On Faltings' method of almost étale extensions

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Contents

1. Introduction

1.1. The last decade has been a time of tremendous activity in p-adic Hodge Theory. We do not intend to give a complete account of these new developments here, but let us at least mention the surveys [2] providing an overview of recent topics in the theory of p-adic representations, the survey of p-adic cohomology [34], the survey of p–adic Hodge theory for torsion coefficients [8], Tsuji's survey of his proof of the C_{st}–conjecture of Fontaine [51], and for an introduction to the main ideas of p-adic Hodge theory the beautiful article [27]. We should also note that in the last decade there has been an increasing interest in non–abelian p-adic Hodge theory and its relationship to motives [20], [35], [41], [52].

Instead of attempting the impossible task of trying to cover all of these topics in one article we will instead discuss one aspect of the theory: Faltings' method of

almost étale extensions. Faltings' method is one of the most powerful tools in p-adic Hodge theory and yields in particular a proof of the C_{st}–conjecture of Fontaine which was first proven by Tsuji in [**49**] using syntomic cohomology. It is also to the author's knowledge the best technology currently available to deal with coefficients and cohomology with compact support of open varieties. Unfortunately for various reasons (not the least of which is the very technical nature of Faltings' theory), the approach to p-adic Hodge theory using "almost mathematics" does not seem to be so widely studied.

1.2. Let us begin by reviewing the main result of p-adic Hodge theory in the simplest situation of good reduction, and the various approaches to the theory in this context.

Let p be a prime, and let V be a complete discrete valuation ring of mixed characteristic $(0, p)$ and perfect residue field k. Let K be the fraction field of V, and fix an algebraic closure $K \hookrightarrow \overline{K}$. Let $W \subset V$ be the ring of Witt vectors of k, and let $K_0 \subset K$ be the field of fractions of W. The ring W comes equipped with a lifting of Frobenius $\sigma : W \to W$ which also induces an automorphism of K_0 which we denote by the same letter.

Let X/V be a smooth proper scheme. The theory of p-adic Hodge theory concerns the comparison between the étale cohomology $H^*(X_{\overline{K}}, \mathbb{Q}_p)$ and the algebraic de-Rham cohomology $H^*_{\text{dR}}(X_K)$ of the generic fiber.

To understand this comparison one must consider the additional structures that these cohomology groups are endowed with. The étale cohomology $H^*(X_{\overline{K}}, \mathbb{Q}_p)$ is a continuous $G_K := \text{Gal}(\overline{K}/K)$-representation, and the de Rham cohomology $H^*_{\text{dR}}(X_K)$ comes equipped with the following data:

(i) A descending filtration $F^\cdot$ (the Hodge filtration).
(ii) A graded K_0-vector space M_0 with a σ-semi-linear automorphism $\varphi :$ $M_0 \to M_0$ (Frobenius) and an isomorphism $K \otimes_{K_0} M_0 \simeq H^*_{\text{dR}}(X_K)$. The K_0-vector space M_0 is given by the crystalline cohomology of the closed fiber of X.

This package is formalized by defining a *filtered φ-module* to be a triple $(D, \varphi, \text{Fil}_D)$ consisting of a finite dimensional K_0-vector space D with a σ-semi-linear automorphism $\varphi : D \to D$ and a descending separated and exhaustive filtration Fil_D on $D_K := D \otimes_{K_0} K$.

To pass between the étale and de Rham cohomologies (with their additional structure) one uses Fontaine's ring $\text{B}_{\text{cris}}(V)$ (see section 11). This is a K_0-algebra which comes equipped with a Frobenius automorphism φ and an action of G_K. The ring $\text{B}_{\text{cris}}(V)$ is a subring of a larger K-algebra $\text{B}_{\text{dR}}(V)$ which is a discrete valuation field. In particular, the valuation on $\text{B}_{\text{dR}}(V)$ defines a filtration on $\text{B}_{\text{dR}}(V)$. If E is a filtered K-vector space then we view $E \otimes_K \text{B}_{\text{dR}}(V)$ as filtered using the product filtration.

For a continuous G_K-representation W define

$$D(W) := (W \otimes_{\mathbb{Q}_p} \text{B}_{\text{cris}}(V))^{G_K}.$$

This is known to be a finite-dimensional $\mathbb{Q}_p$-vector space, and it has a natural structure of a filtered φ-module. Namely, it inherits a semi-linear Frobenius automorphism $\varphi_{D(W)}$ from the automorphism on $\text{B}_{\text{cris}}(V)$, and there is a filtration $\text{Fil}_{D(W)}$ on $D(W)_K := D(W) \otimes_{K_0} K$ induced by the filtration on $D(W)_K \otimes_K \text{B}_{\text{dR}}(V)$.

The representation W is called *crystalline* if the natural map (which is always injective)

$$(1.2.1) \qquad D(W) \otimes_{K_0} B_{\mathrm{cris}}(V) \to W \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V)$$

is an isomorphism. In this case we say that $(D(W), \varphi_{D(W)}, \mathrm{Fil}_{D(W)})$ is *associated* to W. If W is crystalline one can recover W from $(D(W), \varphi_{D(W)}, \mathrm{Fil}_{D(W)})$ as

$$\{x \in D(W) \otimes_{K_0} B_{\mathrm{cris}}(V) | \varphi(x) = x, \quad x \otimes 1 \in \mathrm{Fil}^0(D(W)_K \otimes_K B_{\mathrm{dR}}(V))\}.$$

A filtered φ-module $(D, \varphi, \mathrm{Fil}_D)$ is called *admissible* if it is isomorphic to $(D(W), \varphi_{D(W)}, \mathrm{Fil}_{D(W)})$ for some crystalline representation W.

The main result of p-adic Hodge theory in the present situation is the following:

THEOREM 1.3 ([**14, 18, 49**]). *The G_K-representation $H^*(X_{\overline{K}}, \mathbb{Q}_p)$ is crystalline and is associated to $H^*_{\mathrm{dR}}(X_K)$ with the additional structure (i) and (ii) above.*

REMARK 1.4. In the present good reduction setting, theorem 1.3 was first proven by Fontaine and Messing in [**18**] under the assumption that $p > \dim(X_K)$ and $K_0 = K$. Subsequently Kato and Messing were able to extend this result to the case when $\dim(X_K) < (p-1)/2$ [**31, 32**]. The general result was then proven by Tsuji and Faltings who also proved a comparison result without the assumption of good reduction (where the statement is more complicated).

REMARK 1.5. One application of 1.3, from which the terminology 'p-adic Hodge theory' derives, is the following. As mentioned above the ring $B_{\mathrm{dR}}(V)$ is a discrete valuation field, and its residue field is $\mathbb{C}_p$ (the p-adic completion of $\overline{K}$). Using this one shows that there is an isomorphism of graded rings with G_K-action (the subscript HT stands for *Hodge-Tate*)

$$B_{\mathrm{HT}}(V) := \mathrm{gr}^* B_{\mathrm{dR}}(V) \simeq \oplus_{i \in \mathbb{Z}} \mathbb{C}_p(i),$$

where multiplication on the right is given by the natural isomorphisms $\mathbb{C}_p(i) \otimes \mathbb{C}_p(j) \to \mathbb{C}_p(i+j)$. Now if W is a crystalline representation, then the isomorphism 1.2.1 induces upon tensoring with $B_{\mathrm{dR}}(V)$ and passing to the associated graded modules an isomorphism

$$\mathrm{gr}^*(D(W)_K) \otimes B_{\mathrm{HT}}(V) \simeq W \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}}.$$

In the case when $W = H^m(X_{\overline{K}}, \mathbb{Q}_p)$ the degree 0 part of this isomorphism yields a G_K-equivariant isomorphism

$$H^m(X_{\overline{K}}, \mathbb{Q}_p) \otimes \mathbb{C}_p \simeq \oplus_{i+j=m} H^i(X_K, \Omega^j_{X_K}) \otimes_K \mathbb{C}_p(-j)$$

called the *Hodge-Tate decomposition* of $H^m(X_{\overline{K}}, \mathbb{Q}_p) \otimes \mathbb{C}_p$.

1.6. Let $H^*_{\mathrm{cris}}(X/K_0)$ denote the crystalline cohomology of the closed fiber of X, so we have the Berthelot-Ogus isomorphism

$$H^*_{\mathrm{cris}}(X/K_0) \otimes_{K_0} K \simeq H^*_{\mathrm{dR}}(X/K).$$

The main difficulty in proving 1.3 is to construct maps

$$(1.6.1) \qquad H^*(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}(V) \underset{?}{\overset{?}{\underset{\prec}{\gtrless}}} H^*_{\mathrm{cris}}(X/K_0) \otimes_{K_0} B_{\mathrm{cris}}(V).$$

In fact it suffices to construct a map in one direction which respects the various structures and in addition is compatible with cycle classes. For then it is essentially

formal that it is an isomorphism (as we explain in section 14). Unfortunately there is no known construction of a map relating the two cohomology theories directly, and all known proofs of 1.3 use an intermediate theory to relate the two.

There are three main approaches to proving 1.3. Let $\overline{V} \subset \overline{K}$ be the integral closure of V, and let $X_{\overline{V}}$ be the base change of X.

