SELECTED SOLUTIONS FROM PROBLEM SET 8

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section 7.3, # 6. Using the preceding exercise we can do this as follows. By definition of i, we have $k_i \leq \sqrt{p}$ and $k_{i+1} > \sqrt{p}$. Then by exercise 5 we have

$$|h_i/k_i - u/p| \leq 1/k_ik_{i+1} < 1/k_i\sqrt{p}.$$

Multiplying both sides by k_ip we get

$$|h_ip - uk_i| < \sqrt{p}.$$

Now if $x = k_i$ and $y = h_ip - uk_i$, then we get

$$x^2 + y^2 \equiv k_i^2 - u^2k_i^2 \equiv 0 \pmod{p},$$

where we use the definition of u which gives $u^2 \equiv -1 \pmod{p}$. Therefore $p|x^2 + y^2$. On the other hand, we have $|x^2| \leq p$ since $k_i \leq \sqrt{p}$ and we just showed that $|y| < \sqrt{p}$ so

$$|x^2 + y^2| \leq |x^2| + |y^2| < 2p.$$

Therefore $x^2 + y^2$ is a number between 0 and $2p$ which is divisible by p. We conclude that $p = x^2 + y^2$.

Section 7.4, # 4. Let θ denote the number $\langle b_1, b_2, \dots \rangle$. Then by theorem 7.3 we have

$$\langle a_0, a_1, \dots, a_n, b_1, b_2, \dots \rangle = \langle a_0, a_1, \dots, a_n, \theta \rangle = \frac{\theta h_n + h_{n-1}}{\theta k_n + k_{n-1}}.$$

Let r_n denote $\langle a_0, \dots, a_n \rangle$ and recall (theorem 7.4) that $r_n = h_n/k_n$. then we have

$$\langle a_0, a_1, \dots, a_n, b_1, b_2, \dots \rangle - r_n = \frac{\theta h_n + h_{n-1}}{\theta k_n + k_{n-1}} - \frac{h_n}{k_n}$$

which upon finding a common denominator on the right side gives

$$\begin{aligned}
\langle a_0, a_1, \dots, a_n, b_1, b_2, \dots \rangle - r_n &= \frac{\theta h_n k_n + h_{n-1}k_n - h_n\theta k_n - h_n k_{n-1}}{k_n(\theta k_n + k_{n-1})} \\
&= \frac{h_{n-1}k_n - h_n k_{n-1}}{k_n(\theta k_n + k_{n-1})} \\
&= \frac{(-1)^n}{k_n(\theta k_n + k_{n-1})},
\end{aligned}$$

where the last equality is by theorem 7.5. Since the k_n tend to infinity as n gets large this gives

$$\lim_{n\to\infty} \langle a_0, a_1, \dots, a_n, b_1, b_2, \dots \rangle - r_n = 0,$$

and therefore

$$\lim_{n\to\infty} \langle a_0, a_1, \dots, a_n, b_1, b_2, \dots \rangle = \lim_{n\to\infty} r_n = \xi.$$