, which induces an isomorphism of group objects in the category of fine saturated log schemes over Y.

Theorem 7.2.11. *The functor F is representable by $\underline{X}$. In other words, there exists a unique pair (E_N, i) satisfying conditions 1-3 over X.*

Proof. We apply 6.3.2 to F. Observe first that given two pairs (E_N, i) and (E'_N, i') over Y, there exists at most one isomorphism between them. Two such isomorphisms would define a non-trivial morphism $E_N \to E_N$ over $E \times_X Y$. By passage to the local ring of a point, and then to the completion we may assume that $\underline{Y}$ is isomorphic to the spectrum of a complete local ring. It follows from 7.2.9.3 and 7.2.10.3 that the automorphism induces the trivial automorphism modulo every power of the maximal ideal. By the Grothendieck existence theorem (E_N is proper over E) and 3.3.3 it follows that the automorphism is trivial globally. From this we conclude that the functor F is a sheaf for the étale topology.

Suppose now that $\underline{Y}$ is isomorphic to the spectrum of an artinian local ring and that $E \times_X Y$ is not smooth, and that we have a diagram

$$\begin{tikzcd}
(E^t)^{val}[N] \arrow[d] & E_N \arrow[l, "\simeq"'] \arrow[d, "i"'] & & E'_N \arrow[d, "i'"] \arrow[r, "\simeq"] & (E^{t'})^{val}[N] \arrow[d] \\
E^t & E \times_X Y \arrow[l, "\simeq"'] \arrow[rr, "id"'] & & E \times_X Y \arrow[r, "\simeq"] & E^{t'}.
\end{tikzcd}$$

By the following lemma we must have $t = t'$:

Lemma 7.2.12. *Let A be an artinian local ring, $\mathbb{N} \to A$ a monoid morphism such that the image of 1 is in the maximal ideal of A. Denote by t the image of 1, and let X be the associated log scheme. Then for any unit $u \in A^*$ the Kato-Tate curves E^t and E^{ut} are isomorphic over X if and only if $u = 1$.*

Proof. We can without loss of generality assume that A is strictly henselian. Let J be the maximal ideal of A, and let l be the smallest integer such that $J^l = 0$. We proceed by induction on l.

If $l = 1$, then A is an algebraically closed field k. We can view E^{ut} as being obtained by base change from E^t by an automorphism

$$u : X \to X.$$

Suppose we are given an isomorphism ϕ filling in the diagram

$$\begin{tikzcd}
E^t \arrow[r, "\phi"] \arrow[d] & E^t \arrow[d] \\
X \arrow[r, "u"] & X.
\end{tikzcd}$$

The automorphism group of $\underline{E}^t$ as a pointed curve is $\{\pm 1\}$ and hence using 7.2.3 we may assume that ϕ is the identity map on schemes. Choosing a chart at the singular point x_0 and looking at the completion we see that ϕ^b induces an isomorphism

$$\begin{array}{ccc} \hat{O}_{E^t,x_0} \oplus \mathbb{N}^2 & \xrightarrow{\phi^b} & \hat{O}_{E^t,x_0} \oplus \mathbb{N}^2 \\ \downarrow & & \downarrow \\ k[[x,y]]/(xy) & \xrightarrow{id} & k[[x,y]]/(xy) \end{array}$$

where x and y are the images of the generators $\tilde{x}$ and $\tilde{y}$ of $\mathbb{N}^2$. It follows that ϕ^b is given by

$$\tilde{x} \mapsto \lambda(1 + yf_x) + \tilde{x}, \quad \tilde{y} \mapsto \lambda(1 + xf_y) + \tilde{y}$$

where $f_x, f_y \in k[[x,y]]/(xy)$. This implies that

$$\tilde{x} + \tilde{y} \mapsto \lambda(1 + yf_x + xf_y) + \tilde{x} + \tilde{y}.$$

By assumption $1 + yf_x + xf_y$ is the image of $u \in k^*$ and hence $f_x = f_y = 0$. It follows that $u = 1$. This proves the case $l = 1$.

For general l, note that if $u \in A$ is such that $u \mapsto 1 \in A/J^{l-1}$ then E^{ut} is obtained from E^t by the infinitesimal automorphism defined by $1 \mapsto \lambda(u) + 1$ of X over $X \otimes A/J^{l-1}$. The induction step now follows from 5.3.8. $\square$

Since the automorphism group of E^t/Y is $\{\pm 1\}$, it follows from 7.2.8 and 7.2.9 that there exists a unique isomorphism $E_N \simeq E'_N$ filling in the diagram. This implies that $F(\underline{Y})$ for every $\underline{Y}$ isomorphic to the spectrum of an artinian local ring consists of one element. By passage to the limit using 3.3.3 and the Grothendieck existence theorem, it follows that $F(\underline{Y})$ consists of one element also in the case when $\underline{Y}$ is isomorphic to the spectrum of a complete local ring. This implies all the conditions of 6.3.2 except relative representability.

Thus suppose we are given two collections of data $a, a' : \underline{Y} \to F$. We apply 6.3.2 to the functor $I = \underline{Y} \times_F \underline{Y}$. Since the E_N are finite over $\underline{Y}$, I is a sheaf for the étale topology. In addition, since there exists at most one isomorphism between a and a', I is relatively representable. The remaining conditions of 6.3.2 follow from the fact that $F(A)$ consists of exactly one element for any artinian local ring A combined with the fact that any isomorphism between two collections of data is unique. It follows that the map

$$I \to \underline{Y} \times_{\underline{X}} \underline{Y}$$

is an isomorphism.

This concludes the verification of 6.3.2 and the proof of the theorem. $\square$

Definition 7.2.13. *Let X be a log scheme over S, and let E/X be an object of $\mathcal{M}_1^{log}$. The* **N-torsion points on** *E is the pair (E_N, i) obtained in the case when E/X is solid by theorem 7.2.11. In the general case, it is the pair $(E_N^\sharp, i^\sharp) \times_{X^\sharp} X$ (notation as in 4.3.7).*

7.2.14. E_N is a log étale group log scheme (in the category of fine saturated log schemes) since we assume that N is invertible on the base. Note however that $\underline{E}_N$ is neither a group object nor an étale scheme.

7.3 Full Level Structure.

Lemma 7.3.1. *Let $E/X \in \mathcal{M}_1^{log}$ be an object with X saturated. Then for any morphism $X' \to X$ of connected saturated log schemes, the natural map*

$$E_N(X) \to E_N(X')$$

is injective, and after a log étale base extension $X'' \to X$ independent of X' the induced map

$$E_N(X'') \to E_N(T_i)$$

is bijective where T_i is any connected component of $(X' \times_X X'')^{sat}$.

Proof. Let

$$X' \to X'^\sharp \to X$$

be the canonical factorization of the map $X' \to X$, and suppose $f_1, f_2 \in E_N(X)$ map to the same element in $E_N(X')$. We claim that then $f_1 = f_2$ in $E_N(X'^\sharp)$. To see this note that the underlying morphisms of schemes of f_1 and f_2 are equal. Let us denote this map by f so we have a diagram

$$\mathcal{M}_{X'^\sharp} \to f^*\mathcal{M}_{E_N} \rightrightarrows \mathcal{M}_{X'^\sharp} \to \mathcal{M}_{X'}.$$

By construction of E_N, if $m \in f^*\mathcal{M}_{E_N}$ then Nm is in the image of $\mathcal{M}_{X'^\sharp}$. Since X is saturated this implies that for all sections $m \in \mathcal{M}_{X'^\sharp}$ there exists $\zeta \in \mu_N$ such that

$$f_1^b(m) = f_2^b(m) + \lambda(\zeta).$$

Now if the images of $f_1^b(m)$ and $f_2^b(m)$ in $\mathcal{M}_{X'}$ are equal we must have $\zeta = 1$ and hence they are already equal in $\mathcal{M}_{X'^\sharp}$. Thus it suffices to consider the case when $X' \to X$ is strict.

It follows from the fact that E_N is finite over $\underline{X}$ that the condition that the underlying morphisms of schemes of f_1 and f_2 are equal is represented by a closed subscheme of $\underline{X}$. By the above argument, we see that over this closed subscheme the two maps

$$f_1^b, f_2^b : f^*\overline{\mathcal{M}}_{E_N} \to \overline{\mathcal{M}}_X$$

are equal. From the proof of 3.1.5 it follows that the condition $f_1^b = f_2^b$ is also represented by a closed subscheme (not just a locally closed subscheme). Therefore, the condition $f_1 = f_2$ is represented by a closed subscheme $\underline{S} \subset \underline{X}$.

On the other hand, the fact that $E_N \to X$ is log étale implies that $\underline{S} \to \underline{X}$ is also étale. Hence $\underline{S}$ is also open and since $\underline{X}$ is connected must be either empty or all of $\underline{X}$. From this the first part of the lemma follows.