(i) (*Syntomic cohomology*). This approach has been developed by Fontaine, Messing, Hyodo, Kato, and Tsuji among others (see [**18, 26, 30, 32, 49, 51**]).

Here one defines certain *syntomic cohomology groups* $H^*_{\mathrm{syn}}(X_{\overline{V}}, \mathscr{S}(r)_{\mathbb{Q}_p})$ ($r \geq 0$). There are maps

$$(1.6.2) \qquad\qquad \alpha_r : H^*_{\mathrm{syn}}(X_{\overline{V}}, \mathscr{S}(r)_{\mathbb{Q}_p}) \to H^*(X_{\overline{K}}, \mathbb{Q}_p(r))$$

and

$$(1.6.3) \qquad \beta_r : H^*_{\mathrm{syn}}(X_{\overline{V}}, \mathscr{S}(r)_{\mathbb{Q}_p}) \to H^*_{\mathrm{cris}}(X/K_0) \otimes_{K_0} \mathrm{B}_{\mathrm{cris}}(V).$$

The key result in this approach is to show that for $r >> 0$ the map α_r is an isomorphism. By inverting the map α_r and applying suitable Tate twists one obtains the desired morphism 1.6.1.

The proof that for $r >> 0$ the map α_r is an isomorphism requires a detailed analysis of p-adic vanishing cycles. Let $\overline{Y}$ denote the base change of $X_{\overline{V}}$ to the residue field $\bar{k}$ of $\overline{V}$ so we have a commutative diagram

$$X_{\overline{K}} \xrightarrow{\ \bar{j}\ } X_{\overline{V}} \xleftarrow{\ \bar{i}\ } \overline{Y}.$$

By the proper base change theorem we have

$$H^*(X_{\overline{K}}, \mathbb{Z}/(p^n)(r)) \simeq H^*(\overline{Y}, \bar{i}^* R\bar{j}_* \mathbb{Z}/(p^n)(r)).$$

The syntomic theory provides a crystalline interpretation of the p-adic vanishing cycles $\bar{i}^* R\bar{j}_* \mathbb{Z}/(p^n)(r)$.

Suppose T is a flat and local complete intersection scheme over W (such a scheme T is called *syntomic* over W), and assume that T is quasi-projective. Then we can find an immersion $i : T \hookrightarrow Z$, where Z is a smooth W-scheme such that there exists a morphism $F_Z : Z \to Z$ lifting the Frobenius morphism on $Z \otimes_W k$, and such that the diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{\ F_Z\ } & Z \\
\downarrow & & \downarrow \\
\mathrm{Spec}(W) & \xrightarrow{\ \sigma\ } & \mathrm{Spec}(W)
\end{array}
$$

commutes (for example Z can be chosen to be a suitable projective space). Let

$$i_n : T_n \hookrightarrow Z_n$$

be the morphism over $W_n := W/p^n$ defined by reduction modulo p^n. Let D_n be the divided power envelope of T_n in Z_n and let $J_n^{[r]}$ ($r \geq 0$) be the r-th divided power ideal of D_n (which we view as a sheaf on T_1). For $r < 0$ we define $J_n^{[r]} := \mathcal{O}_{D_n}$. We then get a complex on T_1

$$\mathbb{J}_n^{[r]} : J_n^{[r]} \to J_n^{[r-1]} \otimes \Omega^1_{Z_n} \to J_n^{[r-2]} \otimes \Omega^2_{Z_n} \to \cdots .$$

In case $r = 0$ this is simply the complex computing the crystalline cohomology of T_1/W_n, and in general a crystalline interpretation is given in [**4**, 7.2]. The map F_Z induces a map

$$f : \mathbb{J}_n^{[r]} \to \mathbb{J}_n^{[0]}.$$

Define $\widetilde{\mathscr{S}_n}(r)_T$ to be the mapping fiber of the map of complexes

$$\mathbb{J}_n^{[r]} \xrightarrow{p^r - f} \mathbb{J}_n^{[0]}.$$

This complex $\widetilde{\mathscr{S}_n}(r)_T$ is in fact independent of the choice of $i : T \hookrightarrow Z$ and can be defined globally using hypercovers.

Returning to our smooth proper scheme X/V, note that for every finite extension $V \to V'$ the base change $X_{V'}$ is syntomic over W and therefore one obtains a complex $\widetilde{\mathscr{S}_n}(r)_{X_{V'}}$ on the closed fiber of $X_{V'}$, and hence also by pullback a complex on $\overline{Y}$. By passing to a suitable limit one then obtains a complex $\widetilde{\mathscr{S}_n}(r)_{X_{\overline{V}}}$ on $\overline{Y}$.

Let $\mathbb{Z}/p^n\mathbb{Z}(r)'$ denote $(\frac{1}{p^a a!}\mathbb{Z}_p(r)) \otimes \mathbb{Z}/p^n Z$, where $r = (p-1)a + b$ ($a \in \mathbb{Z}$ and $0 \le b \le p - 2$). The key result of Tsuji is then the following:

THEOREM 1.7 ([**49**, §3]). *There is a canonical G_K-equivariant morphism*

(1.7.1) $$\widetilde{\mathscr{S}_n}(r)_{X_{\overline{V}}} \to \bar{i}^* R\bar{j}_* \mathbb{Z}/p^n\mathbb{Z}(r)'.$$

If $0 \le q \le r$ then there exists an integer N depending only on p, q, and r such that the kernel and cokernel of the map

$$\mathscr{H}^q(\widetilde{\mathscr{S}_n}(r)_{X_{\overline{V}}}) \to \bar{i}^* R^q \bar{j}_* \mathbb{Z}/p^n\mathbb{Z}(r)'$$

is annihilated by p^N.

We define

$$H_{\mathrm{syn}}^*(X_{\overline{V}}, \mathscr{S}(r)_{\mathbb{Q}_p}) := \mathbb{Q}_p \otimes \varprojlim H^*(\overline{Y}, \widetilde{\mathscr{S}_n}(r)_{X_{\overline{V}}}).$$

The map α_r in 1.6.2 is obtained from the map 1.7.1 by passing to the limit in n and tensoring with $\mathbb{Q}_p$. It follows from the second statement in 1.7 that for $r >> 0$ the map α_r is an isomorphism.

On the other hand, by construction there is a natural map

$$H_{\mathrm{syn}}^*(X_{\overline{V}}, \widetilde{\mathscr{S}_n}(r)_{X_{\overline{V}}}) \to H^*(\overline{Y}, \mathbb{J}_n^{[r]}),$$

where on the right we write $\mathbb{J}_n^{[r]}$ for the limit of the complexes obtained from the $X_{V'}$ ($V \subset V'$ a finite extension).

It is fairly easy (at least in comparison to the proof of 1.7) to show that for $r \ge m$ there is a canonical isomorphism

$$\mathrm{B}_{\mathrm{cris}}(V)^+ \otimes_W H_{\mathrm{cris}}^m(X/W) \simeq \mathbb{Q}_p \otimes \varprojlim_n H^*(\overline{Y}, \mathbb{J}_n^{[0]}),$$

where $\mathrm{B}_{\mathrm{cris}}(V)^+$ is defined in 11.1.6 (the ring $\mathrm{B}_{\mathrm{cris}}(V)$ is a localization of $\mathrm{B}_{\mathrm{cris}}(V)^+$). The map β_r in 1.6.3 is defined to be the map induced by the natural inclusion $\mathbb{J}_n^{[r]} \subset \mathbb{J}_n^{[0]}$.

(ii) (*K-theory*). This approach developed by Niziol [**36**, **37**] offers perhaps the most direct relation among the two cohomology groups. The key point is that there are regulator maps

$$c_{ij}^{\mathrm{et}} : \mathrm{gr}_\gamma^i K_j(X_{\overline{K}}, \mathbb{Z}/p^n) \to H^{2i-j}(X_{\overline{K}}, \mathbb{Z}/p^n(i)),$$

where $K_j(X_{\overline{K}}, \mathbb{Z}/p^n)$ denotes algebraic K-theory with coefficients, which is filtered by the γ-filtration. Using Thomason's comparison theorem between algebraic and étale K-theory [48], Niziol then describes the kernel and cokernel of the regulator map c_{ij}^{et}.

There is also a syntomic regulator map

$$c_{ij}^{\mathrm{syn}} : \mathrm{gr}_\gamma^i K_j(X_{\overline{K}}, \mathbb{Z}/p^n) \to H^{2i-j}(\overline{Y}, \widetilde{\mathscr{S}}_n(i)).$$

Upon passing to the limit in n, applying a suitable Tate twist, and using the map 1.6.3 we obtain a diagram

$$\mathbb{Q} \otimes \varprojlim_n \mathrm{gr}_\gamma^i K_j(X_{\overline{K}}, \mathbb{Z}/p^n)(-i)$$

$$c^{\mathrm{cris}} \downarrow \qquad\qquad \xrightarrow{\quad c^{\mathrm{et}} \quad}$$

$$H_{\mathrm{cris}}^{2i-j}(X/K_0) \otimes_{K_0} \mathrm{B}_{\mathrm{cris}}(V) \qquad\qquad H^{2i-j}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{cris}}(V).$$

By analyzing the kernel and cokernel of c^{et} Niziol then shows that this induces the desired homomorphism 1.6.1.

