INDEPENDENCE OF ℓ AND SURFACES

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1. The main theorem

1.1. Let $\mathbb{F}_q$ be a finite field of characteristic p, let k be an algebraic closure of $\mathbb{F}_q$, and let $X/\mathbb{F}_q$ be a separated, normal, 2-dimensional $\mathbb{F}_q$-scheme of finite type. For a prime $\ell \neq p$ and $i \in \mathbb{Z}$ define

$$P^i_\ell := \det(1 - TF|H^i(X_k, \mathbb{Q}_\ell)) \in \mathbb{Q}_\ell[T]$$

and

$$P^i_{c,\ell} := \det(1 - TF|H^i_c(X_k, \mathbb{Q}_\ell)) \in \mathbb{Q}_\ell[T],$$

where $H^i(X_k, \mathbb{Q}_\ell)$ (resp. $H^i_c(X_k, \mathbb{Q}_\ell)$) denotes the étale cohomology of X_k (resp. compactly supported étale cohomology of X_k), and F denotes the Frobenius endomorphism.

Our aim in this note is to prove the following:

Theorem 1.2. *The polynomials P^i_ℓ and $P^i_{c,\ell}$ are in $\mathbb{Q}[T]$, and for any two primes ℓ, ℓ' not equal to p we have*

$$P^i_\ell = P^i_{\ell'}, \quad P^i_{c,\ell} = P^i_{c,\ell'}.$$

1.3. The same argument will show that if X/k is a separated normal surface over an algebraically closed field k, then the dimensions of the cohomology groups

$$H^i(X, \mathbb{Q}_\ell), \quad H^i_c(X, \mathbb{Q}_\ell)$$

are independent of ℓ. We apply this to study the Brauer group in this setting to obtain the following theorem:

Theorem 1.4. *Let k be a separably closed field of characteristic exponent p, and let X/k be a smooth 2-dimensional k-scheme. Let $\widetilde{\mathrm{Br}}(X)$ denote the quotient of the Brauer group of X by its p-torsion, and let*

$$\widetilde{\mathrm{Br}}(X)_{\mathrm{div}} \subset \widetilde{\mathrm{Br}}(X)$$

be the subgroup of divisible elements. Then the quotient

$$\widetilde{\mathrm{Br}}(X)/\widetilde{\mathrm{Br}}(X)_{\mathrm{div}}$$

is a finite group, and there exists an integer r such that

$$\widetilde{\mathrm{Br}}(X)_{\mathrm{div}} \simeq F_p^{\oplus r},$$

where F_p denotes the quotient of $\mathbb{Q}/\mathbb{Z}$ by its p-torsion subgroup.

Remark 1.5. The results of this note seem well-known to experts (in particular the independence of ℓ for the Betti numbers of surfaces is tacitly indicated in [Ill, 1.4]), but no reference appears available.

1.6. **Notation.** For an abelian group A and an integer N we write $A[N]$ for

$$\mathrm{Ker}(\cdot N : A \to A).$$

For a prime ℓ we write $T_\ell A$ for the $\mathbb{Z}_\ell$-module

$$\varprojlim_n A[\ell^n]),$$

where the projective limit is taken with respect to the maps

$$\cdot\ell : A[\ell^{n+1}] \to A[\ell^n].$$

We write $V_\ell(A)$ for the $\mathbb{Q}_\ell$-vector space

$$V_\ell(A) := T_\ell(A) \otimes \mathbb{Q}.$$

Observe that $V_\ell(-)$ is a left exact functor in the sense that if

$$0 \to A \to B \to C$$

is a short exact sequence then the resulting sequence

$$0 \to V_\ell A \to V_\ell B \to V_\ell C$$

is also exact. Note also that if A is a finitely generated abelian group, then $V_\ell A = 0$.

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2. Kummer theory

2.1. Let k be an algebraically closed field, and let $\overline{X}/k$ be a proper smooth k-scheme of dimension 2. Let $E, D \subset \overline{X}$ be divisors with simple normal crossings, and assume

$$E \cap D = \emptyset.$$

Let $X \subset \overline{X}$ be the complement of E.

2.2. Recall that the Picard group $\mathrm{Pic}(\overline{X})$ of $\overline{X}$ sits in an extension

$$0 \to \mathrm{Pic}^0(\overline{X}) \to \mathrm{Pic}(\overline{X}) \to NS(\overline{X}) \to 0,$$

where $\mathrm{Pic}^0(\overline{X})$ is the points of an abelian variety over k (and in particular is a divisible group), and $NS(\overline{X})$ (the *Neron-Severi group of* $\overline{X}$) is a finitely generated abelian group.

Set

$$\mathrm{Pic}(\overline{X}, D) = \mathrm{Ker}(\mathrm{restriction} : \mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(D)),$$
$$\mathrm{Pic}^0(\overline{X}, D) := \mathrm{Pic}(\overline{X}, D) \cap \mathrm{Pic}^0(\overline{X}),$$

and

$$NS(\overline{X}, D) := \mathrm{Pic}(\overline{X}, D)/\mathrm{Pic}^0(\overline{X}, D).$$

We then have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Pic}^0(\overline{X}, D) & \longrightarrow & \mathrm{Pic}(\overline{X}, D) & \longrightarrow & NS(\overline{X}, D) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Pic}^0(\overline{X}) & \longrightarrow & \mathrm{Pic}(\overline{X}) & \longrightarrow & NS(\overline{X}) & \longrightarrow & 0,
\end{array}$$

where the vertical maps are inclusions. In particular the group $NS(\overline{X}, D)$ is finitely generated.

Remark 2.3. This definition of $\mathrm{Pic}(\overline{X}, D)$ differs from the one in [MVW], though the two are closely related.

2.4. Observe also that since

$$\mathrm{Pic}^0(\overline{X}, D) = \mathrm{Ker}(\mathrm{Pic}^0(\overline{X}) \to \mathrm{Pic}^0(D)),$$

the group $\mathrm{Pic}^0(\overline{X}, D)$ is the k-points of a proper finite type k-group scheme. In particular, this group has a canonical subgroup of finite index

$$\mathrm{Pic}^{00}(\overline{X}, D) \subset \mathrm{Pic}^0(\overline{X}, D)$$

which is the k-points of an abelian variety over k.

