FOURIER-MUKAI PARTNERS OF K3 SURFACES IN POSITIVE CHARACTERISTIC

MAX LIEBLICH AND MARTIN OLSSON

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1. INTRODUCTION

In this paper we establish several basic facts about Fourier-Mukai equivalence of K3 surfaces over fields of positive characteristic and develop some foundational material on deformation and lifting of Fourier-Mukai kernels, including the study of several "realizations" of Mukai's Hodge structure in standard cohomology theories (étale, crystalline, Chow, etc.).

In particular, we prove the following theorem, extending to positive characteristic classical results due to Mukai, Oguiso, Orlov, and Yau in characteristic 0 (see [Or] and [HLOY]). For a scheme Z of finite type over a field k, let $\mathrm{D}(Z)$ denote the bounded derived category with coherent cohomology. For a K3 surface X over an algebraically closed field k, we have algebraic moduli spaces $\mathrm{M}_X(v)$ of sheaves with fixed Mukai vector v (see Section 3.11 for the precise definition) that are stable with respect to a suitable polarization.

Theorem 1.1. *Let X be a K3 surface over an algebraically closed field k of characteristic $\neq 2$.*

(1) *If Y is a smooth projective k-scheme such that there exists an equivalence of triangulated categories $\mathrm{D}(X) \simeq \mathrm{D}(Y)$, then Y is a K3-surface isomorphic to $\mathrm{M}_X(v)$ for some Mukai vector v such that there exists a Mukai vector w with $\langle v, w\rangle = 1$.*
(2) *There exist only finitely many smooth projective varieties Y with $\mathrm{D}(X) \simeq \mathrm{D}(Y)$.*

(3) *If X has Picard number at least 11 and Y is a smooth projective scheme with $\mathrm{D}(Y) \simeq \mathrm{D}(X)$, then $X \simeq Y$. In particular, any Shioda-supersingular K3 surface is determined up to isomorphism by its derived category.*

The classical proofs of these results in characteristic 0 rely heavily on the Torelli theorem and lattice theory, so a transposition into characteristic p is necessarily delicate. We present here a theory of the "Mukai motive", generalizing the Mukai-Hodge structure to other cohomology theories, and use various realizations to aid in lifting derived-equivalence problems to characteristic 0.

These techniques also yields proofs of several other results. The first answers a question of Mustaţă and Huybrechts, while the second establishes the truth of the variational crystalline Hodge conjecture [MP, Conjecture 9.2] in some special cases. (In the course of preparing this manuscript, we learned that Huybrechts discovered essentially the same proof of Theorem 1.2, in ℓ-adic form.)

Theorem 1.2. *If X and Y are K3 surfaces over a finite field $\mathbf{F}$ of characteristic $\neq 2$ such that $\mathrm{D}(X)$ is equivalent to $\mathrm{D}(Y)$, then X and Y have the same zeta-function. In particular, $\#X(\mathbf{F}) = \#Y(\mathbf{F})$.*

Theorem 1.3. *Suppose X and Y are K3 surfaces over an algebraically closed field k of characteristic $\neq 2$ with Witt vectors W, and that $\mathcal{X}/W$ and $\mathcal{Y}/W$ are lifts, giving rise to a Hodge filtration on the F-isocrystal $H^4_{\text{cris}}(X \times Y/K)$. Suppose $Z \subset X \times Y$ is a correspondence coming from a Fourier-Mukai kernel. If the fundamental class of Z lies in $\mathrm{Fil}^2 H^4_{\text{cris}}(X \times Y/K)$ then Z is the specialization of a cycle on $\mathcal{X} \times \mathcal{Y}$.*

Throughout this paper we consider only fields of characteristic $\neq 2$.

1.4. **Outline of the paper.** Sections 2 and 3 contain foundational background material on Fourier-Mukai equivalences. In Section 2 we discuss variants in other cohomology theories (étale, crystalline, Chow) of Mukai's original construction of a Hodge structure associated to a smooth even dimensional proper scheme. In Section 3 we discuss various basic material on kernels of Fourier-Mukai equivalences. The main technical tool is Proposition 3.3, which will be used when deforming kernels. The results of these two sections are presumably well-known to experts.

In Section 4 we discuss the relationship between moduli of complexes and Fourier-Mukai kernels. This relationship is the key to the deformation theory arguments that follow and appears never to have been written down in this way. The main result of this section is Proposition 4.4.

Section 5 contains the key result for the whole paper (Theorem 5.1). This result should be viewed as a derived category version of the classical Torelli theorem for K3 surfaces. It appears likely that this kind of reduction to the universal case via moduli stacks of complexes should be useful in other contexts. Using this we prove statement (1) in Theorem 1.1 in Section 6.

In Section 7 we prove statement (2) in Theorem 1.1. Our proof involves deforming to characteristic 0, which in particular is delicate for supersingular K3 surfaces.

In section 8 we prove Theorem 1.2, and in section 9 we prove Theorem 1.3.

Finally there is an appendix containing a technical result about versal deformations of polarized K3 surfaces which is used in section 7. The main result of the appendix is Theorem A.7 concerning the Picard group of the general deformation of a fixed K3 surface from characteristic p to characteristic 0.

1.5. **Acknowledgments.** Lieblich partially supported by the Sloan Foundation, NSF grant DMS-1021444, and NSF CAREER grant DMS-1056129, Olsson partially supported by NSF CAREER grant DMS-0748718. We thank Daniel Huybrechts for very helpful comments on an earlier version of this manuscript.

2. Mukai motive

2.1. **Mukai's original construction over C: the Hodge realization.** Suppose X is a smooth projective variety of even dimension $d = 2\delta$. The singular cohomology $\mathrm{H}^i(X, \mathbf{Z})$ carries a natural pure Hodge structure of weight i, and the cup product defines a pairing of Hodge structures

$$\mathrm{H}^i(X, \mathbf{Z}) \times \mathrm{H}^{2\delta - i}(X, \mathbf{Z}) \to \mathrm{H}^{2\delta}(X, \mathbf{Z}) = \mathbf{Z}(-\delta),$$

where $\mathbf{Z}(-1)$ is the usual Tate Hodge structure of weight 2.

Define the *(even) Mukai-Hodge structure of* X to be the pure Hodge structure of weight d given by

$$\widetilde{H}(X, \mathbf{Z}) := \bigoplus_{i=-\delta}^{\delta} \mathrm{H}^{2\delta+2i}(X, \mathbf{Z})(i).$$

The cup product and the identification $\mathrm{H}^{2d}(X, \mathbf{Z}) \cong \mathbf{Z}(-d)$ yield the *Mukai pairing*

$$\widetilde{H}(X, \mathbf{Z}) \times \widetilde{H}(X, \mathbf{Z}) \to \mathbf{Z}(-d)$$

defined by the formula

$$\langle (a_{-\delta}, a_{-\delta+1}, \ldots, a_{\delta-1}, a_\delta), (a'_{-\delta}, a'_{-\delta+1}, \ldots, a'_{\delta-1}, a'_\delta) \rangle := \sum_{i=-\delta}^{0} (-1)^i a_{-i} \cdot a_i.$$

One of the main features of the Mukai Hodge structure is its compatibility with correspondences. In particular, let Y be another smooth projective variety of dimension d. A perfect complex P on $X \times Y$ induces a map of Hodge lattices

$$\Psi_P : \widetilde{H}(X, \mathbf{Z}) \to \widetilde{H}(Y, \mathbf{Z})$$

given by adding the maps

$$\Psi_P^{i,j} : H^{2\delta+2i}(X, \mathbf{Z})(i) \to H^{2\delta+2j}(Y, \mathbf{Z})(j)$$

defined as the composite

$$\begin{array}{ccc} H^{2\delta+2i}(X, \mathbf{Z})(i) & \xrightarrow{\mathrm{pr}_1^*} & H^{2\delta+2i}(X \times Y, \mathbf{Z})(i) \\ & & \downarrow {\scriptstyle \cup \mathrm{ch}^{j-i+d_X}(P)} \\ H^{2\epsilon+2j}(Y, \mathbf{Z})(j) & \xleftarrow{\mathrm{pr}_{2*}} & H^{2\delta+2j+2d_X}(X \times Y, \mathbf{Z})(j + d_X). \end{array}$$

One checks that the Mukai pairing is respected by this transformation. Note also that this map depends only on the image of P in the Grothendieck group $K(X \times Y)$.

Unfortunately, the map Ψ_P is not compatible with the Grothendieck-Riemann-Roch theorem without a modification. Thus, it is usual to let $\Phi_P = \Psi_{P\sqrt{\mathrm{Td}_{X\times Y}}}$, but now one must (in general) restrict to rational coefficients:

$$\Phi_P : \widetilde{H}(X, \mathbf{Q}) \to \widetilde{H}(Y, \mathbf{Q}).$$

Mukai's original work was on the cohomology of K3 surfaces. For such a surface X, the Mukai Hodge structure is

$$\mathrm{H}^0(X, \mathbf{Z})(-1) \oplus \mathrm{H}^2(X, \mathbf{Z}) \oplus \mathrm{H}^4(X, \mathbf{Z})(1)$$

(colloquially rendered as "place H^0 and H^4 in $\widetilde{H}^{1,1}$"), and the Mukai pairing takes the form

$$\langle (a, b, c), (a', b', c')\rangle = bb' - ac' - a'c.$$

Moreover, the class $\sqrt{\mathrm{Td}_{X\times Y}}$ lies in $K(X\times Y)$ (i.e., it has integral components), so that for all pairs of K3 surfaces X and Y, any $P \in K(X \times Y)$ induces a map of rank 24 lattices

$$\Phi_P : \widetilde{H}(X, \mathbf{Z}) \to \widetilde{H}(Y, \mathbf{Z}).$$

As Mukai and Orlov proved in their seminal work, the Mukai Hodge structure of a K3 surface uniquely determines its derived category up to (non-canonical) equivalence.

In the rest of this section, we will discuss the realizations of the "Mukai motive" that exist in all characteristics.

2.2. **Crystalline realization.** Let k be a perfect field of characteristic $p > 0$, let W be its ring of Witt vectors, and let K denote the field of fractions of W. For a proper smooth scheme X/k we write

$$H^i(X/K)$$

for the crystalline cohomology

$$H^i(X/K) := H^i((X/W)_{\mathrm{cris}}, \mathscr{O}_{X/W}) \otimes_W K.$$

This is an F-isocrystal over K and we write

$$\varphi_X : H^{\cdot}(X/K) \to H^{\cdot}(X/K)$$

for the Frobenius action induced by the Frobenius morphism $F_X : X \to X$.

Let $K(1)$ denote the F-isocrystal whose underlying vector space is K, and whose Frobenius action is given by multiplication by $1/p$. If M is another isocrystal and n is an integer we write $M(n)$ for the tensor product $M \otimes K(1)^{\otimes n}$ (with the usual convention that if n is negative then $K(1)^{\otimes n}$ denotes the $-n$-th tensor power of the dual of $K(1)$).

Let X/k be a proper smooth scheme, $A^*(X)$ the Chow ring of algebraic cycles on X modulo rational equivalence, and $K(X)$ the Grothendieck group of vector bundles on X. There is a cycle map (see [G-M])

$$\eta : A^*(X) \to H^{2*}(X/K),$$

which upon composing with the Chern character

$$\mathbf{ch} : K(X) \to A^*(X)$$

defines a map

$$\mathbf{ch}_{\mathrm{cris}} : K(X) \to H^{2*}(X/K),$$

which we call the *crystalline Chern character*. For an integer i we write $\mathrm{ch}^i_{\mathrm{cris}}$ for the $2i$-th component of $\mathrm{ch}_{\mathrm{cris}}$.

Lemma 2.3. *For any integer i and $E \in K(X)$, we have*

$$\varphi_X(\mathrm{ch}^i_{\mathrm{cris}}(E)) = p^i \mathrm{ch}^i_{\mathrm{cris}}(E).$$

Equivalently, cup product with $\mathrm{ch}^i_{\mathrm{cris}}(E)$ defines a morphism of F-isocrystals

$$H^j(X/K) \to H^{j+2i}(X/K)(-i)$$

for every i.

Proof. Fix a vector bundle E on X, and let $p : X' \to X$ be the full flag scheme of E over X. Then the map

$$p^* : H^{2i}(X/K) \to H^{2i}(X'/K)$$

is injective and compatible with the Frobenius actions, so it suffices to verify the result for p^*E. We may therefore assume that E admits a filtration

$$0 = F^n \subset F^{n-1} \subset F^{n-2} \subset \cdots \subset F^1 \subset F^0 = E$$

such that the successive quotients

$$L_s := F^s/F^{s+1}$$

are line bundles on X. Let $a_s \in H^2(X/K)$ be the first Chern class of L_s. Then

$$\mathrm{ch}^i_{\mathrm{cris}}(E)$$

is equal to a sum of terms of the form

$$a_1^{\alpha_1} \cdots a_n^{\alpha_n},$$

with

$$\sum \alpha_s = i.$$

It therefore suffices to prove the result in the case when E is a line bundle L where it follows from the fact that $F_X^* L \simeq L^{\otimes p}$. □

If X/k is proper and smooth of dimension d, then we have an isomorphism (given by the top Chern class of a point)

$$H^{2d}(X/K) \simeq K(-d), \tag{2.3.1}$$

and the cup product pairing

$$H^i(X/K) \times H^{2d-i}(X/K) \to H^{2d}(X/K) \simeq K(-d)$$

is perfect, thereby inducing an isomorphism

$$H^i(X/K) \simeq (H^{2d-i}(X/K)(d))^\vee,$$

where the right side denotes the dual of $H^{2d-i}(X/K)(d)$.