(iii) (*Almost mathematics*). This approach, which is the subject of the rest of the paper, was developed by Faltings in a series of papers [13, 14, 15]. To indicate the flavor of Faltings' approach, let us give a brief sketch of his argument in the simplest case of constant coefficients and a proper scheme.

Faltings' approach is in some sense also based on a calculation of a kind of p-adic vanishing cycles. The key result is the *almost purity theorem* (see 2.17) which enables one to compute Galois cohomology in the local situation. Let R be a smooth V-algebra and assume given an étale morphism

$$(1.7.2) \qquad\qquad \mathrm{Spec}(R) \to \mathrm{Spec}(V[T_1^{\pm}, \ldots, T_d^{\pm}])$$

for some integer d. Assume that $\mathrm{Spec}(R/pR) \neq \emptyset$ and that $R \otimes_V \overline{V}$ is an integral domain. Let R_∞ be the algebra obtained by adjoining all the p^s-roots of T_i in $R \otimes_V \overline{V}$ for all i and s. Let $\overline{R} \subset \overline{K}$ denote the normalization of R in the maximal field extension $\mathrm{Frac}(R) \subset L$ which is unramified over $R \otimes_V K$. We then have an inclusion $R_\infty \subset \overline{R}$. Let Δ (resp. Δ_∞) denote the Galois group of $\overline{R}$ (resp. R_∞) over $R_{\overline{V}}$. The main implication (see 3.8) of Faltings' almost purity theorem is that the natural map of Galois cohomology groups

$$H^*(\Delta_\infty, R_\infty/pR_\infty) \to H^*(\Delta, \overline{R}/p\overline{R})$$

is an *almost isomorphism*, which means that the kernel and cokernel is annihilated by any element in the maximal ideal of $\overline{V}$. On the other hand, the Galois group Δ_∞ is easy to describe and in fact is isomorphic to $\mathbb{Z}_p(1)^d$. From this an elementary computation shows that there is an isomorphism

$$(1.7.3) \qquad\qquad R_{\overline{V}} \otimes_{\overline{V}} \overset{\bullet}{\bigwedge}(\overline{V}/p\overline{V})^d(-1) \to H^*(\Delta_\infty, R_\infty/pR_\infty).$$

This (almost) description of $H^*(\Delta, \overline{R}/p\overline{R})$ of course depends on the choice of chart 1.7.2. It can be made canonical by using the basis dT_i/T_i for $\Omega_{R/V}^1$ to identify the map 1.7.3 with an almost isomorphism

$$\overset{\bullet}{\bigwedge}(\Omega_{R/V}^1 \otimes_R R_{\overline{V}}/pR_{\overline{V}}(-1)) \to H^*(\Delta, \overline{R}/p\overline{R})$$

which is independent of choices.

These local calculations enable one to define a new cohomology theory which plays a similar role to the syntomic cohomology. First Faltings defines a ringed topos $(\mathscr{X}_{\overline{K}}, \mathscr{O}_{\mathscr{X}_{\overline{K}}})$ which maps to X_{et} and such that locally on X when one has a map 1.7.2 the cohomology of the structure sheaf modulo p in $\mathscr{X}_{\overline{K}}$ is given by the group cohomology described above. Using these local descriptions one sees that the cohomology groups $H^*(\mathscr{X}_{\overline{K}}, \mathscr{O}_{\mathscr{X}_{\overline{K}}}/(p))$ are almost finitely generated over $\overline{V}/p\overline{V}$, satisfy Poincaré duality, and there is a theory of Chern classes. There is also a natural map

$$(1.7.4) \qquad H^*(X_{\overline{K}}, \mathbb{Z}/(p)) \otimes \overline{V} \to H^*(\mathscr{X}_{\overline{K}}, \mathscr{O}_{\mathscr{X}_{\overline{K}}}/(p))$$

which one then shows is an almost isomorphism.

To go from this isomorphism to a comparison isomorphism between étale and de Rham cohomology then requires a number of steps. Principal among them are the following:

(1) Recall (see section 11) that the ring $\mathrm{B}_{\mathrm{cris}}(V)$ is obtained by localization from a W-algebra $\mathrm{A}_{\mathrm{cris}}(V)$ with G_K-action and Frobenius automorphism. The ring $\mathrm{A}_{\mathrm{cris}}(V)$ contains certain ideals $\mathfrak{m}_\alpha$ (where α runs over positive elements of $\mathbb{Z}[1/p]$), and one can consider almost mathematics over the ring $\mathrm{A}_{\mathrm{cris}}(V)$ (and hence also over $\mathrm{B}_{\mathrm{cris}}(V)$). A map of $\mathrm{A}_{\mathrm{cris}}(V)$-modules $M \to N$ is an almost isomorphism if the kernel and cokernel are annihilated by every $\mathfrak{m}_\alpha$ for all α.

To pass from a mod p theory to a p-adic theory, one first replaces the sheaf $\mathscr{O}_{\mathscr{X}_{\overline{K}}}/(p)$ in the above with a suitable sheaf $\mathscr{A}_{\mathrm{cris}}$ defined using the same construction as the construction of $\mathrm{A}_{\mathrm{cris}}(V)$, and then shows that there is an almost isomorphism of $\mathrm{A}_{\mathrm{cris}}(V)$-modules

$$H^*(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} (\mathrm{A}_{\mathrm{cris}}(V)_{\mathbb{Q}_p}) \to H^*(\mathscr{X}_{\overline{K}}, \mathscr{A}_{\mathrm{cris}}) \otimes \mathbb{Q}.$$

In fact this is not quite correct, and one has to work instead with the formal completion $\widehat{X}$ of X along the closed fiber and a variant topos $\widehat{\mathscr{X}_{\overline{K}}}$ instead of $\mathscr{X}_{\overline{K}}$.

(2) Construct a map

$$H^*_{\mathrm{cris}}(X/K_0) \otimes_{K_0} \mathrm{B}_{\mathrm{cris}}(V) \to H^*(\mathscr{X}_{\overline{K}}, \mathscr{A}_{\mathrm{cris}}) \otimes_{\mathrm{A}_{\mathrm{cris}}(V)} \mathrm{B}_{\mathrm{cris}}(V)$$

thereby obtaining an almost morphism of $\mathrm{B}_{\mathrm{cris}}(V)$-modules

$$H^*_{\mathrm{cris}}(X/K_0) \otimes_{K_0} \mathrm{B}_{\mathrm{cris}}(V) \to H^*(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{cris}}(V)$$

which by the usual argument using Chern classes will be an almost isomorphism.

(3) Lift this almost isomorphism to an actual isomorphism over $\mathrm{B}_{\mathrm{cris}}(V)$.

1.8. Our aim in this paper is to give a detailed account of Faltings' method in the case of a variety with good reduction (and also an open variant) using the basics of almost mathematics and the almost Purity theorem as a "black box" (the interested reader should consult the very thorough treatment of the necessary almost mathematics in [**21**]). We pay particular attention to the $\mathbb{Q}_p$–theory which is not discussed in detail in the literature and consider non-constant coefficients (the ability to deal with nonconstant coefficients is one of the big advantages of Faltings' approach).

1.9. The paper is organized as follows.

In section 2 we give the basic definitions of almost mathematics and state the purity theorem in the form needed in this paper. The main result is 2.17.

As mentioned above, the importance of the purity theorem is that it enables one to (almost) compute certain Galois cohomology groups. We perform these computations in sections 3 and 4. The main results are 3.9 and 4.7. We also consider in this section cohomology with compact support using the theory of logarithmic structure. These computations are the key local results needed for Faltings' approach to p-adic Hodge theory.

We then begin the necessary foundational work for our study of the topos $\mathscr{X}^o_{\overline{K}}$ which will serve as the intermediary between étale and crystalline cohomology.

It is well-known (see [**1**, XI §3]) that if $Y/\overline{K}$ is a smooth $\overline{K}$-scheme, then for every point $y \in Y(\overline{K})$ there exists an open neighborhood U of y such that U is a $K(\pi, 1)$. This means that for any locally constant sheaf of torsion abelian groups the natural map (see section 5 for a description of this map)

$$(1.9.1) \qquad\qquad H^*(\pi_1(U, y), F) \to H^*(U_{\mathrm{et}}, F)$$

is an isomorphism, where $H^*(\pi_1(U, y), F)$ denotes the continuous group cohomology of F viewed as a representation of $\pi_1(U, y)$. In Faltings' approach a generalization of this result is needed. Namely, if X/V is a smooth proper scheme then we need that for any geometric point $\bar{x} \to X$ of the closed fiber, there exists an étale neighborhood U of $\bar{x}$ such that the geometric generic fiber $U_{\overline{K}}$ of U is a $K(\pi, 1)$. We prove this result in section 5 as well as variants for open varieties.