To prove the second statement, it suffices to consider the case when $X = X^\sharp$. Moreover, the question is étale local so we may assume that $\underline{X}$ is isomorphic to the spectrum of a strictly henselian local ring A. In this case the log structure on A is given by $\mathbb{N} \to A$. Consider the log étale base change induced from

$$\times N : \mathbb{N} \to \mathbb{N}$$

and replace X by the result. Then $E_N(X \otimes A/\mathfrak{m}_A^k)$ consists of N^2 elements for all k. Now since $\underline{E}_N$ is a finite affine scheme over A, to give a section of $\underline{E}_N \to \underline{X}$ is equivalent to solving some polynomial equations in A. By assumption we have N^2 solutions over $\hat{A}$, and it then follows from the Artin approximation theorem ([4]) that we get N^2 sections over A. Given one such section $i : \underline{X} \to \underline{E}_N$, we want to find a map

$$i^*\mathcal{M}_{E_N} \to \mathcal{M}_X.$$

If $\bar{x} \to \underline{X}$ denotes the closed point of $\underline{X}$, then since $\underline{X}$ is isomorphic to the spectrum of a strictly henselian local ring it suffices to construct a map

$$i^*\mathcal{M}_{E_N,\bar{x}} \to \mathcal{M}_{X,\bar{x}}.$$

It follows from the construction of E_N that there exists an isomorphism

$$i^*\mathcal{M}_{E_N,\bar{x}} \simeq \mathbb{N} \oplus_{\mathbb{N}} \mathcal{M}_{X,\bar{x}}$$

where the map $\mathbb{N} \to \mathbb{N}$ is given by multiplication by some integer l which divides N, and $\mathbb{N} \to \mathcal{M}_{X,\bar{x}}$ is the composite

$$\mathbb{N} \to \mathcal{M}_{X^\sharp} \to \mathcal{M}_{X,\bar{x}}.$$

Therefore to give a morphism $i^*\mathcal{M}_{E_N,\bar{x}} \to \mathcal{M}_{X,\bar{x}}$ of monoids such that the composite

$$\mathcal{M}_{X,\bar{x}} \to i^*\mathcal{M}_{E_N,\bar{x}} \to \mathcal{M}_{X,\bar{x}}$$

is equal to the identity is equivalent to giving an element of $\mu_l(A)$. The map $\mu_l(A) \to \mu_l(\hat{A})$ is bijective and hence there exists a unique isomorphism inducing the given one over $\hat{A}$. $\square$

Definition 7.3.2. *Let E/X be an object of $\mathcal{M}_1^{log}$ with X connected and saturated. A* **full level-N-structure** *on E is an isomorphism*

$$\epsilon : (\mathbb{Z}/N\mathbb{Z})^2 \to E_N(X).$$

In general, a full level-N-structure is given by a full level-N-structure on each connected component of X.

Remark 7.3.3. It follows from from the lemma that there is a natural notion of pullback of a full level-N-structure for a morphism of saturated log schemes $f : X' \to X$.

7.3.4. Denote by $FLog^{sat} \subset FLog$ the full subcategory of saturated log schemes. The strict étale topology on $FLog$ restricts to give a topology on $FLog^{sat}$. Thus it makes sense to speak of stacks over $FLog^{sat}$. Let $\mathcal{M}'_N$ be the fibered category over $FLog^{sat}$ whose objects over a log scheme X consists of pairs (E, ϵ) where E is an object in $\mathcal{M}_{1,X}^{log}$ and ϵ is a full level-N-structure on E.

Theorem 7.3.5. *$\mathcal{M}'_N$ is canonically isomorphic to $\mathcal{M}_N^{log}$ (defined in chapter 6) restricted to $FLog^{sat}$.*

Forgetting the level structure defines a functor

$$\begin{array}{c} \mathcal{M}'_N \\ {\scriptstyle \pi}\downarrow \\ \mathcal{M}_1^{log}. \end{array}$$

If E/X is an object of $\mathcal{M}_1^{log}$ and $X_0 \hookrightarrow X$ is a strict closed immersion defined by a nilpotent ideal, then it follows from the fact that E_N is log étale that

$$E_N(X) \to E_N(X_0)$$

is bijective. This implies that the functor π is log étale.

Lemma 7.3.6. *Let $X, Y \in FLog^{sat}$ be log schemes, $x : X \to \mathcal{M}'_N$, $y : Y \to \mathcal{M}'_N$ objects corresponding to log elliptic curves E_X and E_Y with full level structure. Let $X \times_{\mathcal{M}'_N} Y$ be the functor on $FLog$ which to any $Z \in FLog$ associates the set of triples (f, g, α) where $f : Z \to X$, $g : Z \to Y$ and α is an isomorphism*

$$E_X \times_X Z \simeq E_Y \times_Y Z$$

such that the diagram

$$\begin{array}{ccc} (\mathbb{Z}/N\mathbb{Z})^2 & \longrightarrow & E_{X,N}(X) \\ \downarrow & & \downarrow \\ E_{Y,N}(Y) & \longrightarrow & E_{X,N}(Z). \end{array}$$

commutes. Then $X \times_{\mathcal{M}'_N} Y$ is representable by an object of $FLog$.

Proof. Let $X \to \mathcal{M}'_N$, $Y \to \mathcal{M}'_N$ be two objects, and let

$$P = \pi(X) \times_{\mathcal{M}_1^{log}} \pi(Y).$$

Denote by

$$\begin{array}{ccc} E_{X,N} & \xrightarrow{\simeq} & E_{Y,N} \\ \downarrow & & \downarrow \\ E_X & \xrightarrow{\simeq} & E_Y \end{array}$$

the tautological isomorphism over P. For each $a \in (\mathbb{Z}/N\mathbb{Z})^2$ we get two sections $a_X, a_Y : P \to E_{X,N}$. Let P_a denote the fiber product

$$\begin{array}{ccc} P_a & \longrightarrow & P \\ \downarrow & & \downarrow{\scriptstyle a_X \times a_Y} \\ E_{X,N} & \xrightarrow{\Delta} & E_{X,N} \times_P E_{X,N}. \end{array}$$

P_a represents the functor of isomorphisms between E_X and E_Y such that the induced sections a_X and a_Y are the same. It follows that the functor

$$X \times_{\mathcal{M}'_N} Y$$

is represented by the log scheme obtained by taking the product over P of all the P_a. $\square$

Lemma 7.3.7. *An object $X \to \mathcal{M}'_N$ is relatively strict (i.e. $X \times_{\mathcal{M}'_N} Y \to Y$ is strict for all Y) if and only if étale locally there exists a chart for the morphism $X \to X^\sharp$ (4.3.7)*

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{\times N} & \mathbb{N} \\ \downarrow & & \downarrow \\ \Gamma(X^\sharp, \mathcal{M}_{X^\sharp}) & \longrightarrow & \Gamma(X, \mathcal{M}_X). \end{array}$$

Proof. Let $Y \to \mathcal{M}'_N$ be an object. To prove that

$$X \times_{\mathcal{M}'_N} Y \to Y$$

is strict it suffices to consider the case when X and Y both have underlying scheme isomorphic to the spectrum of a separably closed field. Therefore we can assume that isomorphisms between the objects over X and Y with the Kato-Tate curve have been chosen.

We compute the schemes P_a of the preceding lemma. The scheme $\underline{P}$ consists of a finite set of points (two in fact), and so since the construction of P_a is local, we may replace $\underline{P}$ by a single point. It suffices to construct P_a for two generators of $(\mathbb{Z}/N\mathbb{Z})^2$, and we can choose these generators such that one has corresponding section in $[N,0]$ and the other has section in $[N,1]$. Suppose first the section is in $[N,0]$. In this case, the map

$$[N,0] \to [N,0] \times_P [N,0]$$

is strict and so the map $P_a \to P$ is also strict. In the case when a_X is a section to $[N,1]$, either a_Y maps to some other $[N,j]$ with $j \neq 1$ and P_a is empty, or a_Y also maps to $[N,1]$. In the latter case, the diagram

$$\begin{tikzcd} & P \arrow[d, "a_X \times a_Y"] \\ E_{X,N} \arrow[r] & E_{X,N} \times_P E_{X,N} \end{tikzcd}$$

admits a chart

$$\begin{tikzcd} & \mathcal{M}_P \\ \mathcal{M}_P \oplus_{\mathbb{N}} \mathbb{N} & \mathcal{M}_P \oplus_{\mathbb{N}} Q \arrow[l] \arrow[u] \end{tikzcd}$$

where Q is the pushout of the diagram

$$\begin{tikzcd} \mathbb{N} \arrow[r, "\times N"] \arrow[d, "\times N"'] & \mathbb{N} \\ \mathbb{N} & . \end{tikzcd}$$

The log structure $\mathcal{M}_P$ is given by

$$\mathcal{M}_X \oplus_{\mathbb{N}} \mathcal{M}_Y$$

and it follows from this that the log structure on P_a is given by

$$\mathcal{M}_X \oplus_{\mathbb{N}} \mathcal{M}_Y$$

modulo the relation $(a,0) = (0,b)$ where $a \in \mathcal{M}_X$, $b \in \mathcal{M}_Y$ are elements such that Na and Nb are the images of the generator of $\mathbb{N}$. This proves the lemma. $\square$

Lemma 7.3.8. *$\mathcal{M}'_N$ is naturally equivalent to the stack defined by the stack of relatively strict objects with its natural morphism to $LOG_{Spec(\mathbb{Z}[1/N])}$ in the sense of 3.5.2.*

Proof. Given a morphism $s : X \to E_N$ over $X^\sharp$, we get a diagram of inclusions of log structures on X

$$\mathcal{M}_{X^\sharp} \hookrightarrow s^*\mathcal{M}_{E_N} \hookrightarrow \mathcal{M}_X.$$

From the construction of the log schemes E_N we see that if we take the union over sections s giving X a full level N-structure

$$\bigcup_s s^*\mathcal{M}_{E_N} \subset \mathcal{M}_X$$

then we get a sublog structure which satisfies the assumptions in lemma 7.3.7. This amounts to saying that for every geometric point $\bar{x} \to \underline{X}$

$$\bigcup_s s^{-1}\overline{\mathcal{M}}_{E_N,\bar{x}} \subset \overline{\mathcal{M}}_{X,\bar{x}}$$

is a monoid isomorphic to $\mathbb{N}$ such that N times the generator is the image of the generator of $\mathcal{M}_{X^\sharp}$. To prove this it suffices to check it in the case when $\underline{X}$ is isomorphic to the spectrum of a separably closed field. In this case it follows from the structure of the log schemes $[N, j]$ described in 7.2.5. $\square$

Proof of 7.3.5 We have to show that the stack of solid objects with its natural morphism to $LOG_{\mathbb{Z}[1/N]}$ is algebraic. It follows from 7.3.7 that it is relatively representable. Thus it suffices to exhibit a smooth cover.