In particular, if $f : X \to Y$ is a morphism of proper smooth k-schemes of dimensions d_X and d_Y respectively, then pullback maps

$$f^* : H^i(Y/K) \to H^i(X/K)$$

are adjoint to maps of F-isocrystals

$$H^{2d_X - i}(X/K)(d_X) \to H^{2d_Y - i}(Y/K)(d_Y).$$

We write

$$f_* : H^s(X/K) \to H^{s+2(d_Y - d_X)}(Y/K)(d_Y - d_X)$$

for these maps of F-isocrystals.

If X/k is proper and smooth of even dimension $d_X = 2\delta$, set

$$\widetilde{H}^i(X/K) := H^{2\delta+2i}(X/K)(i), \quad -\delta \leq i \leq \delta,$$

and define the *Mukai isocrystal of* X/K to be the F-isocrystal

$$\widetilde{H}(X/K) := \oplus_i \widetilde{H}^i(X/K).$$

Just as for the Hodge realization, there is a pairing

(2.3.2) $$\langle \cdot, \cdot \rangle : \widetilde{H}(X/K) \times \widetilde{H}(X/K) \to K(-d)$$

given by

$$\langle (a_{-\delta}, a_{-\delta+1}, \ldots, a_{\delta-1}, a_\delta), (a'_{-\delta}, a'_{-\delta+1}, \ldots, a'_{\delta-1}, a'_\delta) \rangle := \sum_{i=-\delta}^{0} (-1)^i a_{-i} \cdot a_i.$$

Here $a_i \in H^{2\delta+2i}(X/K)(i)$, and we identify $H^{4\delta}(X/K)$ with $K(-d)$ using (2.3.1). Note that this pairing is compatible with the F-isocrystal structure.

Now suppose Y is a second smooth proper k-scheme of the same dimension as X. An object $P \in K(X \times Y)$ defines a morphism of Mukai isocrystals

$$\Psi_P : \widetilde{H}(X/K) \to \widetilde{H}(Y/K)$$

as follows. This map is the sum of maps

$$\Psi_P^{i,j} : H^{2\delta+2i}(X/K)(i) \to H^{2\delta+2j}(Y/K)(j)$$

defined as the composite

$$\begin{array}{ccc} H^{2\delta+2i}(X/K)(i) & \xrightarrow{\mathrm{pr}_1^*} & H^{2\delta+2i}(X \times Y/K)(i) \\ & & \downarrow \cdot \mathrm{ch}_{\mathrm{cris}}^{j-i+d_X} \\ H^{2\epsilon+2j}(Y/K)(j) & \longleftarrow & H^{2\delta+2j+2d_X}(X \times Y/K)(j+d_X) \end{array}$$

To conform with the more standard transformation on the Mukai lattice in Hodge theory, for an object $P \in K(X)$ we define

(2.3.3) $$\Phi_P : \widetilde{H}(X/K) \to \widetilde{H}(Y/K)$$

to be the map defined by the product of the $\mathrm{ch}(P)$ and the square root of the Todd class $\sqrt{\mathrm{Todd}(X \times Y)}$. Since the map

$$K(X \times Y)_{\mathbf{Q}} \to A^*(X \times Y)_{\mathbf{Q}}$$

is an isomorphism, the map Φ is still a morphism of F-isocrystals.

2.4. **Étale realization.** Let k be a field of characteristic p, and fix a prime ℓ distinct from p. Fix also a separable closure $k \hookrightarrow \bar{k}$, and let G_k denote the Galois group of $\bar{k}$ over k. The étale realization of the Mukai motive is given by the G_k-module

$$\widetilde{H}(X, \mathbf{Z}_\ell) := \oplus H^{2\delta+2i}(X_{\bar{k}}, \mathbf{Z}_\ell)(i), -\delta \leq i \leq \delta.$$

The cycle class maps $\mathrm{CH}^i(X) \to \mathrm{H}^{2i}(X_{\bar{k}}, \mathbf{Z}_\ell(i))$, Gysin maps, etc., yield identical funtorialities to the crystalline case, and the usual formula yields a Mukai pairing.

When X is defined over a finite field $\mathbf{F}_q$, the qth-power Frobenius gives an action of the arithmetic (and geometric) Frobenius on $\widetilde{H}(X \otimes \overline{\mathbf{F}}_q, \mathbf{Z}_\ell)$. Given X, Y, and $P \in \mathrm{D}(X \times Y)$, all defined over $\mathbf{F}_q$, we get a Frobenius invariant map

$$\Psi : \widetilde{H}(\overline{X}, \mathbf{Q}_\ell) \to \widetilde{H}(\overline{Y}, \mathbf{Q}_\ell).$$

In particular, the characteristic polynomial of Frobenius on the ℓ-adic Mukai lattice is preserved by Fourier-Mukai equivalence.

2.5. **Chow realization.** For a scheme X proper and smooth over a field k, let $\mathrm{CH}(X)$ denote the graded group of algebraic cycles on X modulo numerical equivalence.

If X and Y are two smooth proper k-schemes of the same even dimension $d = 2\delta$, and if $P \in \mathrm{D}(X \times Y)$ is a perfect complex, then we can consider the class $\beta(P) := \mathrm{ch}(P) \cdot \sqrt{\mathrm{Td}_{X \times Y}} \in \mathrm{CH}(X \times Y)$. This class induces a map

$$\Phi_P^{\mathrm{CH}} : \mathrm{CH}(X) \to \mathrm{CH}(Y),$$

defined by the formula

$$\Phi_P^{\mathrm{CH}}(\alpha) = \mathbf{pr}_{2*}(\mathbf{pr}_1^*(\alpha) \cup \beta(P)).$$

In the case when k is a perfect field of positive characteristic, the cycle class map defines maps

$$\mathbf{cl}_X : \mathrm{CH}(X) \to \widetilde{H}(X/K), \quad \mathbf{cl}_Y : \mathrm{CH}(Y) \to \widetilde{H}(Y/K)$$

and

$$\mathbf{cl}_X : \mathrm{CH}(X) \to \widetilde{H}(X, \mathbf{Z}_\ell), \quad \mathbf{cl}_Y : \mathrm{CH}(Y) \to \widetilde{H}(Y, \mathbf{Z}_\ell).$$

Proposition 2.6. *The diagrams*

$$\begin{array}{ccc} \mathrm{CH}(X) & \xrightarrow{\Phi_P^{\mathrm{CH}}} & \mathrm{CH}(Y) \\ \downarrow{\scriptstyle \mathbf{cl}_X} & & \downarrow{\scriptstyle \mathbf{cl}_Y} \\ \widetilde{H}(X/K) & \xrightarrow[\Phi_P]{} & \widetilde{H}(Y/K) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{CH}(X) & \xrightarrow{\Phi_P^{\mathrm{CH}}} & \mathrm{CH}(Y) \\ \downarrow{\scriptstyle \mathbf{cl}_X} & & \downarrow{\scriptstyle \mathbf{cl}_Y} \\ \widetilde{H}(X, \mathbf{Z}_\ell) & \xrightarrow[\Phi_P]{} & \widetilde{H}(Y, \mathbf{Z}_\ell) \end{array}$$

commute.

Proof. This follows from the fact that the cycle class map commutes with smooth pullback, proper pushforward, and cup product. □

2.7. It will be useful to consider the codimension filtration $F_X^{\cdot}$ on

$$\mathrm{CH}(X) = \oplus_i \mathrm{CH}^i(X)$$

given by

$$F_X^s := \oplus_{i \geq s} \mathrm{CH}^i(X).$$

If X and Y are smooth proper k-schemes, and $P \in \mathrm{D}(X \times Y)$ is a perfect complex, then we say that

$$\Phi_P^{\mathrm{CH}} : \mathrm{CH}(X) \to \mathrm{CH}(Y)$$

is *filtered* if it preserves the codimension filtration. We will also sometimes refer to the functor

$$\Phi_P : \mathrm{D}(X) \to \mathrm{D}(Y)$$

as being filtered, meaning that Φ_P^{CH} is filtered (this apparently abusive terminology is justified by the theorem of Orlov recalled in 3.4 below, which implies that P is determined by the equivalence $\Phi_P : \mathrm{D}(X) \to \mathrm{D}(Y)$).

Observe that in the case when X and Y are surfaces, we have

$$F_X^0 = \mathrm{CH}(X), F_X^1 = \mathrm{CH}^1(X) \oplus \mathrm{CH}^2(X),\ F_X^2 = \mathrm{CH}^2(X),$$

and F_X^1 is the subgroup of elements orthogonal to F_X^2. Therefore in the case of surfaces, Φ_P^{CH} is filtered if and only if

$$\Phi_P^{\mathrm{CH}}(F_X^2) = F_Y^2.$$

2.8. **De Rham realization.** For a field k of characteristic 0, and a proper smooth k-scheme of even dimension 2δ, we can also consider the de Rham version of the Mukai isocrystal, as we now explain.

For a smooth proper k-scheme X/k, let $H^s_{\mathrm{dR}}(X/k)$ denotes the s-th de Rham cohomology group of X. Recall that this is a filtered vector space with filtration $\mathrm{Fil}_{\mathrm{dR}}$ defined by the Hodge filtration.

In general, if $\mathscr{V} = (V, F^\cdot)$ is a vector space with a decreasing filtration $F^\cdot$, define for an integer n the *n-th Tate twist of $\mathscr{V}$*, denoted $\mathscr{V}(n)$, to be the filtered vector space with the same underlying vector space V, but whose filtration in degree s is given by F^{s+n}.

This Tate twist operation on filtered vector spaces is as usual necessary to formula Poincare duality. Namely, if X/k is a proper smooth k-scheme of dimension d, then the Chern class of a point defines an isomorphism of filtered vector spaces

$$k \to H^{2d}(X/k)(d),$$

where k is viewed as a filtered vector space with $F^s = k$ for $s \geq 0$ and $F^s = 0$ for $s < 0$. Poincare duality then gives a perfect pairing in the category of filtered vector spaces

$$H^i_{\mathrm{dR}}(X/k) \otimes H^{2d-i}_{\mathrm{dR}}(X/k) \to K(-d).$$

Now in the case when X is of even dimension 2δ, we set

$$\widetilde{H}_{\mathrm{dR}}(X/k) := \oplus_{i=-\delta}^{\delta} H^{2\delta+2i}_{\mathrm{dR}}(X/k)(i).$$

This has an inner product, called the ***Mukai pairing***, taking values in $k(-d)$ defined by the same formula as in (2.3.2).

Remark 2.9. In the case when X is a surface, so we have

$$\widetilde{H}_{\mathrm{dR}}(X/k) = H^0_{\mathrm{dR}}(X/k)(-1) \oplus H^2_{\mathrm{dR}}(X/k) \oplus H^4_{\mathrm{dR}}(X/k)(1),$$

the filtration is given by

$$\mathrm{Fil}^2 = \mathrm{Fil}^2 H^2_{\mathrm{dR}}(X/k),\ \ \mathrm{Fil}^1 = H^0_{\mathrm{dR}}(X/k) \oplus \mathrm{Fil}^1 H^2_{\mathrm{dR}}(X/k) \oplus H^4_{\mathrm{dR}}(X/k),$$

and Fil^s is equal to $\widetilde{H}_{\mathrm{dR}}(X/k)$ (resp. 0) for $s \geq 0$ (resp. $s < 2$). Note that this also shows that Fil^1 is the orthogonal complement under the Mukai pairing of Fil^2.

2.10. **Crystalline and de Rham comparison.** Consider now a complete discrete valuation ring V with perfect residue field k and field of fractions K. Let $W \subset V$ be the ring of Witt vectors of k, and let $K_0 \subset K$ be its field of fractions. Let $\mathscr{X}/V$ be a proper smooth scheme of even relative dimension 2δ, and let $\mathscr{X}_s$ (resp. $\mathscr{X}_\eta$) denote the closed (resp. generic) fiber. We then have the Berthelot-Ogus comparison isomorphism

$$H^*_{\mathrm{cris}}(\mathscr{X}_s/K_0) \otimes_{K_0} K \to H^*_{\mathrm{dR}}(\mathscr{X}_\eta/K).$$

This isomorphism induces an isomorphism of graded K-vector spaces

$$\sigma_{\mathscr{X}} : \widetilde{H}_{\mathrm{cris}}(\mathscr{X}_s/K_0) \otimes_{K_0} K \to \widetilde{H}_{\mathrm{dR}}(\mathscr{X}_\eta/K).$$

Because the comparison isomorphism between crystalline cohomology and de Rham cohomology is compatible with cup product and respects the cohomology class of a point, the map σ is compatible with the Mukai pairings on both sides.

Now suppose given two proper smooth V-schemes $\mathscr{X}$ and $\mathscr{Y}$ of the same even dimension, and let X and Y respectively denote their reductions to k. Suppose further given a perfect complex P on $X \times Y$ such that the induced map

$$\Phi_P^{\mathrm{cris}} : \widetilde{H}_{\mathrm{cris}}(X/K_0) \to \widetilde{H}_{\mathrm{cris}}(Y/K_0)$$

is an isomorphism. We then get an isomorphism
(2.10.1)

$$\widetilde{H}_{\mathrm{dR}}(\mathscr{X}_\eta) \xrightarrow{\sigma_{\mathscr{X}}^{-1}} \widetilde{H}_{\mathrm{cris}}(X/K_0) \otimes K \xrightarrow{\Phi_P^{\mathrm{cris}}} \widetilde{H}_{\mathrm{cris}}(Y/K_0) \otimes K \xrightarrow{\sigma_{\mathscr{Y}}} \widetilde{H}_{\mathrm{dR}}(\mathscr{Y}_\eta).$$

Definition 2.11. The families $\mathscr{X}/V$ and $\mathscr{Y}/V$ are called *P-compatible* if the composite morphism (2.10.1) respect the Hodge filtrations.