We then introduce the topos $\mathscr{X}_{\overline{K}}$ and its open variant $\mathscr{X}^o_{\overline{K}}$ in section 6. In the proper case there are morphisms of topoi

$$\nu_X : \mathscr{X}_{\overline{K}} \to X_{\mathrm{et}}$$

and

$$u_X : X_{\overline{K},\mathrm{et}} \to \mathscr{X}_{\overline{K}}.$$

Any locally constant sheaf L of $\mathbb{Z}/(p^r)$-modules on $X_{\overline{K}}$ defines a sheaf $\mathscr{L}$ on $\mathscr{X}_{\overline{K}}$ such that $\nu_X^* \mathscr{L} = L$. Furthermore we show in section 6 that the natural map

$$(1.9.2) \qquad\qquad H^*(\mathscr{X}_{\overline{K}}, \mathscr{L}) \to H^*(X_{\overline{K}}, L)$$

is an isomorphism. On the other hand, the topos $\mathscr{X}_{\overline{K}}$ has a structure sheaf $\mathscr{O}_{\mathscr{X}_{\overline{K}}}$ and there is a natural map

$$(1.9.3) \qquad\qquad H^*(\mathscr{X}_{\overline{K}}, \mathscr{L}) \to H^*(\mathscr{X}_{\overline{K}}, \mathscr{L} \otimes \mathscr{O}_{\mathscr{X}_{\overline{K}}}).$$

Perhaps the main result in the whole theory is 6.17 which implies that this map is an almost isomorphism (and in above situation provides the isomorphism 1.7.4).

The proof of 6.17 occupies the following two sections 7 and 8. The introduction of cohomology with compact support poses some additional technical problems. In particular, we spend some time explaining how to compute compactly supported cohomology using group cohomology in section 7. Then in section 8 we prove 6.17 by developing the necessary theory of Chern classes and verifying that the comparison maps between the various cohomology theories are compatible with Chern classes. For technical reasons we consider in this section a more general theory of cohomology with partial compact support along a boundary.

We then turn to lifting this mod p result to a theory with $\mathbb{Q}_p$-coefficients. In section 9 we introduce a variant $\widehat{\mathscr{X}^o_{\overline{K}}}$ of the topos $\mathscr{X}^o_{\overline{K}}$ which is necessary to deal

with $\mathbb{Q}_p$-coefficients. The main point is to replace X by its p-adic completion. Fortunately the almost purity theorem also holds in this context and as we prove in 9.4 the cohomologies of $\widehat{\mathscr{X}_{\overline{K}}^o}$ and $\mathscr{X}_{\overline{K}}^o$ are almost isomorphic.

Then in sections 10 through 15 we explain Faltings' proof of the C_{crys}–conjecture of Fontaine relating p-adic étale cohomology and crystalline cohomology.

Locally the comparison map is obtained by on the one hand computing crystalline cohomology using an embedding into a smooth scheme, and on the other hand computing étale cohomology using group cohomology. Of course to define the map globally requires simplicial techniques. In section 10 we explain how to compute étale cohomology using group cohomology using simplicial techniques (the only real issue is how to deal with base points for simplicial schemes).

In section 11 we review the necessary facts about Fontaine's rings A_{cris} and B_{cris}. We also consider a certain localization $\widetilde{B}_{\mathrm{cris}}$ of B_{cris} which arises when one considers almost mathematics over B_{cris}.

In section 12 we extend our earlier computations of cohomology in the topos $\mathscr{X}_{\overline{K}}^o$ to cohomology with coefficients in sheafified versions of Fontaine's rings A_{cris} and B_{cris}. The main point is to show that the notion of almost isomorphism behaves well upon taking various inverse limits.

Then in 13 we define the transformation from crystalline to étale cohomology over $\widetilde{B}_{\mathrm{cris}}$ and state the comparison theorem over $\widetilde{B}_{\mathrm{cris}}$ (see 13.21), and then in section 14 we complete the proof of the comparison theorem over $\widetilde{B}_{\mathrm{cris}}$. Again the key point is to show that the comparison map is compatible with Chern classes.

Fnally we explain in section 15 how to pass from the comparison theorem over $\widetilde{B}_{\mathrm{cris}}$ to the comparison theorem over B_{cris}. This follows from some formal properties of these rings and Berger's theorem that any de Rham representation is potentially semistable [**3**, 0.7]. The main result is 15.5.

REMARK 1.10. The results of Faltings discussed in this paper suffice for the applications we studied in [**41**].

REMARK 1.11. Faltings has proven a version of the purity theorem in the substantially more general case of a variety with a certain kind of toroidal reduction [**13**, section 2b] (in particular it applies to varieties with semistable reduction). This enables Faltings to prove the C_{st}-conjecture of Fontaine. While the main ideas in this more general context remain the same, there are several additional technical complications and for the sake of exposition we restrict ourselves to the good reduction case in this paper.

Also as discussed in [**13**, p. 258] there are some additional subtleties in developing the $\mathbb{Q}_p$-theory for coefficients in the semistable reduction case. The main problem lies with the construction of the so-called Hyodo-Kato isomorphism which requires certain finiteness properties of crystalline cohomology over the divided power envelope of the closed immersion $\mathrm{Spec}(V/pV) \hookrightarrow \mathrm{Spec}(W[t])$ obtained by sending t to the class of a uniformizer $\pi \in V$. As this divided power envelope is not noetherian such finiteness of cohomology does not follow from standard techniques. In the case of constant coefficients the necessary finiteness properties follow from a comparison with the de Rham-Witt complex, but for general coefficients this method does not apply.

1.12. **Prerequisites.** In order to study open varieties, we work in this paper with logarithmic geometry in the sense of Fontaine, Illusie, and Kato [**29**]. This is

not necessary for the proper case, so the reader not familiar with this theory can still read the paper restricting themselves to the case of empty boundary divisor. For the discussion of the comparison theorem the reader is assumed familiar with crystalline cohomology as well as the convergent topos [**39**], [**45**].

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2. Almost mathematics and the purity theorem

2.1. Fix a ring $\overline{V}$ together with a sequence of principal ideals $\mathfrak{m}_\alpha \subset \overline{V}$ indexed by the positive elements Λ_+ of some subgroup $\Lambda \subset \mathbb{Q}$ which is dense in $\mathbb{R}$ and contains 1. Let π denote a generator of $\mathfrak{m}_1$, and for $\alpha \in \Lambda_+$ let π^α denote a generator of $\mathfrak{m}_\alpha$. We assume that π^α is not a zero–divisor in $\overline{V}$ and that for every $\alpha, \beta \in \Lambda_+$ there exists a unit $u \in \overline{V}^*$ such that $\pi_\alpha \cdot \pi_\beta = u\pi_{\alpha+\beta}$. We let $\mathfrak{m}$ denote the ideal $\cup_{\alpha>0}\mathfrak{m}_\alpha$.

Observe that since each π^α is a nonzero divisor in $\overline{V}$ the ideal $\mathfrak{m}_\alpha$ is a flat $\overline{V}$-module. This implies that $\mathfrak{m}$ is also a flat $\overline{V}$-module, as it is the filtering direct limit of the flat $\overline{V}$-modules $\mathfrak{m}_\alpha$. This implies in particular that the multiplication map $\mathfrak{m} \otimes_{\overline{V}} \mathfrak{m} \to \mathfrak{m}$ is an isomorphism.

EXAMPLE 2.2. The most important example for this paper is the following. Let V be a complete discrete valuation ring of mixed characteristic $(0, p)$, field of fractions K, and perfect residue field k. Let $\overline{K}$ be an algebraic closure of K and let $\overline{V}$ be the integral closure of V in $\overline{K}$. We then take π to be a uniformizer of V, $\Lambda = \mathbb{Q}$, and $\mathfrak{m}_\alpha$ the ideal of elements of valuation $\geq \alpha$.

2.3. Let $\overline{V} - \mathrm{Mod}$ denote the category of $\overline{V}$–modules, and let $\Sigma \subset \overline{V} - \mathrm{Mod}$ denote the full subcategory of modules annihilated by $\mathfrak{m}$. The category Σ is a Serre subcategory in the sense of [**23**, Chapitre III, §1], and therefore one can form the quotient category of $\overline{V} - \mathrm{Mod}$ by the category Σ. We denote this quotient category by $\overline{V}^a - \mathrm{Mod}$. By [**23**, Chapitre III, Proposition 1] the category $\overline{V}^a - \mathrm{Mod}$ is an abelian category. The objects of $\overline{V}^a - \mathrm{Mod}$ are called *almost $\overline{V}$–modules* and the objects of Σ are called *almost zero*. It follows immediately from the definition that the tensor structure on $\overline{V} - \mathrm{Mod}$ induces a tensor structure on $\overline{V}^a - \mathrm{Mod}$. There is a natural localization functor

$$(2.3.1) \qquad\qquad \overline{V} - \mathrm{Mod} \to \overline{V}^a - \mathrm{Mod}$$

which is compatible with the tensor structure.