2.5. The group $\mathrm{Pic}(\overline{X}, D)$ is related to étale cohomology as follows. Fix a prime ℓ invertible in k, and consider the Kummer sequences (on $\overline{X}$ and D respectively)

$$0 \longrightarrow \mu_{\ell^n, \overline{X}} \longrightarrow \mathbb{G}_{m,\overline{X}} \xrightarrow{\cdot \ell^n} \mathbb{G}_{m,\overline{X}} \longrightarrow 0,$$

and

$$0 \longrightarrow \mu_{\ell^n, D} \longrightarrow \mathbb{G}_{m,D} \xrightarrow{\cdot \ell^n} \mathbb{G}_{m,D} \longrightarrow 0.$$

Taking cohomology we get isomorphisms

$$H^1(\overline{X}, \mu_{\ell^n}) \simeq \mathrm{Pic}(\overline{X})[\ell^n],$$

and

$$H^1(D, \mu_{\ell^n}) \simeq \mathrm{Pic}(D)[\ell^n].$$

Passing to the inverse limit over n and tensoring with $\mathbb{Q}_\ell$ we get isomorphisms

$$H^1(\overline{X}, \mathbb{Q}_\ell(1)) \simeq V_\ell(\mathrm{Pic}(\overline{X})), \quad H^1(D, \mathbb{Q}_\ell(1)) \simeq V_\ell(\mathrm{Pic}(D)).$$

Using the short exact sequence

$$0 \to \mathrm{Pic}(\overline{X}, D) \to \mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(D)$$

and the left exactness of the functor $V_\ell(-)$ (see 1.6), we obtain a canonical isomorphism between $V_\ell(\mathrm{Pic}(\overline{X}, D))$ and the kernel of the restriction map

$$H^1(\overline{X}, \mathbb{Q}_\ell(1)) \to H^1(D, \mathbb{Q}_\ell(1)).$$

Now since $\mathrm{Pic}(\overline{X}, D)$ is an extension of a finitely generated abelian group by $\mathrm{Pic}^{00}(\overline{X}, D)$ we have

$$V_\ell(\mathrm{Pic}(\overline{X}, D)) \simeq V_\ell(\mathrm{Pic}^{00}(\overline{X}, D)).$$

Summarizing:

Corollary 2.6. *There is a natural isomorphism*

$$V_\ell(\mathrm{Pic}^{00}(\overline{X}, D)) \simeq \mathrm{Ker}(H^1(\overline{X}, \mathbb{Q}_\ell(1)) \to H^1(D, \mathbb{Q}_\ell(1))).$$

2.7. We can also consider similar groups for the open scheme X. Define

$$\mathrm{Pic}(X, D) := \mathrm{Ker}(\mathrm{Pic}(X) \to \mathrm{Pic}(D)),$$

let $\mathrm{Pic}^0(X, D) \subset \mathrm{Pic}(X, D)$ be the image of $\mathrm{Pic}^0(\overline{X}, D)$ under the restriction map

$$\mathrm{Pic}(\overline{X}, D) \to \mathrm{Pic}(X, D),$$

and set

$$NS(X, D) := \mathrm{Pic}(X, D)/\mathrm{Pic}^0(X, D).$$

Notice that since $D \subset X$, the group $\mathrm{Pic}(\overline{X}, D)$ is equal to the preimage of $\mathrm{Pic}(X, D)$ under the surjective (since $\overline{X}$ is smooth) map

$$\mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(X).$$

We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}^0(\overline{X}, D) & \longrightarrow & \mathrm{Pic}(\overline{X}, D) & \longrightarrow & NS(\overline{X}, D) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Pic}^0(X, D) & \longrightarrow & \mathrm{Pic}(X, D) & \longrightarrow & NS(X, D) & \longrightarrow & 0, \end{array}$$

where the vertical maps are surjective. In particular $NS(X, D)$ is finitely generated, and $\mathrm{Pic}^0(X, D)$ is an extension of a finite group by a divisible group.

2.8. Let

$$j : \overline{X} - D \hookrightarrow \overline{X}, \quad i : D \hookrightarrow \overline{X}$$

be the inclusions. Then for every $n > 0$ and prime ℓ invertible in k we have a commutative diagram of sheaves on $\overline{X}_{\mathrm{et}}$

(2.8.1)
$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & j_!\mu_{\ell^n} & \longrightarrow & \mu_{\ell^n} & \longrightarrow & i_*\mu_{\ell^n} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{H} & \longrightarrow & \mathbb{G}_m & \longrightarrow & i_*\mathbb{G}_{m,D} & \longrightarrow & 0 \\ & & \downarrow{\cdot \ell^n} & & \downarrow{\cdot \ell^n} & & \downarrow{\cdot \ell^n} & & \\ 0 & \longrightarrow & \mathscr{H} & \longrightarrow & \mathbb{G}_m & \longrightarrow & i_*\mathbb{G}_{m,D} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0, & & \end{array}$$

where the rows and columns are exact sequences and $\mathscr{H}$ is defined to be the kernel of the map

$$\mathbb{G}_m \to i_*\mathbb{G}_{m,D}.$$

Taking cohomology of this diagram restricted to X, we obtain a commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & H^1(X, j_!\mu_{\ell^n}) & \longrightarrow & H^1(X, \mu_{\ell^n}) & \longrightarrow & H^1(D, \mu_{\ell^n}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
(k^*)^{\pi_0(D)} & \longrightarrow & H^1(X, \mathscr{H}) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & \mathrm{Pic}(D) \\
\downarrow{\scriptstyle \cdot\ell^n} & & \downarrow{\scriptstyle \cdot\ell^n} & & \downarrow{\scriptstyle \cdot\ell^n} & & \downarrow{\scriptstyle \cdot\ell^n} \\
(k^*)^{\pi_0(D)} & \longrightarrow & H^1(X, \mathscr{H}) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & \mathrm{Pic}(D) \\
 & & \downarrow & & & & \\
 & & H^2(X, j_!\mu_{\ell^n}). & & & &
\end{array}$$

Since

$$\cdot\ell^n : (k^*)^{\pi_0(D)} \to (k^*)^{\pi_0(D)}$$

is surjective, we get an isomorphism

$$\mathrm{Coker}(\cdot\ell^n : H^1(X, \mathscr{H}) \to H^1(X, \mathscr{H})) \simeq \mathrm{Pic}(X, D)/\ell^n \mathrm{Pic}(X, D) \tag{2.8.2}$$

and an inclusion

$$\mathrm{Pic}(X, D)/\ell^n \mathrm{Pic}(X, D) \hookrightarrow H^2(X, j_!\mu_{\ell^n}).$$

Passing to the projective limit in n and tensoring with $\mathbb{Q}_\ell$ we get an inclusion

$$(\varprojlim_n \mathrm{Pic}(X, D)/\ell^n \mathrm{Pic}(X, D)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \hookrightarrow H^2(X, j_!\mathbb{Q}_\ell(1)). \tag{2.8.3}$$

Proposition 2.9. *The map* (2.8.3) *factors through an inclusion*

$$NS(X, D) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \hookrightarrow H^2(X, j_!\mathbb{Q}_\ell(1)). \tag{2.9.1}$$

Proof. Indeed the kernel $\mathrm{Pic}^0(X, D)$ of the surjection

$$\mathrm{Pic}(X, D) \to NS(X, D)$$

is an extension of a finite group by a divisible group, and therefore there exists an integer N such that for every n the kernel of the surjection

$$\mathrm{Pic}(X, D)/\ell^n \mathrm{Pic}(X, D) \to NS(X, D)/\ell^n NS(X, D)$$

is annihilated by N. In particular we get an isomorphism

$$(\varprojlim_n (\mathrm{Pic}(X, D)/\ell^n \mathrm{Pic}(X, D))) \otimes \mathbb{Q} \simeq (\varprojlim_n NS(X, D)/\ell^n NS(X, D)) \otimes \mathbb{Q}.$$

Since $NS(X, D)$ is a finitely generated abelian group, which implies that

$$(\varprojlim_n NS(X, D)/\ell^n NS(X, D)) \otimes \mathbb{Q} \simeq NS(X, D) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell,$$

we obtain the result. $\square$

3. The main theorem for H^1_c of a smooth surface

3.1. Let $\mathbb{F}_q$ be a finite field of characteristic p, and let $X/\mathbb{F}_q$ be a finite type smooth separated scheme of dimension 2. Let $\mathbb{F}_q \hookrightarrow k$ be an algebraic closure. Our aim in this section is to prove the following special case of 1.2.