2.12. **Mukai vectors of perfect complexes.** Let X be a smooth projective geometrically connected scheme over a field k.

Definition 2.13. Given a perfect complex $E \in \mathrm{D}(X)$, the *Mukai vector* of E is

$$v(E) := \mathrm{ch}(E)\sqrt{\mathrm{Td}_X} \in \mathrm{CH}(X) \otimes \mathbf{Q}.$$

In the case when X is a K3 surface, the Mukai vector of a complex E is given by (see for example [H2, p. 239])

$$v(E) = (\mathbf{rk}(E), c_1(E), \mathbf{rk}(E) + c_1(E)^2/2 - c_2(E)).$$

In particular, using that the Todd class of the tangent bundle of X is $(1, 0, 2)$, one gets by Grothendieck-Riemann-Roch that for two objects $E, F \in \mathrm{D}(X)$ we have

$$\langle v(E), v(F) \rangle = -\chi(E, F).$$

As a consequence, if E is a simple torsion free sheaf on a K3 surface X, the universal deformation of E (keeping X fixed) is formally smooth of dimension $v(E)^2 - 2$, hinting that the Mukai lattice captures the numerology needed to study moduli and deformations. (A review of the standard results in this direction may be found in Section 3.11 below.)

The compatibility of the Chow realization with the crystalline, étale, and de Rham realizations yields Mukai vectors in each of those realizations, satisfying the same rule.

3. Kernels of Fourier-Mukai equivalences

3.1. **Generalities.** Let X and Y be proper smooth schemes over a field k. For a perfect complex P on $X \times Y$, consider the functor

$$\Phi_D^P : \mathrm{D}(X) \to \mathrm{D}(Y)$$

given by

$$\Phi_D^P(K) := \mathrm{pr}_{2*}(P \otimes^{\mathbf{L}} \mathrm{pr}_1^* K).$$

Let $P^\vee$ denote the complex

$$P^\vee := R\mathscr{H}om(P, \mathscr{O}_{X\times Y}),$$

which we view as a perfect complex on $Y \times X$ (switching the factors). Let ω_X (resp. ω_Y) denote the highest exterior power of Ω^1_X (resp. Ω^1_Y).

Let

$$G : \mathrm{D}(Y) \to \mathrm{D}(X) \ \ (\text{resp. } H : \mathrm{D}(Y) \to \mathrm{D}(X)) \tag{3.1.1}$$

denote

$$\Phi_D^{P^\vee \otimes \pi_Y^* \omega_Y[\dim(Y)]} \ \ (\text{resp. } \Phi_D^{P^\vee \otimes \pi_X^* \omega_X[\dim(X)]}),$$

where π_X and π_Y denote the projections from $X \times Y$. From Grothendieck duality one gets:

Proposition 3.2 ([B, 4.5])**.** *The functor G (resp. H) is left adjoint (resp. right adjoint) to Φ_D^P.*

The adjunction maps

$$\eta : G \circ \Phi_D^P \to \mathrm{id}, \ \ \epsilon : \mathrm{id} \to H \circ \Phi_D^P$$

are obtained as follows.

In general if X, Y, and Z are proper smooth k-schemes, P is a perfect complex $X \times Y$, and Q is a perfect complex on $Y \times Z$, then the composite functor

$$\mathrm{D}(X) \xrightarrow{\Phi_D^P} \mathrm{D}(Y) \xrightarrow{\Phi_D^Q} \mathrm{D}(Z)$$

is equal to

$$\Phi_D^{\gamma_{X\times Z*}(\gamma^*_{X\times Y}(P) \otimes^{\mathbf{L}} \gamma^*_{Y\times Z} Q)},$$

where $\gamma_{X\times Z}$, $\gamma_{X\times Y}$, and $\gamma_{Y\times Z}$ are the various projections from $X \times Y \times Z$.

In particular, taking $Z = X$ and $Q = P^\vee \otimes \pi_Y \omega_Y[\dim(Y)]$, we get that the composition $G \circ \Phi_D^P$ is equal to

$$\Phi_D^{\gamma_{X\times X*}(\gamma^*_{X\times Y}(P) \otimes \gamma^*_{Y\times X}(P^\vee \otimes \sigma_Y^* \omega_Y[\dim(Y)])}.$$

The adjunction

$$\eta : G \circ \Phi_D^P \to \mathrm{id}$$

is realized by a map

$$\bar{\eta} : \gamma_{X\times X*}(\gamma^*_{X\times Y}(P) \otimes \gamma^*_{Y\times X}(P^\vee \otimes \sigma_Y^* \omega_Y[\dim(Y)]) \to \Delta_* \mathscr{O}_X, \tag{3.2.1}$$

where we note that

$$\Phi_D^{\Delta_* \mathscr{O}_X} = \mathrm{id}.$$

This map $\bar{\eta}$ is adjoint to the map

$$\begin{array}{c} \Delta^*\gamma_{X\times X*}(\gamma^*_{X\times Y}(P)\otimes\gamma^*_{Y\times X}(P^\vee\otimes\sigma^*_Y\omega_Y[\dim(Y)]) \\ \downarrow \simeq \\ \pi_{X*}(P\otimes P^\vee\otimes\pi^*_Y\omega_Y[\dim(Y)]) \\ \downarrow P\otimes P^\vee\to\mathrm{id} \\ \pi_{X*}\pi^*_Y\omega_Y[\dim(Y)] \\ \downarrow \simeq \\ R\Gamma(Y,\omega_Y[\dim(Y)])\otimes_k\mathscr{O}_X \\ \downarrow \mathrm{tr}\otimes 1 \\ \mathscr{O}_X. \end{array}$$

Similarly, the composite $H\circ\Phi^P_D$ is induced by the perfect complex

$$\gamma_{X\times X*}(\gamma^*_{X\times Y}P\otimes\gamma^*_{Y\times X}(P^\vee\otimes\pi^*_X\omega_X[\dim(X)]))$$

on $X\times X$. There is a natural map

$$\bar{\epsilon}:\Delta_*\mathscr{O}_X\to\gamma_{X\times X*}(\gamma^*_{X\times Y}P\otimes\gamma^*_{Y\times X}(P^\vee\otimes\pi^*_X\omega_X[\dim(X)])) \tag{3.2.2}$$

which induces the adjunction map

$$\epsilon:\mathrm{id}\to H\circ\Phi^P_D.$$

The map $\bar{\epsilon}$ is obtained by noting that

$$\Delta^!\gamma_{X\times X*}(\gamma^*_{X\times Y}P\otimes\gamma^*_{Y\times X}(P^\vee\otimes\pi^*_X\omega_X[\dim(X)]))\simeq\pi_{X*}(P\otimes P^\vee),$$

so giving the map $\bar{\epsilon}$ is equivalent to giving a map

$$\mathscr{O}_X\to\pi_*(P\otimes P^\vee), \tag{3.2.3}$$

and this is adjoint to a map

$$\mathscr{O}_{X\times Y}\to P\otimes P^\vee.$$

Taking the natural scaling map for the latter gives rise to the desired map $\bar{\epsilon}$.

Proposition 3.3. *The functor Φ^P_D is an equivalence if and only if the maps (3.2.1) and (3.2.2) are isomorphisms.*

Proof. For a closed point $x\in X(k)$, let $P_x\in D(Y)$ denote the object obtained by pulling back P along

$$Y\simeq\mathrm{Spec}(k)\times Y\xrightarrow{i_x\times\mathrm{id}}X\times Y,$$

where $i_x:\mathrm{Spec}(k)\hookrightarrow X$ is the closed immersion corresponding to x. If we write $\mathscr{O}_x$ for $i_{x*}\mathscr{O}_{\mathrm{Spec}(k)}$, then we have

$$P_x=\Phi^P_D(\mathscr{O}_x).$$

In particular, if Φ_D^P is fully faithful, then we have for any two closed points $x, x' \in X(k)$

$$RHom\,(P_x, P_{x'}) = \begin{cases} k & \text{if } x = x' \\ 0 & \text{if } x \neq x'. \end{cases} \tag{3.3.1}$$

Moreover, it is shown in [B, 5.1 and 5.4] that Φ_D^P is an equivalence if and only if (3.3.1) holds and moreover

$$P_x \otimes \omega_Y \simeq P_x \tag{3.3.2}$$

for every closed point $x \in X(k)$. To prove the proposition it therefore suffices to show that if (3.3.1) and (3.3.2) hold then the maps (3.2.1) and (3.2.2) are isomorphisms.

Since the cartesian square

$$\begin{array}{ccc} \mathrm{Spec}(k) & \xrightarrow{i_x} & X \\ \downarrow{\scriptstyle i_x} & & \downarrow{\scriptstyle \Delta_X} \\ X & \xrightarrow{i_x \times \mathrm{id}} & X \times X \end{array}$$

is tor-independent, we have

$$(i_x \times \mathrm{id})^* \Delta_* \mathscr{O}_X \simeq \mathscr{O}_x.$$

To show that (3.2.1) and (3.2.2) are isomorphisms, it suffices to show that they induce isomorphisms upon applying $(i_x \times \mathrm{id})^*$ for all closed points $x \in X(k)$ (for then the cones have empty support).

We have

$$(i_x \times \mathrm{id})^*(\gamma_{X\times X*}(\gamma_{X\times Y}^* P \otimes \gamma_{Y\times X}^*(P^\vee \otimes \pi_Y^*\omega_Y[\dim(Y)]))) \simeq \pi_{X*}(P^\vee \otimes \pi_Y^*(P_x \otimes \omega_Y[\dim(Y)])),$$

and

$$(i_x \times \mathrm{id})^*(\gamma_{X\times X*}(\gamma_{X\times Y}^* P \otimes \gamma_{Y\times X}^*(P^\vee \otimes \pi_Y^*\omega_Y[\dim(Y)]))) \simeq \pi_{X*}(\pi_Y^* P_x \otimes P^\vee) \otimes \omega_X[\dim(X)].$$

Under these identifications, the pullback of the map (3.2.1) is identified with the map

$$\pi_{X*}(P^\vee \otimes \pi_Y^*(P_x \otimes \omega_Y[\dim(Y)])) \to i_{x*}k \tag{3.3.3}$$

adjoint to the map

$$i_x^* \pi_{X*}(P^\vee \otimes \pi_Y^*(P_x \otimes \omega_Y[\dim(Y)])) \simeq R\Gamma(Y, P_x \otimes P_x^\vee \otimes \omega_Y[\dim(Y)]) \to k$$

defined by the natural map

$$P_x \otimes P_x^\vee \to \mathscr{O}_Y$$

and the trace map

$$R\Gamma(Y, \omega_Y[\dim(Y)]) \to k.$$

The pullback of the map (3.2.2) is identified with the map

$$i_{x*}k \to \pi_{X*}(\pi_Y^* P_x \otimes P^\vee) \otimes \omega_X[\dim(X)] \tag{3.3.4}$$

adjoint to the map

$$k \to i_x^! \pi_{X*}(\pi_Y^* P_x \otimes P^\vee) \otimes \omega_X[\dim(X)] \simeq R\Gamma(Y, P_x \otimes P_x^\vee (i_x \times \mathrm{id})^! \pi_X^* \omega_X[\dim(X)])$$

induced by the natural map

$$k \to R\Gamma(Y, P_x \otimes P_x^\vee)$$

and the observation that $\pi_X^* \omega_X[\dim(X)] \simeq \pi_Y^! \mathscr{O}_Y$ so that

$$(i_x \times \mathrm{id})^! \pi_X^* \omega_X[\dim(X)] \simeq \mathscr{O}_Y.$$

If $x' \neq x$ is a second closed point of X (possibly equal to x), then we have

$$i_{x'}^* \pi_{X*}(P^\vee \otimes \pi_Y^*(P_x \otimes \omega_Y[\dim(Y)])) \simeq R\Gamma(Y, P_x \otimes P_{x'}^\vee \otimes \omega_Y[\dim(Y)])$$

which is dual to

$$R\mathrm{Hom}(P_x, P_{x'}).$$

Similarly, we have

$$i_{x'}^*(\pi_{X*}(P^\vee \otimes \pi_Y^* P_x) \otimes \omega_X[\dim(X)]) \simeq R\Gamma(Y, P_x \otimes P_{x'}^\vee) \otimes \omega_{X,x}[\dim(X)]$$

which is isomorphic to

$$R\mathrm{Hom}(P_{x'}, P_x) \otimes \omega_{X,x}[\dim(X)].$$

In particular, these complexes are zero for $x \neq x'$, so we conclude that

$$\pi_{X*}(P^\vee \otimes \pi_Y^* P_x) \otimes \omega_X[\dim(X)], \quad \text{and} \quad \pi_{X*}(P^\vee \otimes \pi_Y^*(P_x \otimes \omega_Y[\dim(Y)]))$$

are supported on some infinitesimal neighborhood of x in X. In particular, to prove that (3.3.3) and (3.3.4) are isomorphisms it suffices to show that the induce isomorphisms after applying $i_x^!$ (again consider the cones).