Morphisms in $\overline{V}^a - \mathrm{Mod}$ have the following simple description. If $M \in \overline{V} - \mathrm{Mod}$, then the category of almost isomorphisms $\phi : M' \to M$ has an initial object given by the morphism $\mathfrak{m} \otimes M \to M$. From this and the definition of the localized category $\overline{V}^a - \mathrm{Mod}$ given in [**23**, Chapitre III, §1] we have

$$(2.3.2) \qquad\qquad \mathrm{Hom}_{\overline{V}^a - \mathrm{Mod}}(M, N) = \mathrm{Hom}_{\overline{V} - \mathrm{Mod}}(\mathfrak{m} \otimes M, N)$$

for any $M, N \in \overline{V} - \mathrm{Mod}$.

DEFINITION 2.4. Let R be a $\overline{V}$–algebra and let M be an R–module.
(i) M is *almost projective* if $\operatorname{Ext}_R^i(M, N) \in \Sigma$ for all R–modules N and $i > 0$.
(ii) M is *almost flat* if $\operatorname{Tor}_i^R(M, N) \in \Sigma$ for all R–modules N and $i > 0$.
(iii) M is *almost faithfully flat* if M is almost flat and for any two R–modules N and N' the map

(2.4.1) $$\operatorname{Hom}_R(N, N') \to \operatorname{Hom}_R(N \otimes M, N' \otimes M)$$

has almost zero kernel.
(iv) M is *almost finitely generated* if for every $\alpha \in \Lambda_+$ there exists a finitely generated R–module N_α and maps $\psi_\alpha : N_\alpha \to M$ and $\phi_\alpha : M \to N_\alpha$ such that $\psi_\alpha \circ \phi_\alpha = \pi^\alpha \cdot \operatorname{id}$ and $\phi_\alpha \circ \psi_\alpha = \pi^\alpha \cdot \operatorname{id}$.

LEMMA 2.5. *If* $f : A \to B$ *is a morphism of* $\overline{V}$–*algebras whose underlying morphism of* $\overline{V}$–*modules is almost faithfully flat, then for any* A–*module* M *the natural map* $M \to M \otimes_A B$ *is almost injective.*

PROOF. Let K be the kernel of $M \to M \otimes_A B$. We need to show that the inclusion $K \hookrightarrow M$ is the zero map. For this it suffices to show that $K \otimes_A B \to M \otimes_A B$ is the zero map. This follows from noting that we have a diagram

(2.5.1) $$K \otimes_A B \xrightarrow{\ a\ } M \otimes_A B \xrightarrow{\ b\ } M \otimes_A B \otimes_A B \xrightarrow{\ c\ } M \otimes_A B,$$

where c is the map obtained from multiplication $B \otimes_A B \to B$, $b \circ a$ is the zero map, and $c \circ b$ is the identity. □

2.6. Let A be a $\overline{V}$–algebra, let P be an A–module, and let $\omega_{P/A}$ denote the canonical morphism

(2.6.1) $$P \otimes_A \operatorname{Hom}(P, A) \to \operatorname{Hom}(P, P)$$

By [**21**, 2.4.29 (1b)], if P is almost finitely generated and almost projective then the map $\omega_{P/A}$ is an almost isomorphism. In particular, if

(2.6.2) $$\operatorname{ev}_{P/A} : P \otimes_A \operatorname{Hom}(P, A) \to A$$

is the evaluation morphism then we obtain an *almost trace map* in $\overline{V}^a - \operatorname{Mod}$

(2.6.3) $$\operatorname{Tr}_{P/A} : \operatorname{Hom}(P, P) \to A$$

by sending $\varphi : P \to P$ to $\operatorname{ev}_{P/A}(\omega_{P/A}^{-1}(\varphi))$.

DEFINITION 2.7. A morphism $A \to B$ of $\overline{V}$–algebras is an *almost étale covering* if the following two conditions hold:
(i) B is almost finitely generated, almost faithfully flat, and almost projective as an A–module,
(ii) B is almost finitely generated and almost projective as a $B \otimes_A B$–module.

2.8. Let $A \to B$ be an almost étale covering. For every $b \in B$ multiplication by b induces a morphism of A–modules $\mu_b : B \to B$. Let

(2.8.1) $$\operatorname{tr}_{B/A} : B \to A$$

be the morphism in $\overline{V}^a - \operatorname{Mod}$ sending b to $\operatorname{Tr}_{B/A}(\mu_b)$.

LEMMA 2.9. *Let* $A \to B$ *be an almost étale covering. Then the trace map* $\operatorname{tr}_{B/A} : B \to A$ *induces an epimorphism in* $\overline{V}^a - \operatorname{Mod}$. *In particular, for any* $a \in A$ *and* $\alpha > 0$ *the element* $\pi^\alpha a$ *is in the image of the trace map* $\mathfrak{m} \otimes B \to A$ *(this is the morphism in* $\overline{V} - \operatorname{Mod}$ *obtained from the trace map and 2.3.2).*

PROOF. Let Q denote the cokernel of the trace map $B \to A$. Since $A \to B$ is almost faithfully flat, the vertical arrows in the following diagram

$$(2.9.1) \qquad \begin{array}{ccccccc} B & \xrightarrow{\mathrm{tr}_{B/A}} & A & \longrightarrow & Q & \longrightarrow & 0 \\ {\scriptstyle \mathrm{pr}_1^*}\downarrow & & \downarrow & & \downarrow & & \\ B \otimes B & \xrightarrow{\mathrm{tr}_{B \otimes B/B}} & B & \longrightarrow & Q \otimes_A B & \longrightarrow & 0 \end{array}$$

induce monomorphisms in $\overline{V}^a - \mathrm{Mod}$. It follows that to show that Q is almost zero it suffices to show that the trace map $B \otimes B \to B$ is almost surjective. This is seen by observing that the composite morphism

$$B \xrightarrow{\mathrm{tr}_{B/B \otimes B}} B \otimes B \xrightarrow{\mathrm{tr}_{B \otimes B/B}} B$$

is the identity in $\overline{V}^a - \mathrm{Mod}$. $\qquad\qquad\square$

DEFINITION 2.10. Let $A \to B$ be an almost étale covering, and let G be a finite group of automorphisms of B over A. We say that B/A is *Galois with group G* if the natural map

$$(2.10.1) \qquad B \otimes B \to \prod_{g \in G} B, \quad b \otimes c \mapsto (\dots, b \cdot g(c), \dots)$$

is an almost isomorphism.

2.11. If $A \to B$ is an almost étale Galois covering with group G, then $\mathrm{tr}_{B/A} : B \to A$ in $\overline{V}^a - \mathrm{Mod}$ can be described as follows. Momentarily let $T : B \to B$ denote the morphism $b \mapsto \sum_{g \in G} g(b)$. The map T is a morphism of A–modules. We claim first of all that the induced morphism $T^a : B \to B$ in $\overline{V}^a - \mathrm{Mod}$ factors through A (since $A \to B$ is almost faithfully flat the map $A \to B$ induces a monomorphism in $\overline{V}^a - \mathrm{Mod}$). Indeed let Q denote the cokernel in the category of A–modules of $A \to B$ so that there is a commutative diagram with exact rows

$$(2.11.1) \qquad \begin{array}{ccccccc} A & \longrightarrow & B & \xrightarrow{\kappa} & Q & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle c} & & \\ B & \longrightarrow & B \otimes_A B & \longrightarrow & Q \otimes_A B & \longrightarrow & 0. \end{array}$$

Since $A \to B$ is almost injective, it suffices to show that $c \circ \kappa \circ T$ is almost zero and hence it suffices to consider the morphism $B \to B \otimes_A B$. Since B/A is almost Galois we can replace $B \otimes B$ by $\prod_{g \in G} B$. This reduces the problem to the case when $B = \prod_{g \in G} A$ in which case the statement is clear.

LEMMA 2.12. *The two morphisms* $T^a, \mathrm{tr}_{B/A} : B \to A$ *in* $\overline{V}^a - \mathrm{Mod}$ *are equal.*

PROOF. We think of these morphisms as actual morphisms $\mathfrak{m} \otimes B \to A$. Then using the fact that $A \to B$ is almost faithfully flat as above, one reduces to the case when $B = \prod_{g \in G} A$ in which case the result is immediate. $\qquad\square$

2.13. In addition there is a canonical morphism $e : \overline{V} \to B \otimes B$ in $\overline{V}^a - \mathrm{Mod}$ defined as follows. Namely, since B is almost projective viewed as a $B \otimes B$-module via the diagonal, we obtain a morphism in $\overline{V}^a - \mathrm{Mod}$

$$(2.13.1) \qquad \mathrm{End}_{B \otimes B}(B) \xrightarrow{\omega_{B/B \otimes B}^{-1}} B \otimes_{B \otimes B} \mathrm{Hom}(B, B \otimes B) \xrightarrow{\mathrm{ev}_{B/B \otimes B}} B \otimes B.$$

We define e to be the morphism obtained by composing this map with the natural map $\overline{V} \to \mathrm{End}_{B \otimes B}(B)$. In what follows, we will usually view e as an element of $\mathrm{Hom}(\mathfrak{m}, B \otimes B)$ using 2.3.2. We often write $\pi^\alpha e$ for the value of $e : \mathfrak{m} \to B \otimes B$ on π^α. Note that the composite of e with the multiplication map $B \otimes B \to B$ is simply the structural morphism $\overline{V} \to B$.