Let $\{U_i\}$ be an étale strict cover of $\mathcal{M}_1^{log}$ such that each U_i admits a chart $\mathbb{N} \to \mathcal{M}_{U_i}$. Let $\{U'_i\}$ be the family of log schemes obtained by taking the fibred products

$$\begin{array}{ccc} U'_i & \longrightarrow & Spec(\mathbb{Z}[\zeta_N, 1/N][\mathbb{N}]) \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \times N} \\ U_i & \longrightarrow & Spec(\mathbb{Z}[\mathbb{N}]). \end{array}$$

Then by lemma 7.3.1 after perhaps replacing the U'_i be a strict étale cover $E_N(U'_i)$ consists of N^2 elements. Thus for each isomorphism

$$(\mathbb{Z}/N\mathbb{Z})^2 \simeq E_N(U'_i)$$

we get a solid object $U'_i \to \mathcal{M}'_N$. Let $\{\tilde{U}_i\}$ be the collection of objects of $\mathcal{M}'_N$ obtained from the U'_i by considering all such isomorphisms. Then we claim that

$$U = \coprod \tilde{U}_i \to \mathcal{M}'_N$$

defines an étale surjective cover. For any solid object $Y \to \mathcal{M}'_N$ the induced morphism

$$U \times_{\mathcal{M}'_N} Y \to Y$$

is strict by 7.3.7. It is also étale since the extensions $U_i^! \to U_i$ are log étale. Therefore to check that the map is surjective we can assume that $\underline{Y}$ is isomorphic to the spectrum of a field. In this case it is immediate. This proves that $\mathcal{M}'_N$ is algebraic.

To construct the canonical isomorphism between $\mathcal{M}'_N$ and $\mathcal{M}_N^{log}$ we first construct an isomorphism between the underlying stacks. By ([9], IV.3.5), $\underline{\mathcal{M}}_N^{log}$ is the normalization of $\underline{\mathcal{M}}_1^{log}$ in the substack of $\underline{\mathcal{M}}_N^{log}$ where the log structure is trivial. The morphism

$$\underline{\pi} : \underline{\mathcal{M}}'_N \to \underline{\mathcal{M}}_1^{log}$$

is quasi-finite. If $x : Spec(k) \to \underline{\mathcal{M}}'_N$ is an object with k separably closed, then it follows from 7.2.8 that $Aut(x)$ is trivial unless we are in the case of $N = 2$ in which case $Aut(x) = Aut(\underline{\pi}(x))$. Since objects of $\underline{\mathcal{M}}'_N$ do not admit infinitesimal automorphisms we conclude that $\underline{\pi}$ is relatively representable. If $\underline{X} \to \underline{\mathcal{M}}_1^{log}$ is a 1-morphism then the product

$$P = \underline{\mathcal{M}}'_N \times_{\underline{\mathcal{M}}'_N} \underline{X} \to \underline{X}$$

is flat since it is the underlying morphism of algebraic spaces of a log étale and integral morphism

$$\mathcal{M}'_N \times_{\mathcal{M}_1^{log}} \times X \to X$$

where X is given the pullback log structure from $\mathcal{M}_1^{log}$. $(\mathbb{Z}/N\mathbb{Z})^2$ acts on P and fiber by fiber the action is simply transitive. It follows that P is a $(\mathbb{Z}/N\mathbb{Z})^2$-torsor.

This implies that the morphism $\underline{\pi}$ is finite, and that $\underline{\mathcal{M}}'_N$ is normal. This in turn implies that $\underline{\mathcal{M}}'_N$ is the normalization of $\underline{\mathcal{M}}_1^{log}$ in the open substack of $\mathcal{M}'_N$ where the log structure is trivial. Since the substacks where the log structure is trivial of $\mathcal{M}'_N$ and $\mathcal{M}_N^{log}$ are canonically isomorphic this gives a canonical isomorphism between the underlying stacks.

To extend this to an isomorphism of log stacks, simply observe that both $\mathcal{M}_N^{log}$ and $\mathcal{M}'_N$ are smooth and hence the log structure is canonically isomorphic to the pushforward of the trivial log structure on the open substack where the log structure is trivial. This concludes the proof of 7.3.5. $\square$

7.4 $\mathcal{M}_0(N)^{log}$ and $\mathcal{M}_1(N)^{log}$.

7.4.1. The techniques of 7.3 generalize more easily to the construction of other moduli stacks than those of chapter 6. Since the arguments of this section are essentially the same as the arguments of the preceding one we omit some of the details.

Define $\mathcal{M}_1(N)^{log}$ to be the fibered category over $FLog^{sat}$ whose objects over a saturated log scheme X consists of pairs (E, i) where E/X is an object of $\mathcal{M}_1^{log}$ and $i : X \to E_N$ is a section of exact order N.

$\mathcal{M}_0(N)^{log}$ is the fibered category over $FLog^{sat}$ whose objects over a connected log scheme X consists of pairs (E, H) where E is an object of $\mathcal{M}_1^{log}$ and H is a saturated log subgroup scheme of E_N étale locally on $\underline{X}$ isomorphic to the log subgroup scheme defined by a section $X \to E_N$ of exact order N. By this we mean that if $X \to E_N$ is a section of exact order N, then we obtain a sub-group object of E_N in the category of fine saturated log schemes

$$\coprod_{(\mathbb{Z}/N\mathbb{Z})} X \hookrightarrow E_N.$$

Theorem 7.4.2. *$\mathcal{M}_1(N)^{log}$ (resp. $\mathcal{M}_0(N)^{log}$) is a log algebraic stack, and the underlying algebraic stack $\underline{\mathcal{M}}_1(N)^{log}$ (resp. $\underline{\mathcal{M}}_0(N)^{log}$) is canonically isomorphic to the algebraic stack $\mathcal{M}_{\Gamma_{00}(N)}[1/N]$ (resp. $\mathcal{M}_{\Gamma_0(N)}[1/N]$) of ([9]).*

7.4.3. Suppose X is a log scheme with underlying scheme isomorphic to the spectrum of a separably closed field. Let E/X be an object of $\mathcal{M}_1^{log}$. To each section $a : X \to E_N$ we can attach an integer $e(a) \in \{0, 1, \ldots, N-1\}$ as follows. If E/X is smooth let $e(a)$ be equal to 1. Otherwise, choose an isomorphism of E with the Kato-Tate curve. Such an isomorphism is canonical up to the automorphism " $-$ " of 7.2.8. The section a therefore defines a point of $[N, j]$ for some j. Applying the automorphism " $-$ " has the effect of switching $[N, j]$ and $[N, N-j]$. Consequently the integer

$$e(a) = gcd(N, j)$$

does not depend on the choices. If X is an arbitrary log scheme and E/X is an object of $\mathcal{M}_1^{log}$ together with a section $a : X \to E_N$, then for each geometric point $t \to X$ we get an integer which we denote by $e_a(t)$. If a' is another section which defines the same subgroup of $E_N(X^{val})$ as a then it follows from the construction of E_N that the integer $e_{a'}(t)$ is equal to $e_a(t)$. This implies that for an object of $\mathcal{M}_0(N)^{log}$ we also have a well-defined integer $e(t)$ for each geometric point $t \to \underline{X}$.

Definition 7.4.4. *If X is a log scheme, $x : X \to \mathcal{M}_1(N)^{log}$ (resp. $x : X \to \mathcal{M}_0(N)^{log}$) a 1-morphism, then x is* **solid** *if in an étale neighborhood of every geometric point $t \to X$, there exists a chart for the morphism $X \to X^\sharp$*

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{\times e(t)} & \mathbb{N} \\ \downarrow & & \downarrow \\ \Gamma(X^\sharp, \mathcal{M}_{X^\sharp}) & \longrightarrow & \Gamma(X, \mathcal{M}_X). \end{array}$$

We justify this apparent abuse of the word *solid* with the following:

Lemma 7.4.5. *The stacks $\mathcal{M}_1(N)^{log}$ and $\mathcal{M}_0(N)^{log}$ are naturally equivalent to the stacks defined by the stacks of solid objects with the natural morphisms to*

$$LOG_{Spec(\mathbb{Z}[1/N])}$$

in the sense of 3.5.2.