Proposition 3.2. *For any prime $\ell \neq p$ the characteristic polynomial of Frobenius*

$$\det(1 - TF|H^1_c(X_k, \mathbb{Q}_\ell))$$

is in $\mathbb{Q}[T]$ and is independent of ℓ.

Proof. Fix a compactification $j : X \hookrightarrow \overline{X}$ with $\overline{X}$ smooth and proper over $\mathbb{F}_q$ and such that the complement $i : D \subset \overline{X}$ of X is a divisor with simple normal crossings. Taking cohomology of the exact sequence

$$0 \to j_!\mathbb{Q}_\ell(1) \to \mathbb{Q}_\ell(1) \to i_*\mathbb{Q}_\ell(1) \to 0$$

we get an exact sequence

$$0 \to C_\ell \to H^1_c(X_k, \mathbb{Q}_\ell(1)) \to \mathrm{Ker}(H^1(\overline{X}_k, \mathbb{Q}_\ell(1)) \to H^1(D_k, \mathbb{Q}_\ell(1))) \to 0,$$

where C_ℓ denotes

$$\mathrm{Coker}(\mathbb{Q}_\ell(1)^{\pi_0(\overline{X}_k)} \to \mathbb{Q}_\ell(1)^{\pi_0(D_k)}).$$

To prove the proposition it therefore suffices to show that the characteristic polynomial of Frobenius acting on

$$\mathrm{Ker}(H^1(\overline{X}_k, \mathbb{Q}_\ell(1)) \to H^1(D_k, \mathbb{Q}_\ell(1)))$$

is in $\mathbb{Q}[T]$ and is independent of ℓ. This is clear for by 2.6 this kernel is isomorphic to $V_\ell\mathrm{Pic}^{00}(\overline{X}_k, D_k)$ which is the Tate module of the abelian variety given by the maximal reduced closed subscheme of $\mathrm{Pic}^{00}(\overline{X}, D)$. $\square$

4. The main theorem for $H^1_{D,E}(X)$

4.1. Let $\mathbb{F}_q \hookrightarrow k$ be as in the previous section, and let $\overline{X}/\mathbb{F}_q$ be a smooth proper scheme of dimension 2. Let $D, E \subset \overline{X}$ be two divisors with simple normal crossings, and such that $D \cap E = \emptyset$. Set

$$X := \overline{X} - E,$$

and let

$$j_D : X - D \hookrightarrow X$$

be the inclusion. For integers i and r, and a prime ℓ invertible in k, define

$$H^i_{D,E}(\overline{X}_k, \mathbb{Q}_\ell(r)) := H^i(X_k, j_{D!}\mathbb{Q}_\ell(r)).$$

Proposition 4.2. *For any prime $\ell \neq p$, the characteristic polynomial of Frobenius*

$$\det(1 - TF|H^1_{D,E}(\overline{X}_k, \mathbb{Q}_\ell)))$$

is in $\mathbb{Q}[T]$ and is independent of ℓ.

Proof. Let

$$E = E^{(1)} \cup \cdots \cup E^{(r)}$$

be the irreducible components of E, and define inductively

$$X^{(0)} := \overline{X}, \quad X^{(i)} = X^{(i-1)} - (E^{(i)} \cap X^{(i-1)}).$$

Let

$$j_D^{(i)} : X^{(i)} - D \hookrightarrow X^{(i)}$$

be the inclusions. We show by induction on i that the characteristic polynomial

$$\det(1 - TF | H^1(X_k^{(i)}, j_{D!}^{(i)}\mathbb{Q}_\ell))$$

is in $\mathbb{Q}[T]$ and independent of ℓ.

For $i = 0$ we have

$$H^1(X_k^{(0)}, j_{D!}^{(0)}\mathbb{Q}_\ell) = H_c^1(\overline{X}_k - D_k, \mathbb{Q}_\ell)$$

so the result follows from 3.2.

For the inductive step consider the diagram

$$X^{(i+1)} \xhookrightarrow{\gamma} X^{(i)} \xhookleftarrow{\delta} E^{(i+1)} \cap X^{(i)} .$$

Since $E^{(i+1)} \cap X^{(i)}$ is a smooth divisor in $X^{(i)}$ we have the purity sequence

$$\mathbb{Q}_{\ell, X^{(i)}} \to R\gamma_* \mathbb{Q}_{\ell, X^{(i+1)}} \to \mathbb{Q}_{\ell, E^{(i+1)} \cap X^{(i)}}(-1)[-1].$$

Restricting this sequence to $X^{(i)} - D$ and then applying $j_{D!}^{(i)}$ we get a distinguished triangle (recall that $E \cap D = \emptyset$)

$$j_{D!}^{(i)}\mathbb{Q}_{\ell, X^{(i)}-D} \to R\gamma_* j_{D!}^{(i+1)}\mathbb{Q}_{\ell, X^{(i+1)}-D} \to \mathbb{Q}_{\ell, E^{(i+1)} \cap X^{(i)}}(-1)[-1].$$

Taking cohomology we then get a long exact sequence

$$0 \longrightarrow H^1(X_k^{(i)}, j_{D!}^{(i)}\mathbb{Q}_\ell) \longrightarrow H^1(X_k^{(i+1)}, j_{D!}^{(i+1)}\mathbb{Q}_\ell) \longrightarrow \mathbb{Q}_\ell(-1)^{\pi_0(E_k^{(i+1)} \cap X_k^{(i)})} \xrightarrow{\partial} H^2(X_k^{(i)}, j_{D!}^{(i)}\mathbb{Q}_\ell).$$

By induction it therefore suffices to show that the characteristic polynomial of Frobenius acting on $\mathrm{Ker}(\partial)$ is in $\mathbb{Q}[T]$ and is independent of ℓ.