EXAMPLE 2.14. Suppose $A \to B$ is actually an étale morphism (in the usual sense). Then the diagonal $\mathrm{Spec}(B) \to \mathrm{Spec}(B \otimes B)$ is an open and closed immersion, and hence defined by an idempotent $e \in B \otimes B$. An elementary verification shows that the above defined morphism $\overline{V} \to B \otimes B$ is equal to the map sending 1 to e.

EXAMPLE 2.15. Let $\overline{V}$ be as in 2.2, and let $A \to B$ be a morphism of $\overline{V}$–algebras. Assume that A and B have no p–torsion. Then (2.7 (i)) implies that $A[1/p] \to B[1/p]$ is finite étale. Then for any $\alpha > 0$ the element $\pi^\alpha e \in B \otimes B$ is equal in $B \otimes_A B[1/p]$ to π^α times the idempotent defining the diagonal.

2.16. We can now state the version of the almost purity theorem needed for the good reduction case (see [**13**, §2b] for the most general statement). Let V be a complete discrete valuation ring of mixed characteristic $(0, p)$, let K be the field of fractions of V, and assume that the residue field k of V is perfect. Let $d \geq 0$ be an integer, and consider a flat, formally étale morphism of rings

(2.16.1) $$V[T_1, \ldots, T_d] \to R$$

with R noetherian, and $\mathrm{Spec}(R/pR) \neq \emptyset$. Assume further that

$$R_{\overline{V}} := R \otimes_V \overline{V}$$

is an integral domain. Let $\overline{V}[T_1^{1/p^\infty}, \ldots, T_d^{1/p^\infty}]$ denote the $V[T_1, \ldots, T_d]$–algebra obtained by first extending scalars to the normalization $\overline{V}$ of V in $\overline{K}$, and then adjoining p^n–th roots T_i^{1/p^n} of T_i for all i and n. Denote by R_∞ the base change

(2.16.2) $$R_\infty := R \otimes_{V[T_1, \ldots, T_d]} \overline{V}[T_1^{1/p^\infty}, \ldots, T_d^{1/p^\infty}].$$

Let R_∞^o denote the localization

(2.16.3) $$R_\infty^o := R_\infty \otimes_{V[T_1, \ldots, T_d]} V[T_1^\pm, \ldots, T_d^\pm],$$

and let $R_{\infty, \overline{K}}^o$ denote $R_\infty^o \otimes_V K \simeq R_\infty^o \otimes_{\overline{V}} \overline{K}$.

THEOREM 2.17 (Almost Purity). *Let $R_{\infty, \overline{K}}^o \to S_{\overline{K}}^o$ denote a finite étale morphism, and let S_∞ denote the normalization of R_∞ in $S_{\overline{K}}^o$. Then the morphism $R_\infty \to S_\infty$ is an almost étale covering. If $R_{\infty, \overline{K}}^o \to S_{\overline{K}}^o$ is Galois then $R_\infty \to S_\infty$ is also an almost étale Galois covering.*

REMARK 2.18. Throughout the remainder of the paper we will repeatedly return to the setup of 2.16. In what follows, if $T \to \mathrm{Spec}(R)$ is a morphism of schemes, then we write T^o for the fiber product

$$T^o := T \times_{\mathrm{Spec}(V[T_1, \ldots, T_d])} \mathrm{Spec}(V[T_1^\pm, \ldots, T_d^\pm]).$$

If T is affine, say $T = \mathrm{Spec}(A)$, we also write A^o for the coordinate ring of T^o.

2.19. In order to deal systematically with cohomology, it will also be useful to have an almost version of the derived category. Let (T, A) be a ringed topos with A a sheaf of $\overline{V}$–algebras. As in the punctual case discussed above, define an object $M \in A - \mathrm{Mod}$ to be *almost zero* if M is annihilated by π^α for all $\alpha \in \Lambda_+$. Let

$A^a - \mathrm{Mod}$ denote the localization of $A - \mathrm{Mod}$ by the full subcategory of almost zero modules.

Let $D(A)$ denote the derived category of A–modules in T, and let Σ denote the multiplicative set of morphisms in $D(A)$ which induce almost isomorphisms on cohomology sheaves. By [**53**, 10.3.7] one can then form the localized category $\Sigma^{-1}D(A)$ which we denote by $\widetilde{D}(A)$. The category $\widetilde{D}(A)$ is naturally a triangulated category and the localization functor $D(A) \to \widetilde{D}(A)$ is a triangulated functor. Furthermore, the t–structure on $D(A)$ induces a t–structure on $\widetilde{D}(A)$ whose heart is the category $A^a-\mathrm{Mod}$. For any $* \in \{+, -, b, [a, b]\}$ we also have the corresponding subcategories $\widetilde{D}^*(A) \subset \widetilde{D}(A)$.

Let $f : (T', A') \to (T, A)$ be a morphism of ringed topoi with A' also a $\overline{V}$–algebra. If $f : M \to N$ is a morphism in $D(A')$ inducing almost isomorphisms on all cohomology sheaves, then the morphism $Rf_* : Rf_*M \to Rf_*N$ also induces almost isomorphisms on all cohomology sheaves. It follows that the functor Rf_* descends to a functor, which we denote by the same symbols, $Rf_* : \widetilde{D}(A') \to \widetilde{D}(A)$.

REMARK 2.20. As in 2.3.2, for any objects $M, N \in A-\mathrm{Mod}$ there is a canonical isomorphism

(2.20.1) $$\mathrm{Hom}_{A^a-\mathrm{Mod}}(M, N) \simeq \mathrm{Hom}_{A-\mathrm{Mod}}(\mathfrak{m} \otimes M, N).$$

This implies in particular that morphisms in $A^a - \mathrm{Mod}$ can be constructed locally in T by gluing.

We conclude this section with the following three results about projective systems which will be used in what follows.

LEMMA 2.21. Let $\rho_\cdot : \{F_n\} \to \{G_n\}$ be a morphism of projective systems of $\overline{V}$–modules such that for every n the map $\rho_n : F_n \to G_n$ is an almost isomorphism. Then $\rho : \varprojlim F_n \to \varprojlim G_n$ is an almost isomorphism.

PROOF. Say $f = \{f_n\} \in \varprojlim F_n$ maps to zero under ρ, and let $\epsilon \in \Lambda^+$ be an element. Then for every n the element $f_n \in F_n$ is annihilated by π^ϵ whence f is also annihilated by π^ϵ. Therefore ρ is almost injective.

For almost surjectivity, consider an element $g = \{g_n\} \in \varprojlim G_n$. Choose any element $\epsilon \in \Lambda_+$ and write $\epsilon = \epsilon_0 + \epsilon_1$ with ϵ_0, ϵ_1 also in Λ_+. Choose for each n an element $f_n \in F_n$ such that $\rho_n(f_n) = \pi^{\epsilon_0} g_n$. If $\bar{f}_{n+1} \in F_n$ denotes the image of f_{n+1}, then the element $\bar{f}_{n+1} - f_n$ is in the kernel of ρ_n and is therefore annihilated by π^{ϵ_1}. It follows that the sequence $\{\pi^{\epsilon_1} f_n\}$ defines an element of $\varprojlim F_n$ mapping to $\pi^\epsilon g$. $\square$

LEMMA 2.22. Let $\{C_n^\bullet\}$ be a projective systems of bounded below complexes of $\overline{V}$–modules such that for every n and j the map $C_{n+1}^j \to C_n^j$ is almost surjective. Then the cone of the natural map

(2.22.1) $$\varprojlim C_n^\bullet \to R\varprojlim C_n^\bullet$$

is almost zero.

PROOF. The spectral sequence associated to the "stupid filtration" on the complex of projective systems $\{C_n^\bullet\}$ gives a spectral sequence [**10**, 1.4.5]

(2.22.2) $$E_1^{pq} = R^q \varprojlim C_\cdot^p \implies R^{p+q} \varprojlim C_\cdot^\bullet.$$

From this it follows that it suffices to show that $R^i \varprojlim C_\cdot^p$ is almost zero for all p and all $i > 0$. This reduces the proof to the case when $\{C_n^\bullet\}$ is just a projective system of $\overline{V}$–modules with almost surjective transition maps. For every n, let D_n denote the cokernel of the inclusion $\mathfrak{m}C_n \subset C_n$. Then each D_n is almost zero so for every i the map

$$(2.22.3) \qquad R^i \varprojlim \{\mathfrak{m}C_\cdot\} \to R^i \varprojlim \{C_\cdot\}$$

is an almost isomorphism. The projective system $\mathfrak{m}C_\cdot$ has surjective transition maps. Indeed if $\pi^\alpha c \in C_n$ for some $\alpha \in \Lambda_+$, then write $\alpha = \alpha_1 + \alpha_2$ with $\alpha_1, \alpha_2 \in \Lambda_+$. By our assumptions the element $\pi^{\alpha_1}c$ lifts to C_{n+1} so $\pi^\alpha c$ lifts to $\mathfrak{m}C_{n+1}$. It follows that $R^i \varprojlim \{\mathfrak{m}C_\cdot\}$, and hence also $R^i \varprojlim \{C_\cdot\}$, is almost zero for all $i \geq 1$. $\square$

LEMMA 2.23. *Let $\{C_n^\bullet\}$ be a projective system of bounded below complexes of $\overline{V}$–modules such that each of the maps $C_{n+1}^j \to C_n^j$ is almost surjective. Assume that for every integer i the module $R^1 \varprojlim H^i(C_n)$ is almost zero. Then the natural map*

$$(2.23.1) \qquad H^i(\varprojlim C_n) \to \varprojlim H^i(C_n)$$

is an almost isomorphism.