Proof. The argument of 7.3.8 also gives this lemma. □

The proof of 7.3.5 can with slight modifications be carried out to prove 7.4.2. The shortest method, however, is to apply 3.5.13 using the maps

$$\mathcal{M}_0(N)^{log} \to \mathcal{M}_1^{log}$$

and

$$\mathcal{M}_1(N)^{log} \to \mathcal{M}_1^{log}$$

which are log étale. Then the deformation theories of the stacks are the same as the deformation theory of $\mathcal{M}_1^{log}$. Thus condition 3.5.13.1 is immediate. Condition 3.5.13.2 follows from 3.3.3, the usual Grothendieck existence theorem, and 3.5.13.2 for $\mathcal{M}_1^{log}$. Condition 3.5.13.3 follows from 3.3.4 and 5.4.10 and hence the conditions of are verified. □

Remark 7.4.6. The ramification structure of of the maps

$$\underline{\mathcal{M}}_0(N)^{log} \to \underline{\mathcal{M}}_1^{log}, \quad \underline{\mathcal{M}}_1(N)^{log} \to \underline{\mathcal{M}}_1^{log}$$

is given by lemma 7.4.5. The ramification index at a geometric point t is given precisely by the integer $e(t)$ of 7.4.4. The point is that if X is a log scheme with underlying scheme the spectrum of a separably closed field, and $X \to \mathcal{M}_1(N)^{log}$ a 1-morphism with $e(t) = 1$, then the group of automorphisms of X over $X^\sharp$ is isomorphic to $\mu_{e(t)}$. For each automorphism we get a different morphism of log stacks

$$X \to \mathcal{M}_1(N)^{log}$$

but they all induce the same morphism

$$\underline{X} \to \underline{\mathcal{M}}_1(N)^{log}.$$

For example, if N is prime then the description of E_N given in 7.2.5 shows that $\underline{\mathcal{M}}_1(N)^{log}$ has $N-1$ points that are unramified corresponding to the sections in $[N,0]$ and $N-1$ points with ramification index N corresponding to the sections in $[N,j]$, $j \neq 0$. A similar remark applies to $\mathcal{M}_0(N)^{log}$.

Chapter 8

Analytic theory of log elliptic curves

8.1 Classical theory

We review the classical relationship between abelian varieties over $\mathbb{C}$ and families of Hodge structures.

Definition 8.1.1. *A* **pure Hodge structure over $\mathbb{Z}$ of weight** n *is a free $\mathbb{Z}$ module $H_{\mathbb{Z}}$ together with a filtration $F^{\cdot}$ on the complex vector space $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes \mathbb{C}$ such that*

$$H_{\mathbb{C}} = \oplus_{p+q=n} F^p \cap \overline{F}^q$$

where $\overline{F}^q$ denotes the complex conjugate of F^q under the action of $Aut(\mathbb{C})$ via the second factor of $H_{\mathbb{Z}} \otimes \mathbb{C}$.

8.1.2. If we define

$$H^{p,q} = F^p \cap \overline{F}^q, \quad p, q \in \mathbb{Z}$$

then we have a decomposition

$$H_{\mathbb{C}} = \oplus_{p+q=n} H^{p,q}.$$

The *type* of H is the set of pairs (p, q) for which $H^{p,q} \neq 0$.

8.1.3. Suppose R is any $\mathbb{R}$-algebra. We define an action of an element $z \in (R \otimes_{\mathbb{R}} \mathbb{C})^*$ on $H^{p,q} \otimes_{\mathbb{C}} (R \otimes_{\mathbb{R}} \mathbb{C})$ by multiplication by $z^p \bar{z}^q$ where $\bar{z}$ denotes the element of $(R \otimes_{\mathbb{R}} \mathbb{C})^*$ obtained by applying complex conjugation to the second factor.

Because $H^{q,p} = \overline{H}^{p,q}$ this action is invariant under complex conjugation on $H_{R\otimes_{\mathbb{R}}\mathbb{C}}$ and hence defines an action of $(R\otimes_{\mathbb{R}}\mathbb{C})^*$ on H_R. In other words, if $\mathbb{S}$ denotes the algebraic group representing the functor on $\mathbb{R}$-algebras

$$R \mapsto \mathbb{G}_m(R\otimes_{\mathbb{R}}\mathbb{C})$$

then we have a representation

$$\mathbb{S} \to Aut(H_{\mathbb{R}}).$$

In fact, using this construction one can show that to give a Hodge structure is equivalent to giving a free $\mathbb{Z}$-module $H_{\mathbb{Z}}$ together with a representation

$$\mathbb{S} \to Aut(H_{\mathbb{R}}).$$

There is a norm map

$$N : \mathbb{S} \to \mathbb{G}_m$$

of real algebraic groups which sends

$$z \mapsto z\bar{z}, \quad z \in \mathbb{S}(R)$$

for any $\mathbb{R}$-algebra R.

Definition 8.1.4. *For any $n \in \mathbb{Z}$, we define a Hodge structure $\mathbb{Z}(n)$ called the* **Tate Hodge structure** *by taking $H_{\mathbb{Z}} = (2\pi i)^n\mathbb{Z} \subset \mathbb{C}$ and letting $z \in \mathbb{S}(R)$ act via $N(z)^{-n}$.*

8.1.5. If H is a Hodge structure let $C : H_{\mathbb{R}} \to \mathbb{R}$ denote the action of the element $i \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^*$.

Definition 8.1.6. *A* **polarization** *on a Hodge structure H of weight n is a morphism of Hodge structures*

$$\phi : H \times H \to \mathbb{Z}(-n)$$

such that the induced pairing

$$H_{\mathbb{R}} \times H_{\mathbb{R}} \to \mathbb{R}$$

defined by

$$(x, y) \mapsto (2\pi i)^n \phi(x, Cy)$$

is symmetric and positive definite. A Hodge structure H is **polarizable** *if it admits a polarization.*

Remark 8.1.7. In the definition, $H \times H$ denotes the Hodge structure with lattice $H_{\mathbb{Z}} \times H_{\mathbb{Z}}$ and action of $\mathbb{S}$ on $H_{\mathbb{R}} \times H_{\mathbb{R}}$ given by the action on each component. By definition ϕ is a morphism of Hodge structures if ϕ respects the action of $\mathbb{S}$.

Finally we extend these definitions to families. Let S be a complex manifold.

Definition 8.1.8. *A* **family of Hodge structures of weight** n *on* S *consists of a local system* $H_{\mathbb{Z}}$ *of free* $\mathbb{Z}$*-modules on* S *together with a filtration* $F^{\cdot}$ *on* $H_{\mathbb{Z}} \otimes O_S$ *by holomorphic sub-bundles such that for each point* $s \in S$*, the fiber* $(H_{\mathbb{Z},s}, F_s^{\cdot})$ *is a Hodge structure of weight* n*. A polarization on* H *is a morphism of local systems*

$$H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z}(-n)_S$$

which induces a polarization on $(H_{\mathbb{Z},s}, F_s^{\cdot})$ *for each* $s \in S$*.* H *is* **polarizable** *if* H *admits a polarization. For a family of Hodge structures* $(H_{\mathbb{Z}}, F)$ *we let*

$$\nabla : H_{\mathbb{Z}} \otimes O_S \to H_{\mathbb{Z}} \otimes \Omega^1_S$$

be the canonical connection. We say that $(H_{\mathbb{Z}}, F)$ *is a* **variation of Hodge structures** *if for all* $p \in \mathbb{Z}$

$$\nabla(F^p) \subset F^{p-1} \otimes \Omega^1_S.$$

Remark 8.1.9. If $(H_{\mathbb{Z}}, F)$ is a rank 2 family of Hodge structures of weight -1 then $H_{\mathbb{Z}}$ admits a canonical polarization. For locally there are exactly two pairings ϕ and $-\phi$

$$H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z}(1)$$

for which the underlying map of local systems is surjective and $(2\pi i)^{-1}\phi$ is anti-symmetric. One checks from the definition of polarization that for each point $s \in S$ exactly one of ϕ and $-\phi$ defines a polarization. Since the filtration F varies holomorphically the condition that a pairing ϕ defines a polarization at each point is an open condition. Therefore, by the uniqueness the locally constructed polarizations glue to a global one.

8.1.10. Suppose $\underline{S}/\mathbb{C}$ is a smooth scheme of finite type and that $f : \underline{A} \to \underline{S}$ is an abelian variety (with trivial log structure). Define $R_1 f_* \mathbb{Z}$ to be the dual of the local system

$$R^1 f_* \mathbb{Z}$$

on $\underline{S}^{an}$ (the complex analytic space associated to $\underline{S}$). Then $R_1 f_* \mathbb{Z}$ carries a canonical structure of a polarizable Hodge structure of type $\{(-1,0),(0,-1)\}$ and rank $2g$ where g is the relative dimension of $\underline{A}$. The filtration on $R_1 f_* \mathbb{Z}$ is induced from the Hodge filtration on

$$R^1 f_* \mathbb{Z} \otimes O_S \simeq R^1 f_* (\Omega^{\cdot}_{A/S})$$

via the de-Rham isomorphism. The following theorem can be found in ([7]) and it is a version of this theorem for log elliptic curves which is the goal of this chapter:

Theorem 8.1.11. *Let $\underline{S}/\mathbb{C}$ be a smooth scheme of finite type. The functor*

$$\begin{array}{c}(\textit{abelian varieties over } \underline{S}) \\ \downarrow \\ (\textit{polarizable variations of Hodge structures of type } \{(-1,0),(0,-1)\} \textit{ on } S^{an})\end{array}$$

is an equivalence of categories.