Let

$$\tau : \mathbb{Q}^{\pi_0(E_k^{(i+1)} \cap X_k^{(i)})} \to NS(X_k^{(i)}, D) \otimes \mathbb{Q}$$

be the natural map defined by the connected components of $E_k^{(i+1)} \cap X_k^{(i)}$ (each of which is smooth by construction). Then the map ∂ is obtained by tensoring the map τ with $\mathbb{Q}_\ell$, twisting by $\mathbb{Q}_\ell(-1)$, and then applying the injective map 2.9.1 (we leave to the reader the verification that the two definitions of the cycle class used here agree). We conclude that the characteristic polynomial of Frobenius acting on $\mathrm{Ker}(\partial)$ is equal to the polynomial

$$\det(1 - TF | \mathrm{Ker}(\tau) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell(-1)),$$

which is evidently in $\mathbb{Q}[T]$ and independent of ℓ. $\square$

5. Duality

5.1. Let $\overline{X}/\mathbb{F}_q$ be a smooth proper 2-dimensional scheme, and let $D, E \subset \overline{X}$ be two divisors with simple normal crossings such that $D \cap E = \emptyset$.

We then obtain a commutative diagram

$$\begin{array}{ccc} \overline{X} - (D \cup E) & \xhookrightarrow{j'_D} & \overline{X} - E \\ \Big\downarrow{\scriptstyle j'_E} & & \Big\downarrow{\scriptstyle j_E} \\ \overline{X} - D & \xhookrightarrow{j_D} & \overline{X}. \end{array}$$

Let $\mathscr{D}_{\overline{X}}(-)$ be the Verdier duality functor. We then have

(5.1.1) $$\mathscr{D}_{\overline{X}} Rj_{D*} j'_{E!} \mathbb{Q}_\ell \simeq j_{D!} R j'_{E*} \mathscr{D}_{\overline{X}-(D\cup E)} \mathbb{Q}_\ell.$$

Now observe that since $\overline{X}$ is smooth of dimension 2 we have

$$\mathscr{D}_{\overline{X}-(D\cup E)} \mathbb{Q}_\ell = \mathbb{Q}_\ell(2)[4].$$

so the right side of (5.1.1) can be written as

$$j_{D!} R j'_{E*} \mathbb{Q}_\ell(2)[4].$$

On the other hand, since D and E are disjoint there is a natural isomorphism

$$j_{D!} R j'_{E*} \mathbb{Q}_\ell \simeq R j_{E*} j'_{D!} \mathbb{Q}_\ell.$$

Thus we obtain an isomorphism

$$\mathscr{D}_{\overline{X}} R j_{D*} j'_{E!} \mathbb{Q}_\ell \simeq R j_{E*} j'_{D!} \mathbb{Q}_\ell(2)[4].$$

Taking cohomology we obtain for every integer i a Frobenius invariant isomorphism between

$$H^i_{D,E}(\overline{X}_k, \mathbb{Q}_\ell(2))$$

and the dual of

$$H^{4-i}_{E,D}(\overline{X}_k, \mathbb{Q}_\ell).$$

Corollary 5.2. *For every integer i the characteristic polynomial of Frobenius*

$$\det(1 - TF | H^i_{D,E}(\overline{X}_k, \mathbb{Q}_\ell))$$

is in $\mathbb{Q}[T]$ and is independent of ℓ.

Proof. By 4.2 we know the result for $i = 1$, and hence by duality for $i = 3$. The result for $i = 0$ is immediate, and hence again by duality we also have the result for $i = 4$. Since we also know that the product

$$\prod_{i=0}^{4} \det(1 - TF | H^i_{D,E}(\overline{X}_k, \mathbb{Q}_\ell))^{(-1)^i}$$

is in $\mathbb{Q}[T]$ and independent of ℓ this implies the result for $i = 2$. □

This completes the proof of 1.2 in the case when X is smooth. Indeed in this case we can find a compactification $X \hookrightarrow \overline{X}$, where $\overline{X}$ is smooth and proper over $\mathbb{F}_q$ and such that the complement D is a divisor with simple normal crossings. In this case we have

$$H^i(X_k, \mathbb{Q}_\ell) = H^i_{\emptyset,D}(\overline{X}_k, \mathbb{Q}_\ell), \quad H^i_c(X_k, \mathbb{Q}_\ell) = H^i_{D,\emptyset}(\overline{X}_k, \mathbb{Q}_\ell).$$

5.3. Corollary 5.2 can be generalized as follows. Let $\overline{X}/\mathbb{F}_q$ be a smooth proper 2-dimensional $\mathbb{F}_q$-scheme, and suppose given a divisor with simple normal crossings

$$Z \hookrightarrow \overline{X}$$

and a decomposition

$$Z = D \cup E,$$

where D and E are also divisors with simple normal crossings. Let X denote $\overline{X} - E$. We then get inclusions

$$X - (D \cap X) \overset{j}{\hookrightarrow} X \overset{s}{\hookrightarrow} \overline{X},$$

and we define

$$H^i_{D,E}(\overline{X}_k, \mathbb{Q}_\ell) := H^i(X_k, j_!\mathbb{Q}_{\ell, X-(D\cap X)}).$$

5.4. Let

$$b : \overline{Y} \to \overline{X}$$

be the blowup of $E \cap D$, and let $E' \subset \overline{Y}$ (resp. $D' \subset \overline{Y}$) be the strict transform of E (resp. D). Note that

$$E' \cap D' = \emptyset.$$

Let $F' \subset \overline{Y}$ be the exceptional divisor, so we have

$$b^{-1}(D \cup E) = E' \cup D' \cup F'.$$

5.5. Let Y denote $\overline{Y} - E'$, and let

$$u : Y \hookrightarrow \overline{Y}$$

be the inclusion. Also define

$$v : Y - D' \hookrightarrow Y$$

to be the inclusion of the complement of D'.

There is a natural map

$$Rb_*(Ru_*v_!\mathbb{Q}_\ell) \to Rs_*j_!\mathbb{Q}_{\ell, X-(D\cap X)}. \tag{5.5.1}$$

For this note that giving such a map is equivalent by adjunction to specifying a map

$$s^*Rb_*(Ru_*v_!\mathbb{Q}_\ell) \to j_!\mathbb{Q}_{\ell, X-(D\cap X)},$$

and b is an isomorphism away from $D \cap E$.

Proposition 5.6. *The map 5.5.1 is an isomorphism. In particular we have*

$$H^i_{D,E}(\overline{X}_k, \mathbb{Q}_\ell) \simeq H^i_{D',E'}(\overline{Y}_k, \mathbb{Q}_\ell),$$

and the characteristic polynomial of Frobenius

$$\det(1 - TF | H^i_{D,E}(\overline{X}_k, \mathbb{Q}_\ell))$$

is in $\mathbb{Q}[T]$ and is independent of ℓ.

Proof. We can without loss of generality base change to $\bar{k}$. For the rest of the proof we consider only the base changes to $\bar{k}$ (so the points of intersection of $D \cap E$ are defined over the ground field), and omit the reference to $\bar{k}$ from the notation.

Let $x \in D \cap E$ be a point. It suffices to show that the stalks

$$(Rb_*(Ru_*v_!\mathbb{Q}_\ell))_x, \quad (Rs_*j_!\mathbb{Q}_{\ell, X-(D\cap X)})_x$$

are both 0.