PROOF. The stupid filtration on the complex of projective systems $\{C_n\}$ induces a spectral sequence (the "second spectral sequence of hypercohomology") [**10**, 1.4.5]

$$(2.23.2) \qquad E_2^{pq} = R \varprojlim{}^q H^p(C_n) \implies H^{p+q}(R \varprojlim C_n).$$

By our assumptions $R \varprojlim^q H^p(C_n)$ is almost zero unless $q = 0$ so this shows that the right side of 2.23.1 is almost isomorphic to $H^i(R \varprojlim C_n)$. On the other hand, by the preceding lemma $\varprojlim C_n$ is almost isomorphic to $R \varprojlim C_n$ which implies the lemma. $\square$

3. Galois cohomology

The main application of the Almost Purity Theorem is to the computation of Galois cohomology.

The key lemma is the following:

LEMMA 3.1. *Let G be a finite group, A a ring, and $f : A \to B$ an A–algebra with G–action over A. For $b \in B$ define the trace of b, denoted $\mathrm{tr}(b)$, to be the element $\mathrm{tr}(b) := \sum_{g \in G} g(b)$. Let $b \in B$ be an element with $\mathrm{tr}(b)$ in the image of A, and let a map to $\mathrm{tr}(b)$. Then for any B–module M with semi-linear action of G and $i > 0$, the cohomology groups $H^i(G, M)$ are annihilated by a.*

PROOF. First let us make some standard remarks about injective objects in the category of B-modules with semi-linear action of G.

Let $B\{G\}$ be the non–commutative ring with basis $\{e_g\}_{g \in G}$ and multiplication given by $(be_g) \cdot (b'e_{g'}) = bg(b')e_{gg'}$. Then the category of B–modules with semi-linear action of G is canonically equivalent to the category of left $B\{G\}$–modules. In particular, this category has enough injectives. In fact, injective modules can be constructed as follows. The forgetful functor

$$F : (\text{left } B\{G\}\text{-modules}) \to (\text{left } B\text{-modules})$$

has a right adjoint I given by sending a left B-module M to

$$I(M) := \mathrm{Hom}_{\mathrm{left}\ B\text{-modules}}(B\{G\}, M).$$

Here $I(M)$ is given the structure of a left $B\{G\}$ module by having $e \in B\{G\}$ act by sending $\varphi : B\{G\} \to M$ to the map

$$\tau \mapsto \varphi(\tau \cdot e), \quad \tau \in B\{G\}.$$

Since the functor F is exact, the functor I takes injectives to injectives. It follows that if M is a left $B\{G\}$-module, and $F(M) \hookrightarrow N$ is an inclusion of the underlying B-module into an injective B-module, then the resulting inclusion $M \hookrightarrow I(N)$ is an inclusion of M into an injective $B\{G\}$-module. Note also that I is an exact functor. In particular,

Applying this discussion with $B = \mathbb{Z}$, we see that if M is any abelian group then since I is an exact functor taking injectives to injectives we have

$$H^i(G, I(M)) = 0, \quad \text{for } i > 0.$$

Also note that for an arbitrary ring with G-action B and left B-module M, the underlying $\mathbb{Z}\{G\}$-module of $I(M)$ is isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\{G\}, M)$. In particular, if N is an injective object in the category of left $B\{G\}$-modules then

$$H^i(G, N) = 0, \quad \text{for } i > 0.$$

We conclude that if M is a B-module with semi-linear action of G, then $H^*(G, M)$ can be computed as follows: Choose an injective resolution $M \to I^{\cdot}$ in the category of left $B\{G\}$-modules, and let $I^{\cdot, G}$ be the complex obtained by taking G-invariants. Then

$$H^i(G, M) = H^i(I^{\cdot, G}).$$

Returning to the proof of the lemma, let $M \to I^{\cdot}$ be an injective resolution by B–modules with semi-linear G–action. Let $m \in (I^i)^G$ be an element defining a class in $H^i(G, M)$. Then since $I^{\cdot}$ is a resolution there exists an element $m' \in I^{i-1}$ mapping to m. Then $\mathrm{tr}(bm') \in (I^{i-1})^G$. On the other hand, the image of $\mathrm{tr}(bm')$ in I^i is equal to $\mathrm{tr}(bm)$. Since m is G–invariant, we have $\mathrm{tr}(bm) = \mathrm{tr}(b) \cdot m = f(a) \cdot m$. Therefore $f(a) \cdot m$ is a boundary which proves the lemma. $\qquad\square$

3.2. Let $\overline{V}$ be as in 2.2, and let $f : A \to B$ be an almost étale covering. Assume that A and B are integral domains flat over $\overline{V}$ and integrally closed in their fields of fractions. Then since $A \otimes_V K \to B \otimes_V K$ is étale, we get a morphism between the field of fractions $\mathrm{Frac}(A) \to \mathrm{Frac}(B)$. Assume that this field extension is Galois with group G. Then G also acts on B over A since B is integrally closed in its field of fractions.

LEMMA 3.3. *The extension $A \to B$ is Galois with group G.*

PROOF. We have to show that the map

(3.3.1) $$B \otimes_A B \to \prod_{g \in G} B, \quad b \otimes c \mapsto (\dots, bg(c), \dots)_{g \in G}$$

is an almost isomorphism. Let $e_g \in \prod_{g \in G} B$ be the idempotent $(0, \dots, 1, 0, \dots, 0)$ with a 1 in the g-th component and zeros in the other spots. We first show that for every $\alpha \in \Lambda_+$ the element $\pi^\alpha e_g$ is in the image. This will imply that 3.3.1 is almost surjective.

For this note that the map 3.3.1 is naturally a G-equivariant morphism where $h \in G$ acts on $B \otimes_A B$ by $b \otimes c \mapsto b \otimes h(c)$, and on $\prod_{g \in G} B$ by sending $(f_g)_{g \in G}$ to the element with g-component f_{gh}. This implies that it suffices to show that for every $\alpha \in \Lambda_+$ the element $\pi^\alpha e_1$ is in the image, where $1 \in G$ denotes the identity element. This element is in fact equal to the image of the element $\pi^\alpha e$ defined in 2.13. To verify this note that the map $B \to B[1/\pi]$ is injective so it suffices to show that the image of $\pi^\alpha e$ in $\prod_{g \in G} B[1/\pi]$ is equal to $\pi^\alpha e_1$ which follows from the fact that $A[1/\pi] \to B[1/\pi]$ is étale and Galois with group G. This completes the proof of the almost surjectivity of 3.3.1.

For the almost injectivity, let M denote the kernel of 3.3.1, so we have an almost exact sequence

$$0 \to M \to B \otimes_A B \to \prod_{g \in G} B \to 0.$$

Using this sequence and the fact that $B \otimes_A B$ and $\prod_{g \in G} B$ are almost flat over A, one sees that M is almost flat over A.

Now observe that since $A[1/\pi] \to B[1/\pi]$ is étale and Galois in the usual sense, every element of M is annihilated by some element π^α for some $\alpha \in \Lambda_+$. We therefore get that

$$M = \cup_{\alpha \in \Lambda_+} \operatorname{Tor}_1^A(M, A/(\pi^\alpha)).$$

Now since M is almost flat each of the modules $\operatorname{Tor}_1^A(M, A/(\pi^\alpha))$ is almost zero, which implies that M is also almost zero. $\square$

PROPOSITION 3.4. *Let M be a B–module with semi-linear action of G. Then $H^i(G, M)$ is almost zero for all $i > 0$ and M is almost isomorphic to $B \otimes_A M^G$.*

PROOF. By 2.9 and 2.12, for any $\alpha \in \Lambda_+$ the element $\pi^\alpha \in A$ is in the image of the trace map. From this and 3.1 it follows that π^α annihilates all higher cohomology groups.