8.2 Log Hodge structures

We recall some of the theory of log Hodge structures due to K. Kato and S. Usui ([22]).

8.2.1. Let $S/\mathbb{C}$ be a fine saturated log scheme whose underlying scheme is locally of finite type over $\mathbb{C}$. Recall that there is a ringed space (S^{log}, O_S^{log}) attached to S ([21]). We fix an integer n (the weight) and a collection of integers $(h^{p,q})_{p,q \in \mathbb{Z}}$ (the Hodge type) such that $h^{p,q} = 0$ if $p + q \neq n$.

Definition 8.2.2. *A* **pre-log Hodge structure** *on S is a pair $(H_{\mathbb{Z}}, F)$ where*

1. *$H_{\mathbb{Z}}$ is a locally constant sheaf of $\mathbb{Z}$-modules on S^{log} whose stalks are free of rank*

$$\Sigma h^{p,q}.$$

2. *F is a descending filtration on the O_S^{log} module*

$$O_S^{log} \otimes_{\mathbb{Z}} H_{\mathbb{Z}}.$$

3. *Each F^p is the pullback via*

$$\tau_S : S^{log} \to \underline{S}$$

of a locally free O_S-module of rank $\Sigma_{r \geq p} h^{r,w-r}$.

4. *Each F^p is locally on S^{log} an O_S^{log} direct summand of $O_S^{log} \otimes H_{\mathbb{Z}}$.*

8.2.3. The conditions for a pre-log Hodge structure to be a log Hodge structure are defined pointwise. Therefore, assume $\underline{S} = Spec(\mathbb{C})$. For each point $y \in S^{log}$ let $sp(y)$ be the set of $\mathbb{C}$-homomorphisms

$$O_{S,y}^{log} \to \mathbb{C}.$$

Note that if the rank of $\overline{\mathcal{M}}_S^{gp}$ is n then

$$O_{S,y}^{log} \simeq \mathbb{C}[T_1, \ldots, T_n].$$

Let $sp(S)$ be the set of all monoid morphisms

$$\mathcal{M}_S \to \mathbb{C}$$

which induce the identity on $\mathbb{C}^*$. Recall ([29], p. 50) that on S^{log} there is a sheaf $\mathcal{L}$ and an exact sequence

$$0 \to \mathbb{Z}(1) \to \mathcal{L} \to \tau_S^{-1}\mathcal{M}_S^{gp} \to 0.$$

Moreover, there is a natural map

$$\mathcal{L} \to O_S^{log}.$$

(O_S^{log} is constructed using $\mathcal{L}$). For a section $a \in \mathcal{M}_S$ let $log(a)$ denote any lifting of a to $\mathcal{L}_y$. Such a lifting is well-defined up to adding multiplies of $2\pi i$. Given a map

$$s : O_{S,y} \to \mathbb{C}$$

defining a point in $sp(y)$ we get a map

$$\mathcal{M}_S \to \mathbb{C}^*$$

by

$$a \mapsto exp(s(log(a))).$$

This is well defined since $exp(2\pi i) = 1$. We therefore have a canonical map

$$sp(y) \to sp(S).$$

Choosing generators for $\mathcal{M}_S$ realizes $sp(S)$ as a closed subset of

$$\mathbb{C}^n$$

for some n. This defines a topology on $sp(S)$ which one can verify is independent of the choice of the generators.

Definition 8.2.4. *Suppose $\underline{S} = Spec(\mathbb{C})$. A pre-log Hodge structure $(H_{\mathbb{Z}}, F)$ on S is a* **log Hodge structure** *if it satisfies the following additional conditions:*

5. *The map*
$$d \otimes 1 : O_S^{log} \otimes H_{\mathbb{Z}} \to \Omega_S^{1,log} \otimes H_{\mathbb{Z}}$$
sends F^p to $\Omega_S^{1,log} \otimes F^{p-1}$.

6. *If $y \in S^{log}$, then there exists a neighborhood U of $\alpha : \mathcal{M}_S \to \mathbb{C}$ (log structure morphism) in $sp(S)$ such that for any $s \in sp(y)$ which maps to U the pair*
$$(H_{\mathbb{Z},y}, F(s))$$
where $F(s)$ is the induced filtration on
$$\mathbb{C} \otimes H_{\mathbb{Z}} \simeq \mathbb{C} \otimes_{s, O_{S,y}^{log}} \otimes (O_{S,y}^{log} \otimes H_{\mathbb{Z}})$$
is a Hodge structure in the usual sense.

For general S, a pre-log Hodge structure is a **log Hodge structure** *if the restriction to each fiber is a log Hodge structure.*

8.2.5. If H is a variation Hodge structures on $\underline{S}$ in the sense of 8.1.8 then the local system $\tau^{-1} H_{\mathbb{Z}}$ together with the filtration defined by the filtration on $H_{\mathbb{C}} \otimes O_S$ is a log Hodge structure. This is immediate from the definition.

Definition 8.2.6. *For any integer n, define the* **Tate log Hodge structure** *$\mathbb{Z}^{log}(n)$ to be the log Hodge structure induced from $\mathbb{Z}(n)$.*

Definition 8.2.7. *Let $(H_{\mathbb{Z}}, F)$ be a log Hodge structure on S of weight n. A* **polarization** *on $(H_{\mathbb{Z}}, F)$ is a map of local systems on S^{log}*
$$\phi : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z}^{log}(-n)$$
such that in condition 8.2.4.6 we can choose y and U such that for any $s \in sp(y)$ with image in U the triple
$$(H_{\mathbb{Z},y}, F(s), \phi_y)$$
is a polarized Hodge structure in the usual sense. A log Hodge structure is **polarizable** *if it admits a polarization.*

The following proposition will be used in what follows:

Proposition 8.2.8. *Let $\underline{S}$ be a smooth analytic space and let $\underline{Y} \subset \underline{S}$ be a divisor with normal crossings. Denote by S the log analytic space consisting of $\underline{S}$ with the log structure coming from $\underline{Y}$.*

1. *Let $(H^1_{\mathbb{Z}}, F^1)$ and $(H^2_{\mathbb{Z}}, F^2)$ be pre-log Hodge structures on S^{log} and let*

$$\psi : (H^1_{\mathbb{Z}}, F^1)|_{S^*} \to (H^2_{\mathbb{Z}}, F^2)|_{S^*}$$

be an isomorphism. Then ψ extends uniquely to all of S^{log}.

2. *Suppose $(H_{\mathbb{Z}}, F)$ is a pre-log Hodge structure on S^{log} and that a map*

$$\phi : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z}^{log}(-n)$$

is given such that $(H_{\mathbb{Z}}, F, \phi)|_{S^*}$ is a polarized log Hodge structure. Then

$$(H_{\mathbb{Z}}, F, \phi)$$

is a polarized log Hodge structure.

Proof. (1) Let $j^{log} : S^* \hookrightarrow S^{log}$ be the canonical inclusion. Then by ([29], 5.12) the map j^{log} is locally aspheric and hence there are canonical isomorphisms

$$H^1_{\mathbb{Z}} \xrightarrow{\simeq} j^{log}_* j^{log,*} H^1_{\mathbb{Z}} \xrightarrow{j^{log}_*(\psi)} j^{log}_* j^{log,*} H^2_{\mathbb{Z}} \xrightarrow{\simeq} H^2_{\mathbb{Z}}.$$

It is clear that this isomorphism is the unique isomorphism of local systems extending ψ. We must check that it is an isomorphism of pre-log Hodge structures. Let E^i be the locally free sheaf

$$\tau_{S*}(H^i_{\mathbb{Z}} \otimes_{\mathbb{Z}} O^{log}_S).$$

Thus from the isomorphism $H^1_{\mathbb{Z}} \simeq H^2_{\mathbb{Z}}$ we get an isomorphism $E^1 \simeq E^2$. Fix some $q \in \mathbb{Z}$, and let L^{qi} denote the locally free sheaf $\tau_{S*} F^{qi}$ on S. Using the lemma 8.2.9 below and 8.2.2.3 we see that to verify that the extension of ψ preserves the filtrations it suffices to verify that ψ preserves the L^{qi}. Condition 8.2.2.4 implies that the quotient E^i/L^{qi} is locally free. Hence the map

$$L^{q1} \to E^2/L^{q2}$$

is zero since this can be verified on the dense open set S^*. Consequently $L^{q1} \subset L^{q2}$ and by symmetry we get the reverse inclusion. This concludes the proof of (1).

(2) It follows from Schmid's nilpotent orbit theorem ([33], see also [22] 3.2.8) that there exists an extension of $(H_{\mathbb{Z}}, F, \phi)|_{S^*}$ to a polarized log Hodge structure on all of S^{log} (note that by [22], 3.2.7 the monodromy of $(H_{\mathbb{Z}}, F, \phi)|_{S^*}$ is unipotent). By (1) we get an isomorphism of this extension with $(H_{\mathbb{Z}}, F)$. Now two maps

$$\phi_1, \phi_2 : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z}^{log}(-n)$$

which agree on a dense open set must be equal since all the sheaves involved are locally constant. Thus the isomorphism given by (1) is compatible with the polarizations. This implies (2). $\square$

Lemma 8.2.9. *With notation as above, let F be a locally free sheaf on $\underline{S}$. Then the natural map*

$$F \to \tau_{S*}\tau_S^* F$$

is an isomorphism.