To compute $(Rs_*j_!\mathbb{Q}_{\ell, X-(D\cap X)})_x$, let

$$\bar{i} : D \hookrightarrow \overline{X}, \quad i : D \cap X \hookrightarrow X, \quad q : D \cap X \hookrightarrow D$$

be the inclusions. We then have a short exact sequence

$$0 \to j_!\mathbb{Q}_{\ell, X-(D\cap X)} \to \mathbb{Q}_{\ell, X} \to i_*\mathbb{Q}_{\ell, D\cap X} \to 0,$$

which upon applying Rs_* and taking stalks at x gives a distinguished triangle

$$(Rs_*j_!\mathbb{Q}_{\ell, X-(D\cap X)})_x \to (Rs_*\mathbb{Q}_\ell)_x \to (Rq_*\mathbb{Q}_{\ell, D})_x.$$

It therefore suffices to show that

$$(Rs_*\mathbb{Q}_\ell)_x \to (Rq_*\mathbb{Q}_{\ell, D})_x$$

is an isomorphism, which follows from the purity theorem (observe that E is necessarily smooth at x).

To compute $(Rb_*(Ru_*v_!\mathbb{Q}_\ell))_x$, let F_x denote the fiber of b over x, so $F_x \simeq \mathbb{P}^1_k$. By the proper base change theorem we have

$$(Rb_*(Ru_*v_!\mathbb{Q}_\ell))_x \simeq R\Gamma(F_x, (Ru_*v_!\mathbb{Q}_\ell)|_{F_x}).$$

Choose an isomorphism $F_x \simeq \mathbb{P}^1_k$ so that $F_x \cap D' = \{0\}$, and $F_x \cap E' = \{\infty\}$, and let

$$i_0 : \mathrm{Spec}(k) \hookrightarrow \mathbb{P}^1, \quad i_\infty : \mathrm{Spec}(k) \hookrightarrow \mathbb{P}^1$$

be the inclusions. We then have a commutative diagram with cartesian squares

$$\begin{array}{ccccc} \mathbb{P}^1 & \overset{f}{\hookleftarrow} & \mathbb{A}^1 & \overset{g}{\hookleftarrow} & \mathbb{A}^1 - \{0\} \\ \downarrow & & \downarrow & & \downarrow \\ \overline{Y} & \overset{u}{\hookleftarrow} & Y & \overset{v}{\hookleftarrow} & Y - D'. \end{array}$$

Since D' and E' are disjoint, we have

$$(Ru_*v_!\mathbb{Q}_\ell)|_{\mathbb{P}^1} \simeq Rf_*g_!\mathbb{Q}_{\ell, \mathbb{A}^1-\{0\}}.$$

Therefore it suffices to show that

$$R\Gamma(\mathbb{A}^1, g_!\mathbb{Q}_\ell) = 0.$$

Using the short exact sequence

$$0 \to g_!\mathbb{Q}_\ell \to \mathbb{Q}_\ell \to i_{0*}\mathbb{Q}_\ell \to 0$$

this is equivalent to the statement that the restriction map

$$R\Gamma(\mathbb{A}^1, \mathbb{Q}_\ell) \to R\Gamma(\mathrm{Spec}(k), \mathbb{Q}_\ell) = \mathbb{Q}_\ell$$

is an isomorphism, which is standard. $\square$

6. Completion of proof of 1.2

6.1. Suppose $X/\mathbb{F}_q$ is a normal separated $\mathbb{F}_q$-scheme of finite type of dimension 2. Let

$$j : U \hookrightarrow X$$

be the smooth locus, and let

$$i : Z \hookrightarrow X$$

be the complement. Since X is normal the scheme Z is zero dimensional. Consider the short exact sequence of sheaves

$$0 \to j_!\mathbb{Q}_\ell \to \mathbb{Q}_\ell \to i_*\mathbb{Q}_\ell \to 0.$$

Taking cohomology we get that

$$H^i(X_k, \mathbb{Q}_\ell) \simeq H^i(X_k, j_!\mathbb{Q}_\ell), \quad i \geq 2,$$

and an exact sequence

$$0 \to D_\ell \to H^1(X_k, j_!\mathbb{Q}_\ell) \to H^1(X_k, \mathbb{Q}_\ell) \to 0,$$

where D_ℓ denotes

$$\mathrm{Coker}(\mathbb{Q}_\ell^{\pi_0(X_k)} \to \mathbb{Q}_\ell^{\pi_0(Z_k)}).$$

Similarly, if X is not proper over $\mathbb{F}_q$ then we get

$$H^i_c(X_k, \mathbb{Q}_\ell) \simeq H^i_c(U_k, \mathbb{Q}_\ell), \quad i \geq 2$$

and an exact sequence

$$0 \to \mathbb{Q}_\ell^{\pi_0(Z_k)} \to H^1_c(U_k, \mathbb{Q}_\ell) \to H^1_c(X_k, \mathbb{Q}_\ell) \to 0.$$

Since we already know 1.2 in the case when X is smooth (so in particular we know the result for U), to complete the proof of 1.2 for the normal X, it suffices to show that for every i the characteristic polynomial of Frobenius

$$\det(1 - TF|H^i(X_k, j_!\mathbb{Q}_\ell))$$

is in $\mathbb{Q}[T]$ and is independent of ℓ.

6.2. Let

$$b : X' \to X$$

be a proper morphism such that b is an isomorphism over U, $X'/\mathbb{F}_q$ is smooth, and $D := b^{-1}(Z)$ is a divisor with simple normal crossings on X'. Furthermore, choose a dense open immersion

$$X' \hookrightarrow \overline{X}',$$

where $\overline{X}'$ is smooth and proper over $\mathbb{F}_q$ and such that

$$E := \overline{X}' - X'$$

is a divisor with simple normal crossings. Notice that since D is proper over $\mathbb{F}_q$, the composite $D \hookrightarrow X' \hookrightarrow \overline{X}'$ is a closed immersion and $D \cap E = \emptyset$.

Let

$$\tilde{j} : U \hookrightarrow X'$$

be the unique lifting of j. Since b is proper we have $j_! = Rb_*\tilde{j}_!$ and therefore

$$H^i(X_k, j_!\mathbb{Q}_\ell) = H^i(X', \tilde{j}_!\mathbb{Q}_\ell) = H^i_{D,E}(\overline{X}', \mathbb{Q}_\ell).$$

From this and 5.2 we therefore obtain 1.2. □

7. Restriction maps

7.1. Let $X/\mathbb{F}_q$ be a smooth separated finite type $\mathbb{F}_q$-scheme of dimension 2. Let $i : Z \subset X$ be a closed subscheme, and for an integer i let $I^i(X, Z)$ (resp. $I^i_c(X, Z)$) denote the image of the restriction map

$$H^i(X_k, \mathbb{Q}_\ell) \to H^i(Z_k, \mathbb{Q}_\ell), \quad (\text{resp. } H^i_c(X_k, \mathbb{Q}_\ell) \to H^i_c(Z_k, \mathbb{Q}_\ell)).$$