Now we have

$$i_x^! \pi_{X*}(P^\vee \otimes \pi_Y^* P_x \otimes \pi_X^* \omega_X[\dim(X)]) \simeq R\Gamma(Y, P_x \otimes P_x^\vee) \simeq R\mathrm{Hom}(P_x, P_x) \simeq k$$

which implies that (3.3.4) is an isomorphism. To see that (3.3.3) induces an isomorphism upon applying $i_x^!$, note that we have

$$\begin{aligned} \pi_{X*}(P^\vee \otimes \pi_Y^*(P_x \otimes \omega_Y[\dim(Y)])) &\simeq G \circ \Phi_D^P(\mathscr{O}_x) \\ &\simeq H \circ \Phi_D^P(\mathscr{O}_x) \\ &\simeq \pi_{X*}(P^\vee \otimes \pi_Y^* P_x \otimes \pi_X^* \omega_X[\dim(X)]), \end{aligned}$$

and therefore

$$i_x^! \pi_{X*}(P^\vee \otimes \pi_Y^*(P_x \otimes \omega_Y[\dim(Y)])) \simeq k$$

as well, and the map (3.3.3) induces an isomorphism upon applying $i_x^!$. □

Let us also recall the following two results which will be relevant in the following discussion.

Theorem 3.4 (Orlov [Or, 2.2])**.** *Let X and M be smooth projective schemes over a field k, and let*

$$F : \mathrm{D}(X) \to \mathrm{D}(M)$$

be an equivalence of triangulated categories. Then $F = \Phi^P$ for a perfect complex P on $X \times M$, and the complex P is unique up to isomorphism.

Proposition 3.5. *If X is a K3 surface over k and Y is a smooth projective k-scheme such that there is an equivalence $\mathrm{D}(X) \xrightarrow{\sim} \mathrm{D}(Y)$ then Y is a K3 surface.*

Proof. We follow the argument of [B, proof of 5.4]. (If one is willing to accept the HKR-isomorphism then there is a more elegant proof along the lines of Corollary 10.2 of [H2]. We give a proof here that is easily seen to be independent of characteristic 0 methods.)

By Orlov's theorem 3.4, there exists a perfect complex P on $X \times Y$ inducing an equivalence of triangulated categories. Consider the adjoints G and H of Φ^P defined in (3.1.1). Since F is an equivalence, we see that $G \simeq H$, and by the uniqueness part of Orlov's theorem we have

$$P^\vee \otimes \pi_Y^* \omega_Y[\dim(Y)] \simeq P^\vee \otimes \pi_X^* \omega_X[\dim(X)].$$

Taking determinants on both sides and cancelling one factor of $\det(P^\vee)$ we get that

$$\pi_Y^* \omega_Y[\dim(Y)] \simeq \pi_X^* \omega_X[\dim(X)].$$

This implies that $\dim(Y) = \dim(X)$ so Y is a surface. Furthermore, since X is a K3 surface we have $\omega_X \simeq \mathscr{O}_X$. It follows that

$$\pi_Y^* \omega_Y \simeq \mathscr{O}_{X \times Y}.$$

Applying $R^0 \pi_{Y*}$ we get that $\omega_Y \simeq \mathscr{O}_Y$.

Fix a prime ℓ invertible in k. The kernel $P \in \mathrm{D}(X \times Y)$ gives rise to an algebraic class

$$\mathrm{ch}(P)\sqrt{\mathrm{Td}_{X \times Y}} \in \bigoplus_{i \text{ even}} \mathrm{H}^i(X \times Y, \mathbf{Q}_\ell)$$

that induces two isomorphisms of $\mathbf{Q}_\ell$-modules

$$\bigoplus_{i \text{ even}} \mathrm{H}^i(X, \mathbf{Q}_\ell) \xrightarrow{\sim} \bigoplus_{i \text{ even}} \mathrm{H}^i(Y, \mathbf{Q}_\ell)$$

and

$$\bigoplus_{i \text{ odd}} \mathrm{H}^i(X, \mathbf{Q}_\ell) \xrightarrow{\sim} \bigoplus_{i \text{ odd}} \mathrm{H}^i(Y, \mathbf{Q}_\ell).$$

Since X is K3, all odd $\mathbf{Q}_\ell$-adic cohomology vanishes, and we conclude the same for Y.

It follows that Y is a smooth projective surface with trivial canonical sheaf, $b_2 = 22$, and $b_1 = 0$. By definition, this makes Y a K3 surface. (See e.g. [BM].) □

We end this section with a brief review of some standard kernels.

3.6. **Tensoring with line bundles.** Let k be a field and X/k a smooth projective scheme. Let L be a line bundle on X. Then the equivalence of triangulated categories

$$\mathrm{D}(X) \to \mathrm{D}(X), \quad K \mapsto K \otimes^{\mathbf{L}} L$$

is induced by the kernel $P := \Delta_* L$ on $X \times X$, where $\Delta : X \to X \times X$ is the diagonal. In the case when X is a surface the corresponding action on the Mukai-motive is given by the map sending $(a, b, c) \in \mathrm{CH}(X)$ to

$$(a, b + ac_1(L), c + b \cdot c_1(L) + ac_1(L)^2/2). \tag{3.6.1}$$

3.7. **Spherical twist.** Recall the following definition (see e.g., Definition 8.1 of [H2]).

Definition 3.8. A perfect complex $E \in \mathrm{D}(X)$ is *spherical* if

(1) $E \overset{\mathbf{L}}{\otimes} \omega_X \cong E$
(2) $\mathrm{Ext}^i(E, E) = 0$ unless $i = 0$ or $i = \dim(X)$, and in those cases we have $\dim \mathrm{Ext}^i(E, E) = 1$

In other words, $\mathbf{R}\mathscr{H}om(E, E)$ has the cohomology of a sphere. A standard example is given by the structure sheaf of a (-2)-curve in a K3 surface.

The trace map

$$\mathbf{R}\mathscr{H}om(E, E) \to \mathscr{O}_X$$

defines a morphism

$$t : \mathbf{L}p^*\mathscr{E}^\vee \otimes \mathbf{L}q^*\mathscr{E} \to \mathbf{R}\Delta_*\mathscr{O}_X$$

in $\mathrm{D}(X \times X)$, where p and q are the two projections. Define P_E to be the cone over t. The following result dates to work of Kontsevich, Seidel and Thomas.

Theorem 3.9 (Proposition 8.6 of [H2]). *The transform*

$$T_E : \mathrm{D}(X) \to \mathrm{D}(X)$$

induced by P_E is an equivalence of derived categories.

This transform is called a spherical twist. We know that P_E also acts by a reflection on cohomology.

Proposition 3.10 (Lemma 8.12 of [H2]). *Suppose X is a K3 surface and P_E is the complex associated to a spherical object $E \in \mathrm{D}(X)$ as above. Let H be any realization of the Mukai motive described in the preceding sections (étale, crystalline, de Rham, Chow) and let $v \in H$ be the Mukai vector of P_E. Then $v^2 = 2$ and the induced map*

$$\Phi_{P_E} : H(X) \to H(X)$$

is the reflection in v:

$$\Phi_{P_E}(x) = x - (x \cdot v)v.$$

3.11. **Moduli spaces of vector bundles.** Let X be a K3 surface over a field k. One of Mukai and Orlov's wonderful discoveries is that one can produce Fourier-Mukai equivalences between X and moduli spaces of sheaves on X by using tautological sheaves.

If S is a scheme and E is a locally finitely presented quasi-coherent sheaf on $X \times_S$ flat over S, then we get a function on the points of S to $\mathrm{CH}(X)$ by sending a point s to the Mukai vector of the restriction E_s of E to the fiber over s. This function is a locally constant function on S, and so if S is connected it makes sense to talk about the Mukai vector of E, which is defined to be the Mukai vector of E_s for any $s \in S$.

For any ample class h on X, let $\mathscr{M}_h$ denote the algebraic stack of Gieseker-semistable sheaves on X, where semistability is defined using h. (A good summary of the standard results on these moduli spaces may be found in Section 10.3 of [H2], with a more comprehensive treatment in [HL] and some additional non-emptiness results in [M].) If we fix a vector $v \in \mathrm{CH}(X)$, we then get an open and closed substack $\mathscr{M}_h(v) \subset \mathscr{M}_h$ classifying semistable sheaves on X

with Mukai vector v. Since the Mukai vector of a sheaf determines the Hilbert polynomial, the stack $\mathscr{M}_h(v)$ is an algebraic stack of finite type over k. In fact, it is a GIT quotient stack with projective GIT quotient variety.

Theorem 3.12 (Theorem 5.1 of [M], Section 4.2 of [Or])**.** *Suppose X is a K3 surface over an algebraically closed field k.*

(1) *If $v \in \mathrm{CH}(X)$ is primitive and $v^2 = 0$ (with respect to the Mukai pairing) then $\mathscr{M}_h(v)$ is non-empty.*
(2) *If, in addition, there is a vector v' such that $\langle v, v' \rangle = 1$ then every semistable sheaf with Mukai vector v is locally free and geometrically stable (Remark 6.1.9 of* [HL]*), in which case $\mathscr{M}_h(v)$ is a $\boldsymbol{\mu}_r$-gerbe over a smooth projective $K3$ surface such that the associated $\mathbf{G}_m$-gerbe is in fact trivial (in older language, a tautological family exists on $X \times M_h(v)$).*
(3) *In this case, a tautological family $\mathscr{E}$ on $X \times M_h(v)$ induces a Fourier-Mukai equivalence*
$$\Phi_{\mathscr{E}} : \mathrm{D}(M_h(v)) \to \mathrm{D}(X),$$
and thus in the case when $k = \mathbf{C}$ an isomorphism of Mukai Hodge lattices.

Finally, if $k = \mathbf{C}$, then any FM partner of X is of this form.

Remark 3.13. While the non-emptiness uses the structure of analytic moduli of K3 surfaces (and deformation to the Kummer case), it still holds over any algebraically closed field. One can see this by lifting v and h together with X using Deligne's theorem and then specializing semistable sheaves using Langton's theorem.

Remark 3.14. Using the results of [H1], one can restrict to working with moduli of locally free slope-stable sheaves from the start.

4. Fourier-Mukai transforms and moduli of complexes

In this section, we extend the philosophy of Mukai and Orlov and show that fully faithful FM kernels on $X \times Y$ correspond to certain open immersions of Y into a moduli space of complexes on X. We start be reviewing the basic results of [L] on moduli of complexes.

Given a proper morphism of finite presentation between schemes $Z \to S$, define a category fibered in groupoids as follows: the objects over an S-scheme $T \to S$ are objects $E \in \mathrm{D}(\mathscr{O}_{Z_T})$ such that

(1) for any quasi-compact T-scheme $U \to T$ the complex $E_U := \mathbf{L}p^*E$ is bounded, where $p : Z_U \to Z_T$ is the natural map ("E is relatively perfect over T");
(2) for every geometric point $s \to S$, we have that $\mathrm{Ext}^i(E_s, E_s) = 0$ for all $i < 0$ ("E is universally gluable").

By [L, theorem on p. 2] this category fibered in groupoids is an Artin stack locally of finite presentation over S. This stack is denoted $\mathscr{D}_{Z/S}$ (or just $\mathscr{D}_Z$ when S is understood).

Remark 4.1. In [L, Proposition 2.1.9], it is proven that if $f : Z \to S$ is flat then a relatively perfect complex E is universally gluable if and only if the second condition holds after base change to geometric points of S, and it is straightforward to see that it suffices to check at closed points of S when the closed points are everywhere dense (e.g., S is of finite type over a field).

Furthermore, suppose $f : Z \to S$ is smooth and that S is of finite type over a field k. In this case it suffices to verify both conditions for geometric points lying over closed points of S (i.e., we need not even assume E is relatively perfect in order to get a fiberwise criterion). This can be seen as follows. First of all we may without loss of generality assume that k is algebraically closed. Suppose condition (2) holds for all closed points, and let $\bar{\eta} \to S$ be a geometric point lying over an arbitrary point $\eta \in S$. Let $Z \hookrightarrow S$ be the closure of η with the reduced structure. Replacing S by Z we can assume that S is integral with generic point η, and after shrinking further on S we may also assume that S is smooth over k. Consider the complex

$$Rf_*\mathscr{R}Hom(E, E) \tag{4.1.1}$$

on S. The restriction of this complex to $\bar{\eta}$ computes $\mathrm{Ext}^i(E_{\bar{\eta}}, E_{\bar{\eta}})$, so it suffices to show that (4.1.1) is in $D^{\geq 0}(S)$. Since S, and hence also Z is smooth, the complex $\mathscr{R}Hom(E, E)$ is a bounded complex on S. Now by standard base change change results, we can after further shrinking arrange that the sheaves $R^i f_*\mathscr{R}Hom(E, E)$ are all locally free on S, and that their formation commute with arbitrary base change on S. In this case by our assumptions for $i < 0$ these sheaves must be zero since their fibers over any closed point of S is zero.

Recall from [L, 4.3.1] that an object $E \in \mathscr{D}_{Z/S}(T)$ over some S-scheme T is called *simple* if the natural map of algebraic spaces

$$\mathbf{G}_m \to \mathscr{A}ut(E)$$

is an isomorphism. By [L, 4.3.2 and 4.3.3], the substack $s\mathscr{D}_{Z/S} \subset \mathscr{D}_{Z/S}$ classifying simple objects is an open substack, and in particular $s\mathscr{D}_{Z/S}$ is an algebraic stack. Moreover, there is a natural map

$$\pi : s\mathscr{D}_{Z/S} \to \mathrm{sD}_{Z/S}$$

from $s\mathscr{D}_{Z/S}$ to an algebraic space $\mathrm{sD}_{Z/S}$ locally of finite presentation over S which realizes $s\mathscr{D}_{Z/S}$ as a $\mathbf{G}_m$-gerbe.

Fix smooth projective varieties X and Y over a field k.