Proof. The question is local on $\underline{S}$ and hence it suffices to prove the lemma for O_S. Let $s \in \underline{S}$ be a point and let $\tilde{s} \in S^{log}$ be a point in the fiber over s. Then since τ_S is proper we have

$$(\tau_{S*}\tau_S^* O_S)_s \simeq O_{S,\tilde{s}}^{log, \pi_1(S_s^{log}, \tilde{s})}.$$

We can choose isomorphisms

$$O_{S,\tilde{s}}^{log} \simeq O_{S,s}[T_1, \dots, T_n], \quad \pi_1(S_s^{log}, \tilde{s}) \simeq \mathbb{Z}^n$$

such that the i-th generator γ_i of $\mathbb{Z}^n$ acts by

$$T_i \mapsto T_i + 2\pi i, \quad T_j \mapsto T_j \quad j \neq i.$$

If $f \in O_{S,\tilde{s}}^{log}$ we can write

$$f = \sum_i p_i(T_1, \dots, T_{n-1}) T_n^i.$$

By induction we conclude that each $p_i(T_1, \dots, T_{n_1}) \in O_{S,s}$ and hence it suffices to consider the case $n = 1$. In this case we have

$$f(0) = f(2\pi i n)$$

for all $n \in \mathbb{Z}$ and hence $f = f(0)$. $\square$

8.3 The log Hodge structure associated to a log elliptic curve

8.3.1. Let $S/\mathbb{C}$ be a smooth log scheme of finite type and let $f : E \to S$ be an object of $\mathcal{M}_1^{log}$ (a log elliptic curve). Looking at the associated analytic spaces we get a diagram

$$\begin{array}{ccc} E^{log} & \xrightarrow{\tau_E} & E^{an} \\ {\scriptstyle f_{log}}\downarrow & & \downarrow{\scriptstyle f} \\ S^{log} & \xrightarrow{\tau_S} & S^{an}. \end{array}$$

Theorem 8.3.2. *The sheaf $R_1 f_{log,*}\mathbb{Z}$ on S^{log} (the dual of $R^1 f_{log,*}\mathbb{Z}$) has a unique structure of a log Hodge structure of type $\{(-1,0),(0,-1)\}$ inducing the usual Hodge structure over the open set of S where the log structure is trivial.*

The proof occupies the remainder of this section.

8.3.3. We reduce the problem to the universal case. If E'/S' is obtained from a strict étale morphism $S' \to \mathcal{M}_1$ and E/S is obtained by base change from a map $S \to S'$ then we have a diagram

$$\begin{array}{ccc} E^{log} & \xrightarrow{\alpha} & E^{',log} \\ {\scriptstyle f_{log}}\downarrow & & \downarrow{\scriptstyle f'_{log}} \\ S^{log} & \xrightarrow{\beta} & S^{',log} \end{array}$$

which is cartesian by definition. Now the map

$$E^{',log} \to S^{',log}$$

is proper. To see this consider the diagram

$$\begin{array}{ccc} E^{',log} & \xrightarrow{\tau_{E'}} & E^{',an} \\ {\scriptstyle f'_{log}}\downarrow & & \downarrow{\scriptstyle f'} \\ S^{',log} & \xrightarrow{\tau_{S'}} & S^{',an} \end{array}$$

and let $C \subset S^{',log}$ be a compact set. Then

$$f_{log}^{-1}(C) \subset \tau_{E'}^{-1} \circ f^{',-1}(\tau_{S'}(C))$$

and hence is a closed subset of a compact set (since we know that f' and $\tau_{E'}$ are proper). It follows from the proper base change theorem that we have an isomorphism

$$R_1 f_{log,*}\mathbb{Z} \simeq \beta^{-1} R_1 f'_{log,*}\mathbb{Z}.$$

Thus it suffices to prove uniqueness for S and existence for S'.

8.3.4. The uniqueness follows from 8.2.8. For the assumptions on S ensure that the log structure on S is of the form assumed in 8.2.8 ([17], II.2.1.13 and [30]).

8.3.5. From now on we therefore assume that E/S is a universal family. That is, that the map

$$S \to \mathcal{M}_1$$

is strict and étale.

Consider the de-Rham cohomology with its Hodge filtration

$$0 \longrightarrow R^0 f_* \Omega^1_{E/S} \longrightarrow R^1 f_*(\Omega^{\cdot}_{E/S}) \longrightarrow R^1 f_* O_E \longrightarrow 0.$$

Just as in the classical case, the de-Rham cohomology comes with a Gauss-Manin connection

$$\nabla : R^1 f_*(\Omega^{\cdot}_{E/S}) \to \Omega^1_{S/\mathbb{C}} \otimes R^1 f_*(\Omega^{\cdot}_{E/S})$$

which extends the Gauss-Manin connection on S^*. The explicit description of this connection in ([23], appendix) shows that ∇ has unipotent monodromy around the points $S - S^*$. Thus $(R^1 f_*(\Omega^{\cdot}_{E/S}), \nabla)$ is an object of $D_{nilp}(S)$ in the notation of ([21]). Applying ([21], 0.5) and taking duals we conclude that there exists a unique local system of $\mathbb{C}$-vector spaces $H_{\mathbb{C}}$ on S^{log} extending the local system on S^* coming from singular homology. In addition, the Hodge filtration on $R^1 f_*(\Omega^{\cdot}_{E/S})$ induces a filtration F on $H_{\mathbb{C}} \otimes O^{log}_S$. Now since the map

$$j_{log} : S^* \hookrightarrow S^{log}$$

is locally aspheric, we have an isomorphism

$$H_{\mathbb{C}} \simeq j_{log*} j^*_{log} H_{\mathbb{C}}.$$

This implies that $H_{\mathbb{C}}$ admits a unique integral structure $H_{\mathbb{Z}}$ extending that on S^*. For the same reason the canonical polarization on $(H_{\mathbb{Z}}, F)|_{S^*}$ extends uniquely to a pairing on all of $H_{\mathbb{Z}}$. Therefore from 8.2.8 we see that $(H_{\mathbb{Z}}, F, \phi)$ is a log Hodge structure of type $\{(-1,0),(0,-1)\}$.

8.4 Elliptic log Hodge structures

8.4.1. Let $S/\mathbb{C}$ be a smooth log scheme with $\underline{S}/\mathbb{C}$ smooth. In the previous section we defined a functor

$$\begin{array}{c}(\text{log elliptic curves over } S)\\ \downarrow \\ (\text{rank 2 log Hodge structures of type } \{(-1,0),(0,-1)\} \text{ on } S^{log}).\end{array}$$

In this section we describe a subcategory of the category of log Hodge structures on S^{log} which we will show in the next section is the image of the above functor.

8.4.2. Let $(H_{\mathbb{Z}}, F)$ be a rank 2 log Hodge structure of type

$$\{(-1,0),(0,-1)\}$$

on S^{log} and suppose $\tilde{s} \in S^{log}$ is a point mapping to $s \in S^{an}$. The fundamental group $\pi_1(S_s^{log}, \tilde{s})$ of the fiber over s has the following explicit description. On S^{log} there is an exact sequence of sheaves ([29], p.50)

$$0 \longrightarrow \mathbb{Z}(1) \longrightarrow \mathcal{L} \longrightarrow \tau_S^{-1}\mathcal{M}_S^{gp} \longrightarrow 0$$

which induces a map

$$\mathcal{M}_{S,s}^{gp} \to H^1(S_s^{log}, \mathbb{Z}(1)).$$

By ([29], 5.15) this map induces an isomorphism

$$\overline{\mathcal{M}}_{S,s}^{gp} \simeq H^1(S_s^{log}, \mathbb{Z}(1)) \simeq Hom(\pi_1(S_s^{log}, \tilde{s}), \mathbb{Z}(1))$$

and hence an isomorphism

$$\alpha : Hom(\overline{\mathcal{M}}_{S,s}^{gp}, \mathbb{Z}(1)) \simeq \pi_1(S_s^{log}, \tilde{s}).$$

Definition 8.4.3. *Let $\mathbb{Z}(1)_{\leq 0} = (2\pi i)(-\mathbb{N}) \subset 2\pi i\mathbb{Z}$. A rank 2 log Hodge structure $(H_{\mathbb{Z}}, F)$ on S^{log} of type*

$$\{(-1,0),(0,-1)\}$$

is an **elliptic log Hodge structure** *if for each $\tilde{s} \in S^{log}$ the image of the map*

$$Hom(\overline{\mathcal{M}}_{S,s}, \mathbb{Z}(1)_{\leq 0}) \to Aut(H_{\mathbb{Z},\tilde{s}})$$

is contained in $(id + \mathbb{Q}_{\geq 0} \cdot N)$ for some nilpotent endomorphism N of $H_{\mathbb{Z},\tilde{s}}$.

Remark 8.4.4. If $(H_{\mathbb{Z}}, F)$ comes from a log elliptic curve E/S then it is elliptic. For in this case $(H_{\mathbb{Z}}, F)$ is obtained by pullback from the universal curve, and in the universal case $\overline{\mathcal{M}}_{S,s}$ is isomorphic to either $\mathbb{N}$ or 0 and has unipotent monodromy by ([23], appendix).