Set

$$Q^i_\ell := \det(1 - TF | I^i(X, Z)), \quad Q^i_{c,\ell} := \det(1 - TF | I^i_c(X, Z)).$$

Proposition 7.2. *For every integer i and prime ℓ invertible in k, we have*

$$Q^i_\ell, Q^i_{c,\ell} \in \mathbb{Q}[T],$$

and for any two primes ℓ and ℓ' not equal to the characteristic of k we have

$$Q^i_\ell = Q^i_{\ell'}, \quad Q^i_{c,\ell} = Q^i_{c,\ell'}.$$

Proof. Let

$$j : U \hookrightarrow X$$

be the complement of Z. Taking cohomology (resp. compactly supported cohomology) of the short exact sequence

$$0 \to j_!\mathbb{Q}_\ell \to \mathbb{Q}_\ell \to i_*\mathbb{Q}_\ell \to 0$$

we obtain boundary maps

$$\partial^i : H^{i-1}(Z_k, \mathbb{Q}_\ell) \to H^i(X_k, j_!\mathbb{Q}_\ell), \quad \partial^i_c : H^{i-1}_c(Z_k, \mathbb{Q}_\ell) \to H^i_c(U_k, \mathbb{Q}_\ell),$$

whose kernels are $I^{i-1}(X, Z)$ and $I^{i-1}_c(X, Z)$ respectively. We therefore get short exact sequences

$$0 \to H^{i-1}(Z_k, \mathbb{Q}_\ell)/I^{i-1}(X, Z) \to H^i(X_k, j_!\mathbb{Q}_\ell) \to H^i(X_k, \mathbb{Q}_\ell) \to I^i(X, Z) \to 0,$$

$$0 \to H^{i-1}_c(Z_k, \mathbb{Q}_\ell)/I^{i-1}_c(X, Z) \to H^i_c(U_k, \mathbb{Q}_\ell) \to H^i_c(X_k, \mathbb{Q}_\ell) \to I^i_c(X, Z) \to 0.$$

Proceeding by induction on i, we see that it suffices to show that the characteristic polynomials of Frobenius acting on

$$H^i(X_k, j_!\mathbb{Q}_\ell),\ H^i_c(U_k, \mathbb{Q}_\ell),\ H^i(X_k, \mathbb{Q}_\ell),\ H^i_c(X_k, \mathbb{Q}_\ell),\ H^i(Z_k, \mathbb{Q}_\ell), \text{ and } H^i_c(Z_k, \mathbb{Q}_\ell)$$

are in $\mathbb{Q}[T]$ and independent of ℓ. This statement for $H^i(X_k, j_!\mathbb{Q}_\ell)$ follows from the same argument as in 6.2 (and using 5.6), for $H^i_c(U_k, \mathbb{Q}_\ell)$, $H^i(X_k, \mathbb{Q}_\ell)$, and $H^i_c(X_k, \mathbb{Q}_\ell)$ it follows from 1.2, and for the cohomology groups of the lower dimensional Z the result is standard. □

8. Proof of theorem 1.4

As in 1.4, let k be a separably closed field and X/k a smooth 2-dimensional k-scheme. Let p denote the characteristic of k. In what follows ℓ denotes a prime not equal to p.

Lemma 8.1. *Suppose $H^3(X, \mathbb{Z}_\ell(1))_{\text{tors}} = 0$. Then for any $n \geq 1$, if $x \in \text{Br}(X)[\ell^n]$, then x is ℓ-divisible.*

Proof. From the morphism of Kummer sequences

(8.1.1)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mu_{\ell^{n+1}} & \longrightarrow & \mathbb{G}_m & \xrightarrow{\cdot \ell^{n+1}} & \mathbb{G}_m & \longrightarrow & 0 \\ & & \downarrow{\cdot \ell} & & \downarrow{\cdot \ell} & & \downarrow{\text{id}} & & \\ 0 & \longrightarrow & \mu_{\ell^n} & \longrightarrow & \mathbb{G}_m & \xrightarrow{\cdot \ell^n} & \mathbb{G}_m & \longrightarrow & 0, \end{array}$$

we get a commutative diagram

$$\begin{array}{ccccc} H^2(X, \mu_{\ell^{n+1}}) & \longrightarrow & \text{Br}(X)[\ell^{n+1}] & \longrightarrow & 0 \\ \downarrow & & \downarrow{\cdot \ell} & & \\ H^2(X, \mu_{\ell^n}) & \longrightarrow & \text{Br}(X)[\ell^n] & \longrightarrow & 0. \end{array}$$

It therefore suffices to show that the map

$$H^2(X, \mu_{\ell^{n+1}}) \to H^2(X, \mu_{\ell^n})$$

is surjective. This follows from noting that in fact the map

$$H^2(X, \mathbb{Z}_\ell(1)) \to H^2(X, \mu_{\ell^n})$$

is surjective. Indeed from the short exact sequence

$$0 \longrightarrow \mathbb{Z}_\ell(1) \xrightarrow{\cdot \ell^n} \mathbb{Z}_\ell(1) \longrightarrow \mu_{\ell^n} \longrightarrow 0$$

we get an exact sequence

$$H^2(X, \mathbb{Z}_\ell(1)) \to H^2(X, \mu_{\ell^n}) \to H^3(X, \mathbb{Z}_\ell(1))_{\text{tors}},$$

and

$$H^3(X, \mathbb{Z}_\ell(1))_{\text{tors}} = 0$$

by assumption. □

Lemma 8.2. *Poincare duality defines an isomorphism*

$$\text{Ext}^1(H^3(X, \mathbb{Z}_\ell(1)), \mathbb{Z}_\ell) \simeq H^2_c(X, \mathbb{Z}_\ell(3))_{\text{tors}}.$$

In particular, the group $H^2_c(X, \mathbb{Z}_\ell(1))$ is torsion free if and only if the group $H^3(X, \mathbb{Z}_\ell(1))$ is torsion free.

Proof. To ease notation, write simply K for the complex $R\Gamma(X, \mathbb{Z}_\ell(1))$. Poincare duality defines an isomorphism

$$R\mathcal{H}om(K, \mathbb{Z}_\ell) \simeq \mathbb{R}\Gamma_c(X, \mathbb{Z}_\ell(3))[4].$$

In particular we have

$$H^2_c(X, \mathbb{Z}_\ell(3)) \simeq \text{Ext}^{-2}(K, \mathbb{Z}_\ell).$$

From the distinguished triangle

$$\tau_{\leq 1}K \to K \to \tau_{\geq 2}K$$

we obtain a long exact sequence

$$\cdots \to \mathrm{Ext}^{-3}(\tau_{\leq 1}K, \mathbb{Z}_\ell) \to \mathrm{Ext}^{-2}(\tau_{\geq 2}K, \mathbb{Z}_\ell) \to \mathrm{Ext}^{-2}(K, \mathbb{Z}_\ell) \to \mathrm{Ext}^{-2}(\tau_{\leq 1}K, \mathbb{Z}_\ell) \to \cdots .$$