Notation 4.2. Let $\mathscr{F}$ be the groupoid of perfect complexes $P \in \mathrm{D}(X \times Y)$ such that the transform

$$\Phi_P : \mathrm{D}(X) \to \mathrm{D}(Y)$$

is fully faithful. Let $s\mathscr{D}_Y(X)^\circ$ be the groupoid of morphisms

$$\mu : X \to s\mathscr{D}_Y$$

such that the composed map

$$\begin{array}{ccc} X & \xrightarrow{\mu} & s\mathscr{D}_Y \\ & \searrow^{\bar{\mu}} & \downarrow \\ & & \mathrm{sD}_Y \end{array}$$

is an open immersion.

Lemma 4.3. *Any complex $P \in \mathscr{F}$ defines an object of $s\mathscr{D}_Y(X)$.*

Proof. Since X is smooth the first condition (P has finite Tor-dimension over X) is automatic. It therefore suffices (using Remark 4.1) to show that for any geometric point $\bar{x} \to X$ lying over a closed point of X we have

$$\mathbf{Ext}^i(P_{\bar{x}}, P_{\bar{x}}) = 0$$

for $i < 0$, where $P_{\bar{x}}$ denotes the pullback of P along

$$\bar{x} \times Y \to X \times Y.$$

This follows from equation (3.3.1). □

For a kernel $P \in \mathscr{F}$ let

$$\mu_P : X \to s\mathscr{D}_Y$$

denote the corresponding morphism.

Proposition 4.4. *The functor $P \mapsto \mu_P$ yields an equivalence of groupoids*

$$\Xi : \mathscr{F} \to s\mathscr{D}_Y(X)^\circ.$$

Proof. First we show that for a kernel $P \in \mathscr{F}$ the map $\bar{\mu}_P$ is an open immersion, so that Ξ is well-defined. To show that $\bar{\mu}_P$ is an open immersion it suffices to show that it is an étale monomorphism. For this it suffices in turn to show that for distinct closed points $x_1, x_2 \in X(k)$ the objects P_{x_1} and P_{x_2} are not isomorphic, and that (see [EGA, IV.17.11.1])

$$\Omega^1_{X/s\mathscr{D}_Y} = 0.$$

To see that P_{x_1} and P_{x_2} are not isomorphic for $x_1 \neq x_2$, note that since Φ_P is fully faithful, it yields isomorphisms

$$\mathrm{Ext}^i_X(k(x_1), k(x_2)) \xrightarrow{\sim} \mathrm{Ext}^i_Y(P_{x_1}, P_{x_2})$$

for all i. Thus, if $x_1 \neq x_2$ we have that

$$\mathrm{Hom}_Y(P_{x_1}, P_{x_2}) = 0,$$

so that $P_{x_1} \not\cong P_{x_2}$.

Next we show that the relative differentials vanish.

Recall from [L, 3.1.1] the following description of the tangent space to sD_Y at a point corresponding to a complex E. First of all since

$$\pi : \mathscr{D}_Y \to \mathrm{sD}_Y$$

is a $\mathbf{G}_m$-gerbe, the tangent space $T_{[E]}\,\mathrm{sD}_Y$ to sD_Y at E is given by the set of isomorphism classes of pairs (E', σ), where $E' \in \mathscr{D}_Y(k[\epsilon])$ and $\sigma : E' \otimes^{\mathbf{L}} k \to E$ is an isomorphism. Tensoring the exact sequence

$$0 \to \mathscr{O}_Y \otimes_k (\epsilon) \to \mathscr{O}_{Y_{k[\epsilon]}} \to \mathscr{O}_Y \to 0$$

with E' we see that a deformation (E', σ) of E to $k[\epsilon]$ induces a distinguished triangle

$$E \otimes (\epsilon) \longrightarrow E' \longrightarrow E \xrightarrow{\partial_{(E',\sigma)}} E \otimes (\epsilon)[1].$$

In this way we obtain a map

$$T_{[E]}\,\mathrm{sD}_Y \to \mathrm{Ext}^1_{Y_{k[\epsilon]}}(E, (\epsilon) \otimes E), \quad (E', \sigma) \mapsto [\partial_{(E',\sigma)}]. \tag{4.4.1}$$

This map is a morphism of k-vector spaces. It is injective as the isomorphism class of (E', σ) can be recovered from $\partial_{(E',\sigma)}$ by taking a cone of $\partial_{(E',\sigma)}$ and rotating the resulting triangle.

The image of (4.4.1) can be described as follows. The usual derived adjunction formula gives an isomorphism

$$\mathscr{R}Hom_{\mathscr{O}_{Y_{k[\epsilon]}}}(E, (\epsilon)\otimes E) \simeq \mathscr{R}Hom_{\mathscr{O}_Y}(E\otimes_k (k\otimes^{\mathbf{L}}_{k[\epsilon]} k), (\epsilon)\otimes E).$$

Now we have

$$k\otimes_{k[\epsilon]} k \simeq \oplus_{i\geq 0} k[i],$$

so this gives

$$\mathrm{Ext}^1_{Y_{k[\epsilon]}}(E, (\epsilon)\otimes E) \simeq \oplus_{i\geq 0}\mathrm{Ext}^{1-i}_Y(E, (\epsilon)\otimes E).$$

since E is universally gluable this reduces to an exact sequence

$$0\to \mathrm{Ext}^1_Y(E, (\epsilon)\otimes E)\to \mathrm{Ext}^1_{Y_{k[\epsilon]}}(E, (\epsilon)\otimes E)\to \mathbf{Hom}_Y(E, (\epsilon)\otimes E)\to 0.$$

As explained in [L, proof of 3.1.1], the image of (4.4.1) is exactly

$$\mathrm{Ext}^1_Y(E, (\epsilon)\otimes E)\subset \mathrm{Ext}^1_{Y_{k[\epsilon]}}(E, (\epsilon)\otimes E).$$

Now taking $E = P_x$ for a closed point $x\in X(k)$, we get by applying the fully faithful functor Φ^P an isomorphism

$$\mathrm{Ext}^1_X(k(x), (\epsilon)\otimes_k k(x)) \xrightarrow{\Phi^P} \mathrm{Ext}^1_Y(P_x, (\epsilon)\otimes_k P_x) \xrightarrow{\simeq} T_{[P_x]}\,\mathrm{sD}_Y\,.$$

On the other hand, applying $\mathrm{Hom}_X(-, k(x))$ to the short exact sequence

$$0\to I_x\to \mathscr{O}_X\to k(x)\to 0,$$

where I_x denotes the quasi-coherent sheaf of ideals defining x, we get an exact sequence

$$\mathrm{Hom}_X(\mathscr{O}_X, k(x))\to \mathrm{Hom}_X(I_x, k(x))\to \mathrm{Ext}^1_X(k(x), k(x))\to 0.$$

Since any morphism $\mathscr{O}_X\to k(x)$ factors through $k(x)$, this gives an isomorphism

$$T_xX = \mathrm{Hom}(I_x/I_x^2, k(x))\simeq \mathrm{Ext}^1_X(k(x), k(x)).$$

Putting it all together we find an isomorphism

$$T_xX\simeq T_{[P_x]}\,\mathrm{sD}_Y\,.$$

We leave to the reader the verification that this isomorphism is the map induced by $\bar\mu_P$, thereby completing the proof that $\bar\mu_P$ is étale.

To show that it is an equivalence of groupoids, note that by the definition of the stack $s\mathscr{D}_Y$ the functor Ξ is fully faithful. To see that Ξ is essentially surjective, note that a morphism $X\to s\mathscr{D}_Y$ corresponds to a complex $P\in \mathrm{D}(X\times Y)$ such that for all $x_1, x_2\in X(k)$ we have that

$$\mathrm{Ext}^i_X(k(x_1), k(x_2))\xrightarrow{\sim}\mathrm{Ext}^i_Y(P_{x_1}, P_{x_2}).$$

Indeed, by the calculations above $\overline{\mu}_P$ sets up an isomorphism of exterior algebras

$$\Lambda^*T_xX\xrightarrow{\sim}\Lambda^*T_{[P_x]}\,\mathrm{sD}_Y$$

for each $x\in X$. By [B, 5.1], this implies that Φ^P is fully faithful. □

5. A TORELLI THEOREM IN THE KEY OF D

Fix K3 surfaces X and Y over an algebraically closed field k.

In this section we prove the following derived category version of the Torelli theorem that has no characteristic restrictions. It is similar to the classical Torelli theorem in that it specifies that some kind of "lattice" isomorphism preserves a filtration on an associated linear object.

Theorem 5.1. *If there is a kernel $P \in \mathrm{D}(Y \times X)$ inducing a filtered equivalence $\mathrm{D}(Y) \to \mathrm{D}(X)$ (see Paragraph 2.7) then X and Y are isomorphic.*

We prove some auxiliary results before attacking the proof. By abuse of notation we will write Φ for both Φ_P and Φ_P^{CH}.

Lemma 5.2. *In the situation of Theorem 5.1, we may assume that $\Phi(1,0,0) = (1,0,0)$ and that the induced isometry $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ send the ample cone of X isomorphically to the ample cone of Y.*

Proof. Since Φ is an isometry, $(1,0,0)\cdot(0,0,1) = -1$, and $(1,0,0)^2 = 0$, we see that there is some $b \in \mathrm{Pic}(X)$ such that

$$\Phi(1,0,0) = \left(1, b, \frac{1}{2}b^2\right).$$

Composing Φ by the twist with $-b$ yields a new Fourier-Mukai transformation sending $(0,0,1)$ to $(0,0,1)$ and $(1,0,0)$ to $(1,0,0)$ (by the formula 3.6.1). Since $\mathrm{Pic}(X) = (0,0,1)^\perp \cap (1,0,0)^\perp$ and similarly for $\mathrm{Pic}(Y)$, we see that such a Φ induces an isometry $\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Pic}(Y)$. In particular, Φ induces an isomorphism of positive cones.

By results of Ogus [Og, Proposition 1.10 and Remark 1.10.9], we know that the ample cone is a connected component of the positive cone, and that the group R generated by reflections in (-2)-curves and multiplication by -1 acts simply transitively on the set of connected components. In particular, there is some element $\rho : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ of this group such that the composition $\rho \circ \Phi : \mathrm{Pic}(Y) \xrightarrow{\sim} \mathrm{Pic}(X)$ induces an isomorphism of ample cones.

We claim that there is a representation of R as a group of Fourier-Mukai autoequivalences of X whose induced action on $\mathrm{CH}(X) = \mathbf{Z} \oplus \mathrm{Pic}(X) \oplus \mathbf{Z}$ is trivial on the outer summands and equals the natural reflection action on $\mathrm{Pic}(X)$. This will clearly complete the proof of the Lemma.

To define this embedding of R, suppose $C \subset X$ is a (-2)-curve. The structure sheaf $\mathscr{O}_C$ is a spherical object of $\mathrm{D}(X)$ (see Definition 3.8ff), and the spherical twist $T_{\mathscr{O}_C} : \mathrm{D}(X) \xrightarrow{\sim} \mathrm{D}(X)$ acts on $\mathrm{H}(X)$ by reflecting in the Mukai vector $(0, C, 1)$. Composing this twist with the tensoring equivalence $\otimes \mathscr{O}(C) : \mathrm{D}(X) \to \mathrm{D}(X)$ gives a Fourier-Mukai equivalence whose induced action on $\mathrm{CH}(X) = \mathbf{Z} \oplus \mathrm{Pic}(X) \oplus \mathbf{Z}$ is the identity on the outer summands and the reflection in C on $\mathrm{Pic}(X)$. Similarly, the shift isomorphism $\mathscr{F} \mapsto \mathscr{F}[1] : \mathrm{D}(X) \to \mathrm{D}(X)$ acts on $\mathrm{CH}(X)$ by the identities on the summands and -1 on $\mathrm{Pic}(X)$. This establishes the claim. □

We will assume that our kernel P satisfies the conclusions of Lemma 5.2.

Proposition 5.3. *There is an isomorphism of infinitesimal deformation functors $\delta : \mathrm{Def}_X \to \mathrm{Def}_Y$ such that*

(1) $\delta^{-1}(\mathrm{Def}_{(Y,L)}) = \mathrm{Def}_{(X,\Phi(L))}$ *for all* $L \in \mathrm{Pic}(Y)$*;*
(2) *for each augmented Artinian* W*-algebra* $W \to A$ *and each* $(X_A \to A) \in \mathrm{Def}_{(X,H_X)}(A)$*, there is an object* $P_A \in \mathrm{D}(X_A \times_A \delta(X_A))$ *reducing to* P *on* $X \times Y$.

Proof. Given an augmented Artinian W-algebra $W \to A$ and a deformation $X_A \to A$, let $\mathscr{D}_A$ denote the stack of unobstructed universally gluable relatively perfect complexes with Mukai vector $(0,0,1)$. By Proposition 4.4, the kernel P defines an open immersion $Y \hookrightarrow D_k$ such that the fiber product $\mathbf{G}_m$-gerbe

$$\mathscr{Y} := Y \times_{D_k} \mathscr{D}_k \to Y$$

is trivial.