Definition 8.4.5. *Let N be a positive integer. A* **full level** *N structure on an elliptic log Hodge structure $(H_{\mathbb{Z}}, F)$ is an isomorphism*

$$\alpha : (\mathbb{Z}/N\mathbb{Z})_{S^{log}} \to H_{\mathbb{Z}}/NH_{\mathbb{Z}}.$$

8.4.6. If $S/\mathbb{C}$ is log smooth with $\underline{S}/\mathbb{C}$ smooth, and if E/S is a log elliptic curve with a full level N-structure then the log Hodge structure associated to E has a level-N structure. Over S^* this follows from the classical theory and since $H_{\mathbb{Z}}$ is locally constant and $S^* \to S^{log}$ is locally aspheric there exists a unique extension of this level structure to all of S^{log}.

The following is a special case of the theory of K. Kato and S. Usui ([22]):

Theorem 8.4.7. *The functor*

$$F : (\textit{log smooth analytic spaces } S/\mathbb{C}) \to Set$$

which to any to any $S/\mathbb{C}$ associates the isomorphism classes of elliptic log Hodge structures with level N-structure is representable if $N \geq 3$ by a smooth log analytic space. In addition, the natural map

$$(\mathcal{M}'_N)^{an} \to F$$

is an isomorphism.

Proof. We explain how this follows from their results. Let H_0 be the free $\mathbb{Z}$-module with basis e_0 and e_1. Define an anti-symmetric pairing

$$\phi_0 : H_0 \times H_0 \to (2\pi i)\mathbb{Z}, \quad \phi_0(e_0, e_1) = 2\pi i.$$

Let $\Gamma(N)$ be the subgroup of $SL_2(\mathbb{Z})$ of matrices congruent to the identity modulo N.

If $(H_{\mathbb{Z}}, F)$ is a elliptic log Hodge structure then we claim that there exists a unique polarization ϕ such that the pair $(H_{\mathbb{Z}}, \phi)$ is locally isomorphic to (H_0, ϕ_0). To see this note that on the open dense set S^* where the log structure is trivial $(H_{\mathbb{Z}}, F)$ admits a polarization by 8.1.9. Since the local system $H_{\mathbb{Z}}$ is isomorphic to the pushforward of that on S^* we get a pairing ϕ on $H_{\mathbb{Z}}$. By 8.2.8 this defines a polarization on $(H_{\mathbb{Z}}, F)$. Now there are only two anti-symmetric pairings to $\mathbb{Z}(1)$ which are surjective and they are ϕ and $-\phi$. It follows from the definition of polarization that $-\phi$ is not a polarization and hence the polarization is unique.

Therefore, an elliptic log Hodge structure $(H_{\mathbb{Z}}, F)$ admits a unique polarization ϕ such that $(H_{\mathbb{Z}}, \phi)$ is locally isomorphic to (H_0, ϕ_0). In addition, if $(H_{\mathbb{Z}}, F)$ comes with a level N-structure then we can locally choose an isomorphism such that the induced map

$$(\mathbb{Z}/NH_{\mathbb{Z}})^2 \simeq H_0/NH_0 \to H_{\mathbb{Z}}/NH_{\mathbb{Z}}$$

is that given by the level structure. It follows that the data of the level structure is equivalent to a global section ι of the sheaf

$$\Gamma(N) \setminus Isom((H_{\mathbb{Z}}, \phi), (H_0, \phi_0))$$

over S^{log}.

Now let Σ be the set of monoids

$$\Sigma = \{\mathbb{Q}_{\geq 0}N | N \in M_2(\mathbb{Z}), N \text{ nilpotent}\}.$$

Then we see that an elliptic log Hodge structure is equivalent to the data of a polarized log Hodge structure of type Φ (notation as in ([22])). The result now follows from ([22], 3.3.3) and the construction of the moduli space given there. $\square$

8.5 The main result

Theorem 8.5.1. *Let $S/\mathbb{C}$ be a smooth log scheme with $\underline{S}/\mathbb{C}$ smooth. Then the functor*

$$\begin{array}{c}(\textit{log elliptic curves over } S)\\ \downarrow \\ (\textit{elliptic log Hodge structures on } S^{log})\end{array}$$

is an equivalence of categories.

The proof occupies the remainder of the section.

Proposition 8.5.2. *Let $S/\mathbb{C}$ be a log scheme with $\underline{S}$ reduced, and suppose E_1 and E_2 are log elliptic curves over S. Denote by I the functor on $\underline{S}$-schemes $Isom_{\underline{S}}(\underline{E}_1, \underline{E}_2)$ (ignoring the log structures). Suppose we are given a section $s_U : \underline{U} \to I$ for some dense open set $U \subset \underline{S}$ containing all points where the j-invariants of E_1 and E_2 are equal to 0 or 1728. Then s_U extends to all of $\underline{S}$.*

Proof. By our assumptions we may as well remove the closed subset where the j-invariants are equal to 0 and 1728. In this case, I is a $\{\pm 1\}$ pseudo-torsor finite over S. Thus I consists of two connected components I_1 and I_2. Suppose the image of s_U is contained in I_1. Consider the two maps

$$\underline{S} \rightrightarrows Spec(\mathbb{C}[1/j])$$

defined by E_1 and E_2. Since $\underline{S}$ is reduced and these maps agree on a dense open set they must be equal. It follows that for all $s \in \underline{S}(\mathbb{C})$ the fiber I_s consists of exactly two points and that $I_{1,s}$ consists of a single point. Consider the map of coherent sheaves on $\underline{S}$

$$\eta : O_S \to \pi_* O_{I_1}.$$

Since η is an isomorphism on a dense open set the map is injective. Let Q be the cokernel. Then Q is a coherent sheaf on S such that for all closed points $s \in S$ we have $Q \otimes k(s) = 0$ since $I_1 \times k(s) \simeq Spec(k(s))$. It follows that $Q_s = 0$ by Nakayama's lemma and hence $Q = 0$. Thus the map η is an isomorphism and consequently $I_1 \to S$ is an isomorphism. □

8.5.3. It follows that the functor in 8.5.1 is fully faithful. Let E_1 and E_2 be two log elliptic curves over S and suppose we are given an isomorphism i between the corresponding log Hodge structures. By 5.2.6 we get log structures $\mathcal{M}_1, \mathcal{M}_2 \subset \mathcal{M}_S$ such that E_l is obtained from a special object over $(\underline{S}, \mathcal{M}_l)$. We claim that $\mathcal{M}_1 = \mathcal{M}_2$. In fact we can explicitly determine the image of

$$\overline{\mathcal{M}}_l \subset \overline{\mathcal{M}}_S.$$

Choose a point $\tilde{s} \in S^{log}$ and let $s \in \underline{S}$ be the image. For each $l = 1, 2$ the action of $\pi_1(S_s^{log}, \tilde{s})$ on $H^l_{\tilde{s}}$ (the log Hodge structure associated to E_i) factors through the map

$$\pi_1(S_s^{log}, \tilde{s}) \simeq Hom(\overline{\mathcal{M}}_{S,s}, \mathbb{Z}(1)_{\leq 0}) \to Hom(\overline{\mathcal{M}}_{l,s}, \mathbb{Z}(1)_{\leq 0}).$$

Thus i induces a commutative diagram

$$\begin{array}{ccc}
\pi_1(S_s^{log}, \tilde{s}) & \longrightarrow & Hom(\overline{\mathcal{M}}_{1,s}, \mathbb{Z}(1)_{\leq 0}) \\
\downarrow & & \downarrow \\
Hom(\overline{\mathcal{M}}_{2,s}, \mathbb{Z}(1)_{\leq 0}) & \longrightarrow & Aut(H^1_{\tilde{s}}).
\end{array}$$

For a suitable choice of basis $H^1_{\tilde{s}} \simeq \mathbb{Z}^2$ the image of $\pi_1(S_s^{log}, \tilde{s})$ is contained in the monoid of matrices

$$\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$$

for $p \in \mathbb{N}$. In terms of this basis, the canonical generator of $Hom(\overline{\mathcal{M}}_{l,s}, \mathbb{Z}(1)_{\leq 0})$ must act via the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Note that this is true for some basis by ([23], appendix) and one checks by calculation that then it is true for any basis with the above property. We conclude that the image

$(n_1, \ldots, n_r) \in \mathbb{N}^r$ of the canonical generator of $\overline{\mathcal{M}}_l$ in $\overline{\mathcal{M}}_{S,s}$ is determined by the condition that the map

$$\mathbb{N}^r \to \mathbb{Z}(1)_{\leq 0}, \quad e_i \mapsto -2\pi i, \quad e_j \mapsto 0, \quad j \neq i$$

acts via the matrix

$$\begin{pmatrix} 1 & n_i \\ 0 & 1 \end{pmatrix}$$

From this we see that the images of $\overline{\mathcal{M}}_{l,s}$ in $\overline{\mathcal{M}}_{S,s}$ must be equal and hence $\mathcal{M}_1 = \mathcal{M}_2$. Let $S^\sharp$ be the corresponding log scheme and let $E_l^\sharp$ be the special log elliptic curves inducing the E_l by base change. Now to give an isomorphism $\underline{E}_1^\sharp \simeq \underline{E}_2^\sharp$ over $\underline{S}$ is equivalent to giving an isomorphism between the log schemes $E_1^\sharp \simeq E_2^\sharp$ over $S^\sharp$. For the description in 1.3.13 of the functor

$$Isom_{\underline{S}}(\underline{E}_1^\sharp, \underline{E}_2^\sharp)$$

in terms of the special objects of the log stack $\mathcal{M}_1$ shows that for any isomorphism w : $\underline{E}_1^\sharp \simeq \underline{E}_2^\sharp$ there exists an automorphism $\alpha : S^\sharp \to S^\sharp$ which is the identity on $\underline{S}$ and a commutative diagram

$$\begin{array}{ccc} E_1^\sharp & \xrightarrow{\tilde{w}} & E_2^\sharp \\ \downarrow & & \downarrow \\ S^\sharp & \xrightarrow{\alpha} & S^\sharp \end{array}$$

inducing w. But by our assumptions on S, the morphism α must be the identity and hence we get an isomorphism of log schemes over $S^\sharp$.