The complex

$$RHom(\tau_{\leq 1}K, \mathbb{Z}_\ell)$$

is concentrated in degrees ≥ -1, and therefore

$$\mathrm{Ext}^{-2}(\tau_{\geq 2}K, \mathbb{Z}_\ell) \to \mathrm{Ext}^{-2}(K, \mathbb{Z}_\ell)$$

is an isomorphism. Considering the distinguished triangle

$$\tau_{\leq 3}\tau_{\geq 2}K \to \tau_{\geq 2}K \to H^4(K)[-4]$$

and the fact that (like any $\mathbb{Z}_\ell$-module) $H^4(K)$ has projective dimension ≤ 1, we then get an isomorphism

$$\mathrm{Ext}^{-2}(\tau_{\geq 2}K, \mathbb{Z}_\ell) \simeq \mathrm{Ext}^{-2}(\tau_{\leq 3}\tau_{\geq 2}K, \mathbb{Z}_\ell).$$

Finally from

$$H^2(K)[-2] \to \tau_{\leq 3}\tau_{\geq 2}K \to H^3(K)[-3]$$

we get an exact sequence

$$\mathrm{Ext}^{-3}(H^2(K)[-2], \mathbb{Z}_\ell) \to \mathrm{Ext}^{-2}(H^3(K)[-3], \mathbb{Z}_\ell) \to \mathrm{Ext}^{-2}(\tau_{\leq 3}\tau_{\geq 2}K, \mathbb{Z}_\ell) \to \mathrm{Ext}^{-2}(H^2(K)[-2], \mathbb{Z}_\ell).$$

Since

$$RHom(H^2(K)[-2], \mathbb{Z}_\ell)$$

is concentrated in degrees ≥ -2, this gives an exact sequence

$$0 \to \mathrm{Ext}^1(H^3(K), \mathbb{Z}_\ell) \to \mathrm{Ext}^{-2}(\tau_{\leq 3}\tau_{\geq 2}K, \mathbb{Z}_\ell) \to \mathrm{Hom}(H^2(K), \mathbb{Z}_\ell).$$

In particular we get an isomorphism

$$\mathrm{Ext}^1(H^3(K), \mathbb{Z}_\ell) \simeq \mathrm{Ext}^{-2}(\tau_{\leq 3}\tau_{\geq 2}K, \mathbb{Z}_\ell)_{\mathrm{tors}}.$$

Combining this with the isomorphism

$$\mathrm{Ext}^{-2}(\tau_{\leq 3}\tau_{\geq 2}K, \mathbb{Z}_\ell) \simeq H^2_c(X, \mathbb{Z}_\ell(3))$$

we get the result. □

8.3. The group $H^2_c(X, \mathbb{Z}_\ell(1))$ can be analyzed as follows. Choose a compactification

$$j : X \hookrightarrow \overline{X},$$

where $\overline{X}/k$ is proper and smooth, and the complement $D := \overline{X} - X$ is divisor with simple normal crossings on $\overline{X}$. Let

$$i : D \hookrightarrow \overline{X}$$

be the inclusion, and as in 2.8 set

$$\mathscr{H} := \mathrm{Ker}(\mathbb{G}_{m,\overline{X}} \to i_*\mathbb{G}_{m,D})$$

so that for every n we have an exact sequence (see 2.8.1)

$$0 \longrightarrow j_!\mu_{\ell^n} \longrightarrow \mathscr{H} \xrightarrow{\cdot \ell^n} \mathscr{H} \longrightarrow 0.$$

Taking cohomology over $\overline{X}$ we obtain an exact sequence

$$0 \to H^1(\overline{X}, \mathscr{H})/\ell^n H^1(\overline{X}, \mathscr{H}) \to H^2_c(X, \mu_{\ell^n}) \to H^2(\overline{X}, \mathscr{H})[\ell^n] \to 0. \tag{8.3.1}$$

As in 2.8.2 we have

$$H^1(\overline{X}, \mathscr{H})/\ell^n H^1(\overline{X}, \mathscr{H}) \simeq \mathrm{Pic}(\overline{X}, D)/\ell^n \mathrm{Pic}(\overline{X}, D).$$

Set (notation as in 2.4)

$$\widetilde{NS}(\overline{X}, D) := \mathrm{Pic}(\overline{X}, D)/\mathrm{Pic}^{00}(\overline{X}, D).$$

Then $\widetilde{NS}(\overline{X}, D)$ is an extension of $NS(\overline{X}, D)$ by a finite group (namely the component group of $\mathrm{Pic}^0(\overline{X}, D)$), and therefore is finitely generated, and we have

$$\mathrm{Pic}(\overline{X}, D)/\ell^n \mathrm{Pic}(\overline{X}, D) \simeq \widetilde{NS}(\overline{X}, D)/\ell^n \widetilde{NS}(\overline{X}, D).$$

We can therefore rewrite (8.3.1) as

$$0 \to \widetilde{NS}(\overline{X}, D)/\ell^n \widetilde{NS}(\overline{X}, D) \to H^2_c(X, \mu_{\ell^n}) \to H^2(\overline{X}, \mathscr{H})[\ell^n] \to 0. \tag{8.3.2}$$

Passing to the inverse limit with respect to the morphisms obtained from the diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & j_!\mu_{\ell^{n+1}} & \longrightarrow & \mathscr{H} & \xrightarrow{\cdot \ell^{n+1}} & \mathscr{H} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \cdot \ell} & & \downarrow{\scriptstyle \cdot \ell} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & j_!\mu_{\ell^{n}} & \longrightarrow & \mathscr{H} & \xrightarrow{\cdot \ell^{n}} & \mathscr{H} & \longrightarrow & 0
\end{array}$$

we get an exact sequence

$$0 \to \widetilde{NS}(\overline{X}, D) \otimes \mathbb{Z}_\ell \to H^2_c(X, \mathbb{Z}_\ell(1)) \to T_\ell H^2(\overline{X}, \mathscr{H}) \to 0.$$

Here the exactness on the right follows from the fact that the system

$$\{\widetilde{NS}(\overline{X}, D)/\ell^n \widetilde{NS}(\overline{X}, D)\}$$

satisfies the Mittag-Leffler condition.

Since $T_\ell H^2(\overline{X}, \mathscr{H})$ is ℓ-torsion free, we get an isomorphism

$$(\widetilde{NS}(\overline{X}, D) \otimes \mathbb{Z}_\ell)_{\mathrm{tors}} \simeq H^2_c(X, \mathbb{Z}_\ell(1))_{\mathrm{tors}}.$$

Since $\widetilde{NS}(\overline{X}, D)$ is a finitely generated abelian group, this yields the following corollary:

Corollary 8.4. *For all but finitely many primes ℓ, the group $H^2_c(X, \mathbb{Z}_\ell(1))$ is torsion free.*

8.5. Let $\mathrm{Br}(X)(\ell) \subset \mathrm{Br}(X)$ (resp. $\mathrm{Br}(X)_{\mathrm{div}}(\ell) \subset \mathrm{Br}(X)_{\mathrm{div}}$) be the subgroup of elements whose order is a power of ℓ.