Since $Y \to D_A \otimes k$ is an open immersion and D_A is smooth over A, we see that the open subscheme Y_A of D_A supported on Y gives a flat deformation of Y over A, carrying a $\mathbf{G}_m$-gerbe $\mathscr{Y}_A \to Y_A$. Write $\mathscr{P}_A$ for the perfect complex of $\mathscr{Y}_A \times X_A$-twisted sheaves corresponding to the natural inclusion $\mathscr{Y}_A \to \mathscr{D}_A$. Write $\pi : \mathscr{Y}_A \times X_A \to Y_A \times X_A$

Proposition 5.4. *With the preceding notation, there is an invertible sheaf* $\mathscr{L}$ *on* $\mathscr{Y}_A \times X_A$ *such that the complex*

$$P_A := \mathbf{R}(\pi_* \mathscr{P}_A \otimes \mathscr{L}_A^\vee) \in \mathrm{D}(Y_A \times_A X_A)$$

satisfies

$$\mathbf{L}\iota^* P_A \cong P \in \mathrm{D}(Y \times X),$$

where $\iota : Y \times X \hookrightarrow Y_A \times_A X_A$ *is the natural inclusion.*

Proof. Consider the complex $\mathscr{Q} := \mathscr{P}_A^\vee \otimes \mathrm{pr}_2^* \omega_{X_A}[2]$. Pulling back by the morphism

$$g : Y \times X \to \mathscr{Y}_A \times_A X_A$$

corresponding to P yields the equality

$$\mathbf{L}g^* \mathscr{Q} = P^\vee.$$

It follows that

$$\mathbf{R}(\mathrm{pr}_1)_*(\mathscr{Q})$$

is a perfect complex on $\mathscr{Y}_A$ whose pullback via the section $Y \to \mathscr{Y}$ is $\Phi^{-1}(\mathscr{O}_X)$. Since

$$\Phi(1,0,1) = (1,0,1),$$

this complex has rank 1 and

$$\mathbf{L}g^* \det \mathbf{R}(\mathrm{pr}_1)_*(\mathscr{Q}) = \det \mathbf{R}(\mathrm{pr}_1)_*(P^\vee) \cong \mathscr{O}_Y.$$

It follows that

$$P \cong \mathscr{P} \otimes \det \mathbf{R}(\mathrm{pr}_1)_*(\mathscr{Q}).$$

Setting

$$\mathscr{L} = \det \mathbf{R}(\mathrm{pr}_1)_*(\mathscr{Q})^\vee$$

completes the proof. □

We can now prove part (ii) of Proposition 5.3: the scheme Y_A defined before Proposition 5.4 gives a point of $\mathrm{Def}_Y(A)$, giving the functor

$$\delta : \mathrm{Def}_X \to \mathrm{Def}_Y,$$

and Proposition 5.4 shows that P lifts to

$$P_A \in \mathrm{D}(Y_A \times_A X_A),$$

as desired.

A symmetric construction starting with the inverse kernel $P^\vee$ yields a map

$$\delta' : \mathrm{Def}_Y \to \mathrm{Def}_X$$

and lifts

$$P_A^\vee \in \mathrm{D}(\delta'(Y_A) \times Y_A).$$

Composing the two yields an endomorhpism

$$\eta : \mathrm{Def}_X \to \mathrm{Def}_X$$

and, for each A-valued point of Def_X, lifts of

$$P^\vee \circ P$$

to a complex

$$Q_A \in \mathrm{D}(X_A \times \eta(X_A)).$$

But the adjunction map yields a quasi-isomorphism

$$\mathscr{O}_{\Delta_X} \xrightarrow{\sim} P^\vee \circ P,$$

so Q_A is a complex that reduces to the sheaf $\mathscr{O}_{\Delta_X}$ via the identification

$$\eta(X_A) \otimes k \xrightarrow{\sim} X.$$

It follows that Q_A is the graph of an isomorphism

$$X_A \xrightarrow{\sim} \eta(X_A),$$

showing that $\delta' \circ \delta$ is an automorphism of Def_X, whence δ is an isomorphism.

Now suppose Y_A lies in $\mathrm{Def}_{(Y,L)}$. Applying $P_A^\vee$ yields a complex C_A on X_A whose determinant restricts to $\Phi(L)$ on X. It follows that X_A lies in $\mathrm{Def}_{(X,\Phi(L))}$, as desired.

This completes the proof of Proposition 5.3. □

Proof of Theorem 5.1. Choose ample invertible sheaves H_X and H_Y that are not divisible by p such that $H_X = \Phi(H_Y)$. Deligne showed in [D] that there is a projective lift (X_V, H_{X_V}) of (X, H_X) over a finite extension V of the Witt vectors $W(k)$. For every $n \geq$ let V_n denote the quotient of V by the n-th power of the maximal ideal, and let K denote the field of fractions of V.

By Proposition 5.4, for each n there is a polarized lift (Y_n, H_{Y_n}) of (Y, H_Y) over V_n and a complex

$$P_n \in \mathrm{D}(Y_n \times X_n)$$

lifting P. By the Grothendieck Existence Theorem, the polarized formal scheme (Y_n, H_{Y_n}) is algebraizable, so that there is a lift (Y_V, H_{Y_V}) whose formal completion is (Y_n, H_{Y_n}).

By the Grothendieck Existence Theorem for perfect complexes [L, Proposition 3.6.1], the system (P_n) of complexes is the formal completion of a perfect complex

$$P_V \in \mathrm{D}(Y_V \times_V X_V).$$

In particular, P_V lifts P and Nakayama's Lemma shows that the adjunction maps

$$\Delta_* \mathscr{O}_X \to P_V \circ P_V^\vee$$

and

$$P_V^\vee \circ P_V \to \Delta_* \mathscr{O}_Y$$

are quasi-isomorphisms. It follows that for any field extension K'/K, the generic fiber complex

$$P_{K'} \in \mathrm{D}(Y_{K'} \times_{K'} X_{K'})$$

induces a Fourier-Mukai equivalence

$$\Phi : \mathrm{D}(Y_{K'}) \to \mathrm{D}(X_{K'}),$$

and compatibility of Φ with reduction to k shows that $\Phi(0,0,1) = (0,0,1)$. Choosing an embedding $K \hookrightarrow \mathbf{C}$ yields a filtered Fourier-Mukai equivalence

$$\mathrm{D}(Y_V \otimes \mathbf{C}) \xrightarrow{\sim} \mathrm{D}(X_V \otimes \mathbf{C}).$$

Since Φ is filtered and induces an isometry of integral Mukai lattices, Φ induces a Hodge isometry

$$\mathrm{H}^2(Y_V \otimes \mathbf{C}, \mathbf{Z}) \xrightarrow{\sim} \mathrm{H}^2(X_V \otimes \mathbf{C}, \mathbf{Z})$$

(see e.g. part (i) of the proof of Proposition 10.10 in [H2]), so that $Y_V \otimes \mathbf{C}$ and $X_V \otimes \mathbf{C}$ are isomorphic. Spreading out, we find a finite extension $V' \supset V$ and isomorphisms of the generic fibers $X_{K'} \xrightarrow{\sim} Y_{K'}$. Since X is not birationally ruled, it follows from Corollary 1 of [MM] that X and Y are isomorphic, as desired. □

6. EVERY FM PARTNER IS A MODULI SPACE OF SHEAVES

In this section we prove statement (1) in Theorem 1.1

Fix K3 surfaces X and Y over an algebraically closed field of characteristic exponent p. Suppose P is the kernel of a Fourier-Mukai equivalence $\mathrm{D}(X) \to \mathrm{D}(Y)$. We now show that Y is isomorphic to a moduli space of sheaves on X.

Let $v = (r, L_X, s) = \Phi(0,0,1)$ be the Mukai vector of a fiber P_y (hence all fibers).

Lemma 6.1. *We may assume that r is positive and prime to p and that L_X is very ample.*

Proof. First, if either r or s is not divisible by p then, up to a shift and composition with the standard transform on Y given by the shifted ideal of the diagonal we are done. So we will assume that both r and s are divisible by p and show that we can compose with an autoequivalence of Y to ensure that r is not divisible by p.

Since Φ induces an isometry of numerical Chow groups, we have that there is some other Mukai vector (r', ℓ, s') such that

$$(r, L_X, s)(r', \ell, s') = \ell \cdot L_X - rs' - r's = 1.$$

Thus, since both r and s are divisible by p we have that $\ell \cdot L_X$ is prime to p. Consider the equivalence $D(X) \to D(X)$ given by tensoring with $\ell^{\otimes n}$ for an integer n. This sends the Mukai vector (r, L_X, s) to

$$(r, L_X + rn\ell, s + n\ell \cdot L_X + \frac{n^2}{2}\ell^2).$$

It is elementary that for some n the last component will be non-zero modulo p. After composing with the shifted diagonal and shifting the complex we can swap the first and last components and thus find that r is not divisible by p, as desired.

Changing the sign of r is accomplished by composing with a shift. Making L_X very ample is accomplished by composing with an appropriate twist functor. □

As discussed in Section 3.11, we can consider the moduli space $M_X(v)$ of sheaves on X with Mukai vector v (with respect to the polarization L_X of X), and this is again a K3 surface which is a FM partner of X.

Proposition 6.2. *With the notation from the beginning of this section, there is an isomorphism* $Y \xrightarrow{\sim} M_X(v)$.

Proof. Consider the composition of the equivalences $\mathrm{D}(Y) \to \mathrm{D}(X) \to \mathrm{D}(M_X(v))$ induced by the original equivalence and the equivalence defined by the universal bundle on $X \times M_X(v)$. By assumption we have that $\mathrm{D}(Y) \to \mathrm{D}(M_X(v))$ sends $(0,0,1)$ to $(0,0,1)$, so it is filtered. Theorem 5.1 implies that $Y \cong M_X(v)$, as desired. □

7. Finiteness results

Fix a K3 surface X over the algebraically closed field k. We will prove that X has finitely many Fourier-Mukai partners, but first we record a well-known preliminary lemma.

Lemma 7.1. *Let Y and Z be relative K3 surfaces over a dvr R. If the generic fibers of Y and Z are isomorphic then so are the special fibers.*

In other words, specializations of K3 surfaces are unique.

Proof. Applying Theorem 1 of [MM], any isomorphism of generic fibers yields a birational isomorphism of the special fibers. Since K3 surfaces are minimal, this implies that the special fibers are in fact isomorphic, as desired. □

Note that we are not asserting that isomorphisms extend, only that isomorphy extends!

Proposition 7.2. *The surface X has finitely many Fourier-Mukai partners.*

Proof. First, suppose X has finite height. We know that there is a lift X_W of X over W such that the restriction map $\mathrm{Pic}(X_W) \to \mathrm{Pic}(X)$ is an isomorphism. Since every partner of X has the form $M_X(v)$ for some Mukai vector v, we see that any partner of X is the specialization of a partner of the geometric generic fiber. But the generic fiber has characteristic 0, whence it has only finitely many FM partners by the Lefschetz principle and the known result over **C** (see [BrM]). Since specializations of K3 surfaces are unique by Lemma 7.1, we see that X has only finitely many partners.

Now suppose that X is supersingular. If X has Picard number at most 4 then there is a flat deformation X_t of X over $\operatorname{Spec} k[\![t]\!]$ such that the generic fiber has finite height and the restriction map $\mathrm{Pic}(X_t) \to \mathrm{Pic}(X)$ is an isomorphism. Indeed, choosing generators $g_1, \ldots, g_n$ for $\mathrm{Pic}(X)$, each g_i defines a Cartier divisor G_i in Def_X. Moreover, the supersingular locus of Def_X has dimension 9 by Proposition 14 of [Og99]. Thus, a generic point of the intersection of the G_i lies outside the supersingular locus, and we are done since we can dominate any local ring by $k[\![t]\!]$. The argument of the previous paragraph then implies the result for X.

This leaves the case when X has Picard number ≥ 5. In this case, as explained at the beginning of Section 3 of [LM], X is Shioda-supersingular and has Picard number 22. In this case we will prove that in fact X has a unique FM partner (namely itself).

This we again do by lifting to characteristic 0. The key result is [HLOY, Corollary 2.7 (2)], which implies that if K is an algebraically closed field of characteristic 0 and if Z/K is a K3 surface with Picard number ≥ 3 and square-free discriminant, then any FM partner of Z is isomorphic to Z.

We use this by showing that if X/k is Shioda-supersingular and Y is a FM partner of X, then we can lift the pair (X, Y) to a FM pair $(\mathcal{X}, \mathcal{Y})$ over the ring of Witt vector $W(k)$ such that the Picard lattice of the geometric generic fiber of $\mathcal{X}$ has rank ≥ 3 and square-free discriminant. Then by the result of [HLOY] just mentioned, we conclude that the geometric generic fibers of $\mathcal{X}$ and $\mathcal{Y}$ are isomorphic whence X and Y are isomorphic.

So fix such a pair (X, Y), and let's construct the desired lifting $(\mathcal{X}, \mathcal{Y})$. By Proposition 6.2, we know that Y is isomorphic to $\mathrm{M}_X(v)$ for some primitive Mukai vector $v = (r, \ell, s)$, for which there exists $v' = (r', \ell', s')$ such that $v \cdot v' = 1$. Of course the v for which $Y \simeq \mathrm{M}_X(v)$ is not unique. In fact we have the following:

Lemma 7.3. *For any subgroup $\Gamma \subset \mathrm{NS}(X)$ of non-maximal rank, there exists a primitive Mukai vector $v = (r, \ell, s)$ such that the following hold:*

(1) $Y \simeq \mathrm{M}_X(v)$.
(2) *The map $v \cdot (*) : \mathrm{CH}(X) \to \mathbf{Z}$ is surjective;*
(3) *ℓ is an ample class.*
(4) $\ell \notin p\,\mathrm{NS}(X) + \Gamma$.

Proof. If L is an invertible sheaf on X, then sending a sheaf E to X to $E \otimes L$ defines an isomorphism of moduli spaces

$$\mathrm{M}_X(v) \to \mathrm{M}_X(w),$$

where

$$w = (r, \ell + L, s).$$

Furthermore, since v is the Mukai vector of a perfect complex P on $X \times X$ defining an isometry $\Phi^P : \mathrm{CH}(X) \to \mathrm{CH}(X)$ with $v = \Phi^P(0, 0, 1)$, the same is true of $w = (r, \ell + L, s)$. Namely, $w = \Phi^{\Delta_* L} \circ \Phi^P(0, 0, 1)$. Therefore the condition (2) is preserved upon replacing v by w.