It follows that if we are given an isomorphism between $\underline{E}_1$ and $\underline{E}_2$ then we get an isomorphism between $E_1^\sharp$ and $E_2^\sharp$ and hence an isomorphism between the E_i. Now we get an isomorphism $E_1|_{S^{\sharp *}} \simeq E_2|_{S^{\sharp *}}$ from i by 8.1.11. By 8.5.2 this isomorphism extends to an isomorphism $\underline{E}_1^\sharp \simeq \underline{E}_2^\sharp$ over all of $\underline{S}$.

Let j be the corresponding isomorphism between the log Hodge structures. Then i and j agree on a dense open subset of S^{log} and since they are maps between local systems they must be equal. Conversely, if i and j are two isomorphisms between E_1 and E_2 which induce the same map on the associated log Hodge structures, then by 8.1.10 we must have $i = j$ over S^*. But E_1 and E_2 are reduced and separated and hence if two maps agree on a dense open set they must be equal. This proves that the functor is fully faithful.

8.5.4. To show that the functor is essentially surjective, let $(H_{\mathbb{Z}}, F)$ be an elliptic log Hodge structure on S^{log}. By the full faithfulness the question is Zariski local on S. Thus we may assume that we have a smooth morphism ([17], II.2.1.13)

$$S \to Spec(\mathbb{C}[\mathbb{N}^r])$$

giving a chart for $\mathcal{M}_S$. Let $N \geq 3$ be an integer and consider the map

$$\times N : Spec(\mathbb{C}[\mathbb{N}^r]) \to Spec(\mathbb{C}[\mathbb{N}^r])$$

given by

$$\mathbb{N}^r \to \mathbb{N}^r, \quad \underline{n} \mapsto N\underline{n}.$$

We let S' be the fiber product of

$$\begin{array}{ccc} & & Spec(\mathbb{C}[\mathbb{N}^r]) \\ & & \Big\downarrow{\scriptstyle \times N} \\ S & \longrightarrow & Spec(\mathbb{C}[\mathbb{N}^r]). \end{array}$$

We claim that on S' the local system $H_{\mathbb{Z}}/NH_{\mathbb{Z}}$ on $S^{',log}$ descends to a local system on $S^{',an}$. To see this it suffices to show that the map

$$\tau^{-1}\tau_* H_{\mathbb{Z}}/NH_{\mathbb{Z}} \to H_{\mathbb{Z}}/NH_{\mathbb{Z}}$$

is an isomorphism. Let $s \in S^{',log}$ be a point. Then

$$(\tau^{-1}\tau_* H_{\mathbb{Z}}/NH_{\mathbb{Z}})_s = (H_{\mathbb{Z}}/NH_{\mathbb{Z}})^{\pi_1(S^{',log}_{\tau(s)}, s)}.$$

Now the action of $\pi_1(S_s^{',log}, s) \simeq Hom(\overline{\mathcal{M}}^{gp}_{S',\tau(s)}, \mathbb{Z}(1))$ on $(H_{\mathbb{Z}}/NH_{\mathbb{Z}})_s$ factors through the map

$$Hom(\overline{\mathcal{M}}^{gp}_{S',\tau(s)}, \mathbb{Z}(1)) \to Hom(\overline{\mathcal{M}}^{gp}_{S,\tau(\bar{s})}, \mathbb{Z}(1))$$

where $\bar{s}$ is the image of s in S^{log}. Since the map

$$\overline{\mathcal{M}}^{gp}_{S,\tau(\bar{s})} \to \overline{\mathcal{M}}^{gp}_{S',\tau(s)}$$

is multiplication by N the action is trivial. It follows that the local system $H_{\mathbb{Z}}/NH_{\mathbb{Z}}$ descends to S'. Therefore by replacing S' by a finite étale cover we can assume that $H_{\mathbb{Z}}/NH_{\mathbb{Z}}$ is trivial. In other words, we have found a log smooth cover $S' \to S$ whose underlying morphism is faithfully flat such that $(H_{\mathbb{Z}}, F)$ admits a level-N structure. It follows from 8.4.7 that there exists a log elliptic curve E'/S' whose associated log Hodge structure is isomorphic to $(H_{\mathbb{Z}}, F)$. Let $U' \subset S'$ be the open subset where $\underline{E}'/\underline{S}'$ is smooth. The image of U' in S is an open set U since S'/S is flat. Now for all $s \in U^{log}$ the natural map

$$(\tau^{-1}\tau_* H_{\mathbb{Z}})_s \to H_{\mathbb{Z},s}$$

is an isomorphism. For as before we have

$$(\tau^{-1}\tau_* H_{\mathbb{Z}})_s = (H_{\mathbb{Z},s})^{\pi_1(S^{log}_{\tau(s)},s)}.$$

If $s' \in S'^{,log}$ is a point lying over s then the quotient

$$\pi_1(S^{log}_{\tau(s)}, s)/\pi_1(S^{log}_{\tau(s')}, s')$$

is killed by multiplication by N. Therefore, if $\gamma \in \pi_1(S^{log}_{\tau(s)}, s)$ then $N\gamma$ acts as the identity. But γ acts via a unipotent endomorphism so this implies that γ acts as the identity.

It follows that $(H_{\mathbb{Z}}, F)|_U$ is a Hodge structure in the usual sense. Therefore by 8.1.10 there exists an elliptic curve E_U/U whose associated log Hodge structure is isomorphic to $(H_{\mathbb{Z}}, F)|_U$ such that $E_U \times_S S' \simeq E'|_{U'}$ canonically. Now to prove the theorem we may as well remove the closed set in U where the j-invariant of E_U is 0 or 1728.

Lemma 8.5.5. *$\underline{S}' \times_{\underline{S}} \underline{S}'$ is reduced.*

Proof. Let $T = Spec(\mathbb{C}[\mathbb{N}^r])$ and let $T' \to T$ be the map induced from $\times N : \mathbb{N}^r \to \mathbb{N}^r$. Then $\underline{S}' \times_{\underline{S}} \underline{S}'$ is smooth over $\underline{T}' \times_{\underline{T}} \underline{T}'$. Hence it suffices to consider $\underline{T}' \times_{\underline{T}} \underline{T}'$.

To prove the lemma it therefore suffices to prove the following statement and apply induction. If R is a reduced ring then

$$R[X, Y]/X^N - Y^N$$

is reduced. Since R is reduced it suffices to show that the ring is reduced after reduction mod $\mathfrak{p}$ for each prime $\mathfrak{p} \subset R$. Thus it suffices to consider the case when R is an integral domain and by embedding into the field of fractions we may assume that R is a field extension of $\mathbb{C}$. Let K be an algebraic closure of the field of fractions of $R[X]$. Then $R[X, Y]/(X^N - Y^N)$ embeds into

$$K[Y]/Y^N - a$$

which is isomorphic to the product of N copies of K. It follows that $R[X, Y]/(X^N - Y^N)$ is reduced. $\square$

8.5.6. By the lemma $\underline{S}' \times_{\underline{S}} \underline{S}'$ is reduced and hence by the proposition there exists a unique isomorphism $p_1^* \underline{E}' \simeq p_2^* \underline{E}'$ over $\underline{S}' \times_{\underline{S}} \underline{S}'$ which agrees with the isomorphism over $\underline{U}' \times_{\underline{S}} \underline{U}'$ coming from the isomorphism

$$E_U \times_S S' \simeq E'|_{U'}.$$

Let us verify that this defines descent datum for $\underline{E}'$. This amounts to saying that a map

$$\underline{S}' \times_{\underline{S}} \underline{S}' \times_{\underline{S}} \underline{S}' \to Aut(\underline{E}')$$

which is the identity on a dense open set is equal to the identity. But $Aut(\underline{E}') \simeq \{\pm 1\}$ and hence this is clear. Let $\underline{E}/\underline{S}$ be the pointed curve defined by the descent datum. $\underline{E}$ defines a map

$$e : \underline{S} \to \underline{\mathcal{M}}_1.$$

Now the inverse image of the log structure on $\mathcal{M}_1$ is contained in the subsheaf of functions in O_S which are invertible on the open set where $\underline{E}/\underline{S}$ is smooth. Now this open set contains S^* since this can be verified by base change to S'. By ([17], 3.4.1) the log structure on S is isomorphic to

$$j_*^{log} O_{S^*}^*$$

It follows that

$$e^* \mathcal{M}_{\mathcal{M}_1} \subset \mathcal{M}_S \subset O_S$$

and hence we get a morphism of log stacks

$$e : S \to \mathcal{M}_1$$

giving E/S the structure of a log elliptic curve.

E/S comes together with an isomorphism between the log Hodge structure associated to E and the log Hodge structure we started with over the open set $U \subset S$. By 8.2.8 this isomorphism extends to all of S^{log}. This proves the essential surjectivity of the functor in 8.5.1 and hence concludes the proof of the theorem.

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