Notice that for an element $x \in \mathrm{Br}(X)(\ell)$, the condition of being divisible is equivalent to the condition of being ℓ-divisible.

Combining 8.1, 8.2, and 8.4 we get that for all but a finite number of primes ℓ we have

$$\mathrm{Br}(X)_{\mathrm{div}}(\ell) = \mathrm{Br}(X)(\ell).$$

Set

$$W_\ell := \mathrm{Br}(X)(\ell)/\mathrm{Br}(X)_{\mathrm{div}}(\ell).$$

Since the quotient

$$\widetilde{\mathrm{Br}}(X)/\widetilde{\mathrm{Br}}(X)_{\mathrm{div}} \tag{8.5.1}$$

is isomorphic to the group

$$\oplus_{\ell \neq p} W_\ell,$$

to prove that 8.5.1 is finite it suffices to show that each of the groups W_ℓ is finite.

For this notice that from the Kummer sequence

$$0 \longrightarrow \mu_{\ell^n} \longrightarrow \mathbb{G}_m \xrightarrow{\cdot \ell^n} \mathbb{G}_m \longrightarrow 0$$

we obtain compatible maps

$$\mathrm{Br}(X) \to H^3(X, \mu_{\ell^n})$$

and therefore a map

$$\mathrm{Br}(X) \to H^3(X, \mathbb{Z}_\ell(1)).$$

The kernel of this map is precisely the ℓ-divisible elements of $\mathrm{Br}(X)$, so we get from this an inclusion

$$W_\ell \hookrightarrow H^3(X, \mathbb{Z}_\ell(1)).$$

Since $H^3(X, \mathbb{Z}_\ell(1))$ is a finitely generated $\mathbb{Z}_\ell$-module this implies that W_ℓ is finite.

8.6. It remains to analyze the divisible part $\widetilde{\mathrm{Br}}(X)_{\mathrm{div}}$. Let S_ℓ denote the ℓ-torsion subgroup of $\mathbb{Q}/\mathbb{Z}$. We have

$$F_p \simeq \oplus_{\ell \neq p} S_\ell.$$

Moreover, we have

$$\widetilde{\mathrm{Br}}(X)_{\mathrm{div}} \simeq \oplus_{\ell \neq p} \mathrm{Br}(X)_{\mathrm{div}}(\ell).$$

Therefore it suffices to show that there exists an integer r such that for every $\ell \neq p$ we have

$$S_\ell^{\oplus r} \simeq \mathrm{Br}(X)_{\mathrm{div}}(\ell).$$

Lemma 8.7. *Let A be a divisible ℓ-torsion group, and assume that $A[\ell]$ is a finite dimensional $\mathbb{F}_\ell$-vector space of some rank r. Then*

$$A \simeq S_\ell^{\oplus r}.$$

Proof. Choose a basis $x_1, \ldots, x_r \in A[\ell]$ for $A[\ell]$ as a $\mathbb{F}_\ell$-vector space, and for each $1 \leq i \leq r$ choose a sequence of elements $\{x_i^{(n)}\}_{n \geq 1}$ with

$$x_i^{(1)} = x_i, \quad \ell x_i^{(n)} = x_i^{(n-1)}.$$

This is possible since A is divisible. Let

$$\pi_i : S_\ell \to A$$

be the map sending $1/\ell^n$ to $x_i^{(n)}$, and let

$$\pi : S_\ell^{\oplus r} \to A$$

be the direct sum of the π_i. We claim that π is an isomorphisms. Since both sides are torsion groups, to verify that π is an isomorphism it suffices to show that π induces an isomorphism on ℓ^n-torsion for every n. This we show by induction on n. For $n = 1$ the result is immediate. For the inductive step consider the commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_\ell^{\oplus r}[\ell] & \longrightarrow & S_\ell^{\oplus r}[\ell^n] & \xrightarrow{\cdot\ell} & S_\ell^{\oplus r}[\ell^{n-1}] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A[\ell] & \longrightarrow & A[\ell^n] & \xrightarrow{\cdot\ell} & A[\ell^{n-1}] & \longrightarrow & 0. \end{array}$$

□

8.8. Now observe that from the Kummer sequence we have a surjection

$$H^2(X, \mu_\ell) \to \mathrm{Br}(X)[\ell],$$

and therefore $\mathrm{Br}(X)[\ell]$ is a finite dimensional $\mathbb{F}_\ell$-vector space. Hence $\mathrm{Br}(X)_{\mathrm{div}}[\ell]$ is also a finite dimensional $\mathbb{F}_\ell$-vector space. It follows that there exists an integer r_ℓ such that

$$\mathrm{Br}(X)_{\mathrm{div}}(\ell) \simeq S_\ell^{\oplus r_\ell}.$$

8.9. We now show that all the r_ℓ are equal.

Let $\mathrm{Pic}^0(X) \subset \mathrm{Pic}(X)$ be the subgroup of divisible elements, and set

$$NS(X) := \mathrm{Pic}(X)/\mathrm{Pic}^0(X).$$

Notice that if $X \hookrightarrow \overline{X}$ is a compactification, with $\overline{X}/k$ a smooth proper surface, then we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}^0(\overline{X}) & \longrightarrow & \mathrm{Pic}(\overline{X}) & \longrightarrow & NS(\overline{X}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & NS(X) & \longrightarrow & 0. \end{array}$$

Since $\overline{X}$ is smooth, the restriction map

$$\mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(X)$$

is surjective. Therefore $NS(X)$ is a quotient of $NS(\overline{X})$ and in particular is a finitely generated abelian group.

Note also that since $\mathrm{Pic}^0(X)$ is a divisible group, we have

$$\mathrm{Pic}(X)/\ell^n\mathrm{Pic}(X) \simeq NS(X)/\ell^n NS(X).$$

From the Kummer sequences

$$0 \longrightarrow \mu_{\ell^n} \longrightarrow \mathbb{G}_m \xrightarrow{\cdot\ell^n} \mathbb{G}_m \longrightarrow 0$$

we get short exact sequences

$$0 \to NS(X)/\ell^n NS(X) \to H^2(X, \mu_{\ell^n}) \to \mathrm{Br}(X)[\ell^n] \to 0,$$

which upon passing to the inverse limit gives a short exact sequence

$$0 \to NS(X) \otimes \mathbb{Z}_\ell \to H^2(X, \mathbb{Z}_\ell(1)) \to T_\ell\mathrm{Br}(X) \to 0.$$

Since

$$T_\ell\mathrm{Br}(X) \simeq T_\ell\mathrm{Br}(X)_{\mathrm{div}}(\ell) \simeq \mathbb{Z}_\ell^{r_\ell},$$

we conclude that

$$r_\ell = \dim_{\mathbb{Q}_\ell} H^2(X, \mathbb{Q}_\ell) - \dim_{\mathbb{Q}_\ell}(NS(X) \otimes \mathbb{Q}_\ell),$$

which is independent of ℓ by 1.2.

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