To prove the lemma, it therefore suffices to show that we can add a suitable ample class to ℓ to ensure conditions (3) and (4). This is immediate since every element of

$$\mathrm{NS}(X)/(p\,\mathrm{NS}(X) + \Gamma)$$

can be represented by an ample class. □

Let N be the Picard lattice of X. By our assumption that X is supersingular, N has the following properties (see for example [Og, 1.7]):

(1) N has rank 22.

(2) Let $N^\vee$ denote the dual of N, and let $N \hookrightarrow N^\vee$ be the inclusion defined by the nondegenerate pairing on N. Then the quotient $N^\vee/N$ is annihilated by p and has dimension as a $\mathbf{F}_p$-vector space $2\sigma_0$ for some integer σ_0 between 1 and 10 (σ_0 is the *Artin invariant*).

(3) The quadratic space N/pN decomposes as an orthogonal sum $A \perp B$ with A totally isotropic and B anisotropic such that $\dim B \geq 2$

Let $F \subset N$ be a rank 2 sublattice reducing to a rank two subspace of B. Applying lemma 7.3 with $\Gamma = F$, we can assume that $Y = M_X(v)$ with $v = (r, \ell, s)$ for $\ell \in N$ with nonzero image in $N/(pN + F)$. Let E be the saturation of $F + \mathbf{Z}\ell$ in N. By construction, the map $F/pF \to F^\vee/pF^\vee = (F/pF)^\vee$ is an isomorphism.

There is a natural diagram

$$\begin{array}{ccc} E & \longrightarrow & N \\ \downarrow & & \downarrow \\ E^\vee & \longleftarrow & N^\vee \\ \downarrow & & \downarrow \\ E^\vee/E & \longleftarrow & N^\vee/N \end{array}$$

in which all four arrows in the bottom square are surjective. In particular, $E^\vee/E$ is an $\mathbf{F}_p$-vector space. Also if Q denotes the quotient E/F, then Q has rank 1 and we have an exact sequence

$$0 \to Q^\vee/(Q^\vee \cap E) \to E^\vee/E \to F^\vee/(\mathrm{Im}(E \to F^\vee)) \to 0.$$

Since the quotient $F^\vee/F$ is already 0, this shows that $E^\vee/E$ is isomorphic to 0 or $\mathbf{Z}/p\mathbf{Z}$. In particular, E has rank 3 and square-free discriminant.

As explained in the appendix (in particular A.7), there is a codimension at most 3 formal closed subscheme of the universal deformation space $D := \mathrm{Spf}\, W[\![t_1, \ldots, t_{20}]\!]$ of X over which E deforms. The universal deformation is algebraizable (as E contains an ample class) and a geometric generic fiber is a $K3$ surface over an algebraically closed field of characteristic 0 with Picard lattice isomorphic to E. Let

$$\mathfrak{X} \to \mathrm{Spec}\, R$$

be a relative $K3$ surface with special fiber X and geometric generic Picard lattice E. Write

$$\mathscr{M} \to \mathrm{Spec}\, R$$

for the stack of sheaves with Mukai vector (r, ℓ, s) stable with respect to a sufficiently general polarization. We know that $\mathscr{M}$ is a $\boldsymbol{\mu}_r$-gerbe over a relative $K3$ surface

$$\mathcal{M} \to \mathrm{Spec}\, R,$$

and by assumption we have that the closed fiber of $\mathscr{M}$ is ismorphic to $Y \times \mathrm{B}\boldsymbol{\mu}_r$. Since r is relatively prime to p, we have that the Brauer class associated to the gerbe $\mathscr{M} \to \mathcal{M}$ is trivial. In particular, there is an invertible $\mathscr{M}$-twisted sheaf $\mathscr{L}$ on $\mathscr{M}$ (see [L2] for basic results on twisted sheaves).

Now let $\mathscr{V}$ be the universal twisted sheaf on $\mathscr{M} \times_R \mathcal{X}$ and V the tautological sheaf on $Y \times X$. Write

$$\pi : \mathscr{M} \times \mathcal{X} \to \mathcal{M} \times \mathcal{X}$$

for the natural projection and let

$$\mathscr{W} := \pi_* (\mathscr{V} \otimes \mathscr{L}^\vee) .$$

There is an invertible sheaf $\mathscr{N}$ on Y such that

$$\mathscr{W}|_{Y \times X} \cong V \otimes \mathrm{pr}_1^* \mathscr{N},$$

the kernel of another equivalence between $\mathrm{D}(X)$ and $\mathrm{D}(Y)$. It follows from the adjunction argument in the proof of Proposition 5.3 that $\mathscr{W}$ also gives a Fourier-Mukai equivalence between the geometric generic fibers of $\mathcal{M}$ and $\mathcal{X}$ over R. By [HLOY], we have that $\mathcal{M}_{\overline{\eta}}$ and $\mathcal{X}_{\overline{\eta}}$ are isomorphic. By specialization (using Lemma 7.1), we see that $Y \cong X$, as desired. □

8. LIFTING KERNELS USING THE MUKAI ISOCRYSTALS

Let k be a perfect field of characteristic $p > 0$, let W be the ring of Witt vectors of k, and let K be the field of fractions of W.

Fix K3 surfaces X and Y over k with lifts X_W and Y_W over W. The Hodge filtrations on the de Rham cohomology of X_W/W and Y_W/W give subspaces $\mathrm{Fil}^2_X \subset \mathrm{H}^2(X/K) \subset \widetilde{H}(X/K)$ and $\mathrm{Fil}^2_Y \subset \mathrm{H}^2(Y/K) \subset \widetilde{H}(Y/K)$, where $\widetilde{H}(X/K)$ and $\widetilde{H}(Y/K)$ denote the crystalline realizations of the Mukai motives.

Theorem 8.1. *Suppose $P \in \mathrm{D}(X \times Y)$ is a kernel whose associated functor $\Phi : \mathrm{D}(X) \to \mathrm{D}(Y)$ is fully faithful. If*

$$\Phi^{\widetilde{H}} : \widetilde{H}(X/K) \to \widetilde{H}(Y/K)$$

sends Fil^2_X to Fil^2_Y then P lifts to a perfect complex $P_W \in \mathrm{D}(X_W \times_W Y_W)$.

Proof. We claim that it suffices to prove the result under the assumption that $\Phi(0,0,1) = (0,0,1)$. Indeed, fix a W-ample divisor β on Y_W. Suppose

$$\Phi(0,0,1) = (r, \ell, s)$$

with $r > 0$. Since Φ preserves the Hodge filtration we see that $\ell \in \mathrm{Fil}^1_Y \mathrm{H}^2(Y/K)$, whence ℓ is unobstructed on Y. Similarly,

$$\Phi(1,0,0) = (r', \ell', s')$$

such that

$$\ell \cdot \ell' - rs' - r's = 1,$$

and ℓ' must also lie in $\mathrm{Fil}^1_Y \mathrm{H}^2_{\mathrm{cris}}(Y/K)$, so that ℓ' lifts over Y_W. Thus, the moduli space $\mathrm{M}_Y(r, \ell, s)$ lifts to a relative moduli space $M_{Y_W}(r, \ell, s)$, and there is a tautological sheaf $\mathscr{E}_W$ on $M_{Y_W}(r, \ell, s) \times_W Y$ defining a relative FM equivalence. This induces an isometry of F-isocrystals

$$\Phi_{\mathscr{E}} : \widetilde{H}(M_Y(r, \ell, s)/K) \xrightarrow{\sim} \widetilde{H}(Y/K)$$

that sends $(0,0,1)$ to (r, ℓ, s). The composition yields a FM equivalence

$$\Phi_Q : \mathrm{D}(X) \to \mathrm{D}(M_Y(v))$$

sending $(0,0,1)$ to $(0,0,1)$ and preserving the Hodge filtrations on Mukai isocrystals. In addition, since $\mathscr{E}$ lifts to $\mathscr{E}_W$, we see that P lifts if and only if Q lifts. Thus, we may assume that $\Phi(0,0,1)=(0,0,1)$, as claimed.

Since Φ is an isometry, it follows that

$$\Phi^{-1}(1,0,0)=\left(1,b,\frac{1}{2}b^2\right)$$

for some $b\in\operatorname{Pic}(X)$. By Proposition 4.4, the kernel P corresponds to a morphism

$$\mu_P : X\to s\mathscr{D}_Y$$

whose image in sD_Y is an open immersion. More concretely, if $\mathscr{P}$ denotes the universal complex on $s\mathscr{D}_Y\times Y$, we have that

$$P=\mathbf{L}(\mu_P\times\mathrm{id})^*\mathscr{P}.$$

Write

$$\mathscr{M}=X\times_{\mathrm{sD}_Y}s\mathscr{D}_Y,$$

so that μ_P defines morphisms

$$X\to\mathscr{M}\to X$$

making X a $\mathbf{G}_m$-torsor over $\mathscr{M}$. The associated invertible sheaf is $\mathscr{M}$-twisted.

Since $s\mathscr{D}_Y$ is smooth over W, there is a canonical formal lift $\mathfrak{X}$ of X over W, with a corresponding formal gerbe $\mathscr{G}\to\mathfrak{X}$ lifting $\mathscr{M}$ such that there is a perfect complex of coherent twisted sheaves $\mathfrak{P}\in\mathrm{D}(\mathscr{G}\times\widehat{Y}_W)$ lifting $\mathscr{P}|_{\mathscr{M}}$. (Indeed, $\mathfrak{X}$ is just the open subspace of the formal completion $\widehat{\mathrm{sD}}_Y$ supported on $\mu(X)$.)

The complex $\mathbf{R}(\mathrm{pr}_1)_*\mathscr{P}$ is an invertible $\mathscr{G}$-twisted sheaf, defining an equivalence

$$\mathrm{D}(\mathfrak{X})\xrightarrow{\sim}\mathrm{D}^{\mathrm{tw}}(\mathscr{G}).$$

Let

$$\mathfrak{Q}\in\mathrm{D}(\mathfrak{X}\times\widehat{Y}_W)$$

be the kernel giving the composition

$$\mathrm{D}(\mathfrak{X})\xrightarrow{\sim}\mathrm{D}^{\mathrm{tw}}(\mathscr{G})\xrightarrow{\sim}\mathrm{D}(\widehat{Y}_W).$$

Since the class of $\mathbf{R}(\mathrm{pr}_1)_*\mathscr{P}$ might differ from the twisted invertible sheaf associated to $X\to\mathscr{M}$, we have that the restriction

$$Q\in\mathrm{D}(X\times Y)$$

differs from P by tensoring with an invertible sheaf $\mathscr{L}$ pulled back from X. One can check that $\Phi_Q(1,0,1)=(1,0,1)$. Since b is the unique invertible sheaf $\mathscr{L}$ on X such that tensoring with $\mathscr{L}$ sends

$$\left(1,b,\frac{1}{2}b^2\right)\text{ to }(1,0,0),$$

we see that

$$Q\cong P\otimes\mathrm{pr}_1^*\mathscr{O}_X(b).$$

We have that $v(b)=\Phi_P^{-1}(v(\mathscr{O}_Y))$ and Φ respects the Hodge filtrations on the Mukai isocrystals; since $\mathscr{O}_Y$ is unobstructed on Y_W, we therefore have that

$$b\in\mathrm{Fil}^1_X\,\mathrm{H}^2_{\mathrm{cris}}(X/K),$$

whence b is unobstructed on X_W. The complex

$$\widehat{P}_W := \mathfrak{Q} \otimes \mathrm{pr}_1^* \mathscr{O}_{\mathfrak{X}}(-b) \in \mathrm{D}(\mathfrak{X} \times \widehat{Y}_W)$$

gives a formal lift of P.

Finally, by construction the isotropic subspace

$$F \subset \mathrm{H}^2(X/K)$$

parametrizing the formal lift $\mathfrak{X}$ is $\Phi^{-1}(\mathrm{Fil}^2 \mathrm{H}^2_Y/K))$. Since

$$\Phi^{-1}(\mathrm{Fil}^2 \mathrm{H}^2_Y/K)) = \mathrm{Fil}^2_X \mathrm{H}^2(X/K),$$

we conclude that

$$\mathfrak{X} = \widehat{X}_W.$$

Applying the Grothendieck Existence Theorem for perfect complexes as in [L], we get the desired lift $P_W \in \mathrm{D}(X_W \times_W Y_W)$. □

Remark 8.2. If we had an integral version of the Mukai isocrystal and an integral version of our results then we could produce the lift X_W from Y_W via $s\mathscr{D}_{Y_W}$. Unfortunately, the Tate twist involved in the formation of $\widetilde{H}(Y/K)$ precludes a naïve extension to integral coefficients.

Remark 8.3. Taking the cycle $Z := \mathrm{ch}(P)\sqrt{\mathrm{Td}_{X\times Y}}$ giving the action on cohomology, we can see that Theorem 8.1 gives a special case of the variational crystalline Hodge conjecture (see e.g. Conjecture 9.2 of [MP]): the fact that Φ_P preserves the Hodge filtrations on the Mukai isocrystals means that

$$[Z] \in \mathrm{Fil}^2 \mathrm{H}^4(X \times Y/K).$$

Lifting the kernel P to P_W lifts the cycle, confirming the conjecture in this case. This could be interpreted as a kind of (weak) "variational crystalline version" of Mukai's original results on the Mukai Hodge structure [M].

9. Zeta functions of FM partners over a finite field

In this section we address a question due to Mustaţă and communicated to us by Huybrechts: do Fourier-Mukai partners over a finite field have the same zeta function?

Theorem 9.1. *Suppose X and Y are K3 surfaces over a finite field k. If there is an equivalence $\mathrm{D}(X) \xrightarrow{\sim} \mathrm{D}(Y)$ of k-linear derived categories then for all finite extensions k'/k we have that*

$$|X(k')| = |Y(k')|.$$

In particular, $\zeta_X = \zeta_Y$.

Proof. By Theorem 3.2.1 of [Or], there is a kernel $P \in \mathrm{D}(X \times Y)$ giving the equivalence. The Leftschetz fixed-pont formula in crystalline cohomology shows that it is enough to see that the trace of Frobenius acting on H^2_{cris} is the same. As in diagram (2.3.3), P induces an isomorphism of F-isocrystals

$$\widetilde{H}(X/K) \xrightarrow{\sim} \widetilde{H}(Y/K).$$

Thus, the trace of Frobenius on both sides is the same. On the other hand, it follows from the definition of the Mukai crystal that

$$\mathrm{Tr}(F|\widetilde{H}(X/K)) = \mathrm{Tr}(F|H^2_{\text{cris}}(X/K)) + 2p$$

and similarly for Y. Thus

$$\mathrm{Tr}(F|H^2_{\text{cris}}(X/K)) = \mathrm{Tr}(F|H^2_{\text{cris}}(Y/K)),$$

giving the desired result. □

APPENDIX A. DEFORMATIONS OF K3'S WITH FAMILIES OF LINE BUNDLES

Throughout this appendix we consider only schemes over $\mathbf{Z}[1/2]$.

A.1. If S is a scheme, a *family of K3 surfaces over* S is a smooth proper morphism of algebraic spaces $f : X \to S$ all of whose geometric fibers are algebraic spaces.

For a family of K3 surfaces X/S, it follows from the methods of [A] that the relative Picard functor $\underline{\mathrm{Pic}}_{X/S}$ is an algebraic space, and for any geometric point $\bar{s} \to S$ the group $\underline{\mathrm{Pic}}_{X/S}(\bar{s})$ of sections $\bar{s} \to \underline{\mathrm{Pic}}_{X/S}$ is canonically isomorphic to the Neron-Severi group $NS(X_{\bar{s}})$ of the fiber.

Definition A.2. Let E be an abelian group and X/S a family of K3 surfaces. An E-*marking* on X/S is a homomorphism of group spaces $\rho : E_S \to \underline{\mathrm{Pic}}_{X/S}$, where E_S denotes the constant group scheme associated to E.

Remark A.3. This definition differs slightly from the one in [Og, 2.1], as we do not consider any inner product here.

A.4. Let S be the spectrum of a complete local ring A with closed point $s \in S$, and let X/S be a family of K3 surfaces over S. Assume that the residue field $k(s)$ is algebraically closed. For any geometric point $\bar{t} \to S$ there is a specialization map (see for example [MP, 3.2])

$$NS(X_{\bar{t}}) \to NS(X_s) \tag{A.4.1}$$

which is injective by [MP, 3.6].

Lemma A.5. *Let $\rho : E \to NS(X_s)$ be an E-marking. There exists a closed subset $Z \hookrightarrow S$ such that a geometric point $\bar{t} \to S$ factors through Z if and only if ρ factors through the specialization map (A.4.1).*

Proof. By [MP, 3.8], there exists a finite decomposition $S = \cup_i S_i$ of S into locally closed subschemes such that for any two geometric points $\bar{t}, \bar{t}'$ mapping to the same S_i and with $\bar{t}$ specializing to $\bar{t}'$, the specialization map (which depends on choices)

$$NS(X_{\bar{t}}) \to NS(X_{\bar{t}'})$$

is an isomorphism. By associativity of the specialization maps, it follows that it suffices to show that if $\bar{t}$ and $\bar{t}'$ are two geometric points of S with $\bar{t}'$ a specialization of $\bar{t}$, and if ρ factors through $NS(X_{\bar{t}})$ then ρ factors through $NS(X_{\bar{t}'})$. This is immediate again by associativity of the specialization maps. □

A.6. Fix now an algebraically closed field k, a K3 surface X/k, and an E-marking $\rho : E \to NS(X)$. Let W denote the ring of Witt vectors of k, and let Art denote the category of artinian local W-algebras with residue field k. Let

$$D_{(X,E)} : \mathrm{Art} \to \mathrm{Set}$$

denote the functor which to any $A \in \mathrm{Art}$ associates the isomorphism classes of pairs (X_A, ρ_A), where X_A is a flat lifting to A of X and $\rho_A : E \to \underline{\mathrm{Pic}}_{X_A/A}$ is a lifting of ρ. It follows immediately from Schlessinger's criterion that the functor $D_{(X,E)}$ is prorepresentable (see for example Section 1 of [D]). We denote by A_E the corresponding complete W-algebra.

If $E' \subset E$ is a subgroup, then there is a forgetful functor

$$D_{(X,E)} \to D_{(X,E')}$$

which is a closed immersion (cf. Proposition 1.5 of [D]). This map corresponds to a surjective map of rings

$$A_E \to A_{E'}.$$

Two cases of particular importance to us are the following:

(1) Taking $E' = 0$ we see that A_E is a quotient of the deformation ring of X, which by Corollary 1.2 of [D] is isomorphic to $W[[t_1, \dots, t_{20}]]$.
(2) If E contains a element $e \in E$ mapping to an ample class $l \in NS(X)$, then taking $E' \subset E$ to be the span of l, we get that A_E is a quotient of the deformation ring of the polarized K3 surface (X, l). In particular, the formal family of K3 surfaces over A_E algebraizes uniquely to a family of K3 surfaces $\mathscr{X}_E/A_E$ with an E-marking $\tilde{\rho} : E \to \underline{\mathrm{Pic}}_{\mathscr{X}_E/A_E}$.

Theorem A.7. *Suppose $\rho : E \to NS(X)$ is injective with p-torsion free cokernel. Let r be the rank of E and assume $10 \geq r \geq 1$.*

(i) Either $A_E \simeq W[[t_1, \dots, t_{20-r}]]$ or $A_E \simeq W[[t_1, \dots, t_{21-r}]]/(q)$, where q is a nonzero element of $W[[t_1, \dots, t_{21-r}]]$ whose image in $k[[t_1, \dots, t_{21-r}]]$ is a nonzero divisor.

In particular, $\mathrm{Spec}(A_E)$ *is flat over W and equidimensional of dimension* $20 - r + \dim(W)$.

(ii) Assume further that E contains an ample class and that the quotient $NS(X)/E$ is torsion free. If $\bar{\eta} \to \mathrm{Spec}(A_E)$, is a geometric generic point, then the map

$$\tilde{\rho}_{\bar{\eta}} : E \to NS(\mathscr{X}_{E,\bar{\eta}})$$

is an isomorphism.

Proof. For (i) we follow the ideas of [Og2, Proof of 2.2].

If we fix an isomorphism $E \simeq \mathbf{Z}^r$, and let L_i be a line bundle representing the class in $NS(X)$ corresponding to the i-th standard generator of $\mathbf{Z}^r$, then $D_{(X,E)}$ is isomorphic to the fiber product of functors

$$D_{(X,L_1)} \times_{D_X} \times \cdots \times_{D_X} D_{(X,L_r)},$$

where for a line bundle L we denote by $D_{(X,L)}$ the deformation functor of the pair (X, L). If A_i denotes the ring prorepresenting $D_{(X,L_i)}$, then we get that

$$A_E \simeq A_1 \otimes_A A_2 \otimes_A \cdots \otimes_A A_r,$$

where A denotes the ring prorepresenting the deformation functor D_X of X (so $A \simeq W[[t_1, \dots, t_{20}]]$). We use this and the careful analysis of the rings A_i in [Og2] to prove (i) as follows.

Namely, let $\overline{A}$ denote $A \otimes_W k$ (so $\overline{A} \simeq k[[t_1, \dots, t_{20}]]$) and let $\mathscr{X}/\overline{A}$ denote the versal deformation (a formal scheme of Spf($\overline{A}$)). We then have the Gauss-Manin connection on the relative de Rham cohomology of $\mathscr{X}/A$ which induces a Kodaira-Spencer map

$$\kappa : H^1(\mathscr{X}, \Omega^1_{\mathscr{X}/\overline{A}}) \to H^0(\mathscr{X}, \Omega^2_{\mathscr{X}/\overline{A}}) \otimes_{\overline{A}} \Omega^1_{\overline{A}/k}.$$

Let $\mathfrak{m} \subset \overline{A}$ be the maximal ideal. Then choosing a generator $\omega \in H^0(\mathscr{X}, \Omega^2_{\mathscr{X}/\overline{A}})$ and evaluating κ at the maximal ideal we get an isomorphism

$$\rho_\omega : H^1(X, \Omega^1_{X/k}) \to \mathfrak{m}/\mathfrak{m}^2.$$

By [Og2, 1.4], the Chern class map

$$c_1 : NS(X) \otimes \mathbf{F}_p \to H^2_{\mathrm{dR}}(X/k)$$

is injective, and has image in the first step of the Hodge filtration. Let

$$c_E : E/pE \hookrightarrow \mathbf{Fil}^1 H^2_{\mathrm{dR}}(X/k)$$

be the resulting injective map, and write

$$\pi : \mathbf{Fil}^1 H^2_{\mathrm{dR}}(X/k) \to \mathbf{gr}^1 H^2_{\mathrm{dR}}(X/k) \simeq H^1(X, \Omega^1_{X/k})$$

for the projection.

The kernel $L \subset E/pE$ of the composite map $\pi \circ c_E : E/pE \to H^1(X, \Omega^1_{X/k})$ injects into the one-dimensional vector space $H^0(X, \Omega^2_{X/k})$ and therefore has dimension either 0 or 1. We consider these two cases separately.

Case 1: $L = 0$**.** In this case choose any isomorphism $\lambda : E \simeq \mathbf{Z}^r$ and for $i = 1, \ldots, r$ let f_i be a generator of the kernel of $A \to A_i$. As explained in [Og2, 1.14], the image of f_i in the k-vector space $\mathfrak{m}/\mathfrak{m}^2$ is up to scalar $\rho_\omega c_e(\lambda^{-1}(e_i))$, where $e_i \in \mathbf{Z}^r$ denotes the i-th standard generator. It follows that the images of the f_i in $\overline{A}$ extend to a system of parameters for $\overline{A}$, which implies that we can find an isomorphism

$$W[[t_1, \ldots, t_{20}]] \to A$$

sending t_i to f_i for $i = 1, \ldots, r$. This implies (i) in this case.

Case 2: $L \neq 0$**.** In this case L is 1-dimensional, and we can choose an isomorphism $E \simeq E' \oplus \mathbf{Z}$ such that the restriction of $\pi \circ c_E$ to E'/pE' is injective. If L denotes a line bundle representing the class of the standard generator of $\mathbf{Z}$, we have

$$A_E \simeq A_{E'} \otimes_A B.$$

where B denotes the deformation ring of the pair (X, L). By the first case, we can choose an isomorphism $A \simeq W[[t_1, \ldots, t_{20}]]$ such that the ideal of A_E in A is given by $(t_1, \ldots, t_{r-1})$. Furthermore, by [Og2, proof of 2.2] if we choose our ω such that its image in $H^0(X, \Omega^2_X)$ is equal to the class of L, then the ideal of B in A is generated by a single element $q \in A$ whose image Q in $\mathfrak{m}^2/\mathfrak{m}^3$ is a quadratic form which for a suitable choice of basis $v_1, \ldots, v_{10}, w_1, \ldots, w_{10}$ for $\mathfrak{m}/\mathfrak{m}^2$ can be written as

$$\sum_{j=1}^{10} v_j w_j.$$

Let V denote the dual of $\mathfrak{m}/\mathfrak{m}^2$, and view Q as a quadratic form on V. The explicit description of Q shows that the dimension of a maximal nullspace of (V, Q) is 10. This implies that if $\mathfrak{n} \subset A_E \otimes_W k$ denotes the maximal ideal, then the image of Q in $\mathfrak{n}^2/\mathfrak{n}^3$ is nonzero, since otherwise the dual of $\mathfrak{n}/\mathfrak{n}^2$ would give a nullspace in V of dimension $20 - r + 1$ which is greater than 10 by our assumption that $r \leq 10$. Statement (i) in this case follows.

To see (ii), let N denote the Neron-Severi group $NS(\mathscr{X}_{E,\bar{\eta}})$, and let $\gamma : N \to NS(X)$ be the N-marking induced by specialization. Consider the resulting map of deformation rings

$$A_E \twoheadrightarrow A_N.$$

Then $\operatorname{Spec}(A_N) \subset \operatorname{Spec}(A_E)$ is the maximal closed subscheme over which the N-marking γ extends, so it follows that $\operatorname{Spec}(A_N) \subset \operatorname{Spec}(A_E)$ contains an irreducible component of $\operatorname{Spec}(A_E)$. By (i), it follows that E and N have the same rank and since the quotient $NS(X)/E$ is torsion free this implies that $E = N$. □

Remark A.8. The assumption that $r \leq 10$ is necessary. If X/k is Shioda supersingular so that $NS(X)$ has rank 22, we can choose $E \subset NS(X)$ of rank 11 such that (with notation as in the proof) we have $E \simeq E' \oplus \mathbf{Z}$ where $\mathbf{Z}$ maps to $\mathrm{Fil}^2 H^2_{\mathrm{dR}}(X/k)$ and E' maps under the map $\rho_\omega \circ \pi \circ c_E$ to the basis elements $v_1, \ldots, v_{10}$.

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