For $\alpha \in \Lambda_+$, let $\pi^\alpha e \in B \otimes B$ be the element defined in 2.13. Write $\pi^\alpha e = \sum_i b_i \otimes c_i$ with $b_i, c_i \in B$. Since $\pi^\alpha e$ maps to π^α in B under multiplication we have

(3.4.1) $$\pi^\alpha = \sum_i b_i \cdot c_i.$$

On the other hand,

(3.4.2) $$\sum_i b_i \cdot g(c_i) = 0$$

for any nontrivial $g \in G$. Indeed this can be verified after tensoring with K in which case it is immediate. Define a map

(3.4.3) $$r : M \to B \otimes_A M^G, \quad m \mapsto \sum_i b_i \otimes \operatorname{tr}(c_i \cdot m),$$

and let $s : B \otimes_A M^G \to M$ be the natural map. Then

(3.4.4) $$r \circ s(b \otimes m) = \sum_i b_i \otimes \operatorname{tr}(c_i b m) = \sum_i b_i \otimes \operatorname{tr}(c_i b) m,$$

since for $m \in M^G$ we have $\operatorname{tr}(bm) = \operatorname{tr}(b)m$. Now for any $\epsilon \in \Lambda_+$ we have $\pi^\epsilon \operatorname{tr}(c_i b) \in A$ so we also have

$$\pi^\epsilon \sum_i b_i \otimes \operatorname{tr}(c_i b) m = \pi^\epsilon \sum_i b_i \operatorname{tr}(c_i b) \otimes m.$$

On the other hand, using 3.4.2 we have
(3.4.5)
$$\sum_i b_i \cdot \mathrm{tr}(c_i b) = \sum_i b_i \cdot (\sum_g g(c_i) \cdot g(b)) = \sum_g \sum_i b_i \cdot g(c_i)g(b) = \sum_i b_i \cdot c_i b = \pi^\alpha \cdot b.$$

This shows that $\pi^\epsilon r \circ s$ is equal to multiplication by $\pi^{\epsilon+\alpha}$ for any $\alpha, \epsilon \in \Lambda_+$. The same argument shows that r is B–linear. Computing we also find that

$$(3.4.6) \quad s \circ r(m) = \sum_i b_i \cdot \mathrm{tr}(c_i \cdot m) = \sum_i \sum_g b_i \cdot g(c_i)g(m) = \sum_i b_i \cdot c_i m = \pi^\alpha m.$$

$\square$

To conclude this section, we record some calculations which will be used later. Let $\overline{K}$ denote an algebraically closed field of characteristic 0, let p be a prime, and for an integer $n > 0$ let μ_{p^n} denote the group of p^n–th roots of unity in $\overline{K}$, and let $\mu_{p^\infty} := \cup_n \mu_{p^n} \subset \overline{K}$. We view μ_{p^∞} as a topological group with the discrete topology.

Raising to the p–th power defines a homomorphism $\mu_{p^{n+1}} \to \mu_{p^n}$ and we let $\mathbb{Z}_p(1)$ denote $\varprojlim_n \mu_{p^n}$. We view $\mathbb{Z}_p(1)$ as a topological group with the profinite topology. Let $d \geq 1$ be an integer, and let Δ denote $\mathbb{Z}_p(1)^d$. Let $\overline{V} \subset \overline{K}$ denote a subring containing μ_{p^∞}. If M is a $\overline{V}$–module and n is an integer we write $M(n)$ for $M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)^{\otimes n}$ (if $n < 0$ then $\mathbb{Z}_p(1)^{\otimes n}$ is defined to be the continuous dual of $\mathbb{Z}_p(1)^{\otimes -n}$).

PROPOSITION 3.5. *Let $\mu : \Delta \to \mu_{p^\infty}$ be a continuous homomorphism, r a positive integer, and let L_μ denote the corresponding rank 1 module over $\overline{V}/p^r\overline{V}$ with continuous action of Δ.*
(i) If μ is trivial, then the $\overline{V}/p^r\overline{V}$–module $H^*(\Delta, L_\mu)$ is isomorphic to the exterior algebra $\bigwedge^\bullet (\overline{V}/p^r\overline{V})^d(-1)$.
(ii) If μ is non–trivial, then there exists an integer $n > 0$ such that if ζ is a primitive p^n–th root of 1 then $H^*(\Delta, L_\mu)$ is annihilated by $\zeta - 1$.

PROOF. For $1 \leq i \leq d$ let $\mu_i : \mathbb{Z}_p(1) \to \mu_{p^\infty}$ denote the restriction of μ to the i–th factor of $\mathbb{Z}_p(1)^d$. Then L_μ is isomorphic to tensor product of the representations L_{μ_i} of $\mathbb{Z}_p(1)$. By the Künneth formula A.7 it therefore suffices to prove the proposition in the case when $d = 1$.

Since μ is continuous there exists an integer n such that the image of $\mathbb{Z}_p(1)$ is contained in μ_{p^n}. Let $\mathfrak{g}_s$ denote the quotient $\mathbb{Z}_p(1)/p^{ns}\mathbb{Z}_p(1)$. For each s the action of $\mathbb{Z}_p(1)$ on L_μ factors through $\mathfrak{g}_s$ so

$$(3.5.1) \qquad\qquad H^i(\mathbb{Z}_p(1), L_\mu) = \varinjlim_s H^i(\mathfrak{g}_s, L_\mu).$$

Choose a generator $\sigma \in \mathbb{Z}_p(1)$, and let $D : L_\mu \to L_\mu$ be the map sending $l \in L_\mu$ to $\sigma(l) - l$. Let $N_s : L_\mu \to L_\mu$ denote the map $l \mapsto \sum_{g \in \mathfrak{g}_s} g(l)$. Then as in [**43**, VIII, §4] for $i > 0$ the cohomology groups $H^i(\mathfrak{g}_s, L_\mu)$ are equal to the i–th cohomology group of the complex

$$(3.5.2) \qquad L_\mu \xrightarrow{\ D\ } L_\mu \xrightarrow{\ N_s\ } L_\mu \xrightarrow{\ D\ } L_\mu \xrightarrow{\ N_s\ } \cdots .$$

Furthermore, the maps $H^i(\mathfrak{g}_s, L_\mu) \to H^i(\mathfrak{g}_{s+1}, L_\mu)$ occuring in the direct limit 3.5.1 are given by the morphisms of complexes

(3.5.3)

$$
\begin{array}{ccccccccccc}
L_\mu & \xrightarrow{D} & L_\mu & \xrightarrow{N_s} & L_\mu & \xrightarrow{D} & L_\mu & \xrightarrow{N_s} & L_\mu & \xrightarrow{D} & \cdots \\
{\scriptstyle\text{id}}\downarrow & & {\scriptstyle\text{id}}\downarrow & & {\scriptstyle\times p^n}\downarrow & & {\scriptstyle\times p^n}\downarrow & & {\scriptstyle\times p^{2n}}\downarrow & & \\
L_\mu & \xrightarrow{D} & L_\mu & \xrightarrow{N_{s+1}} & L_\mu & \xrightarrow{D} & L_\mu & \xrightarrow{N_{s+1}} & L_\mu & \xrightarrow{D} & \cdots .
\end{array}
$$

Since L_μ is annihilated by a power of p this implies that $H^i(\mathbb{Z}_p(1), L_\mu) = 0$ for all $i \geq 2$.

If L_μ is the trivial representation, then N_s is multiplication by p^{ns} and D is the zero map. Since L_μ is annihilated by a power of p it follows that $H^1(\mathbb{Z}_p(1), L_\mu) = \overline{V}/p^r\overline{V}$. Now this isomorphism depends on the choice of generator $\sigma \in \mathbb{Z}_p(1)$. We can make this more canonical as follows. An element $\tau \in \mathbb{Z}_p(-1)$ can be thought of as a morphism $\tau : \mathbb{Z}_p(1) \to \mathbb{Z}_p$. For any such morphism τ, we obtain an extension of $\mathbb{Z}_p(1)$–modules

(3.5.4) $$0 \to \mathbb{Z}_p \to E \to \mathbb{Z}_p \to 0$$

where E is the $\mathbb{Z}_p$–space $\mathbb{Z}_p^2$ with action of $\sigma \in \mathbb{Z}_p(1)$ given by $(a, b) \mapsto (a, b + a \cdot \tau(\sigma))$. This defines an isomorphism $\mathbb{Z}_p(-1) \to H^1(\mathbb{Z}_p(1), L_\mu)$ proving (i).

For (ii), let $\sigma \in \mathfrak{g}_s$ be a generator, let $a \in L_\mu$ be a nonzero element, and let ζ be a primitive p^n–th root of 1. By choosing n appropriately we have $\sigma(a) = \zeta^\alpha a$ for some positive integer α prime to p (since L_μ is nontrivial). If $u \in \overline{V}^*$ denotes the unit $1 + \zeta + \cdots + \zeta^{\alpha-1}$ then we can also write this as $D(a) = (\zeta - 1)ua$. It follows that $(\zeta - 1)a$ is equal to $D(u^{-1}a)$. In particular for any element $a \in L_\mu$ in the kernel of N_s the corresponding cohomology class is killed by $(\zeta - 1)$. $\qquad\square$

3.6. Now let $\overline{V}$, R, etc. be as in 2.16. Recall that $\operatorname{Spec}(R_K)$ is assumed geometrically connected. Let $k(R)$ denote the field of fractions of R, fix an algebraic closure $k(R) \subset \overline{k(R)}$, let $k(R)^u \subset \overline{k(R)}$ denote the union of the finitely generated subfields $k(R) \subset L \subset \overline{k(R)}$ for which the normalization of R_K^o in L is étale over R_K^o. Let $\overline{R}$ denote the normalization of R in $k(R)^u$.

Let $\eta \to \operatorname{Spec}(R_K)$ denote the geometric generic point $\operatorname{Spec}(\overline{k(R)})$ and let Δ denote the fundamental group $\pi_1(\operatorname{Spec}(R_{\overline{K}}^o), \eta)$. Also let Δ_∞ denote the Galois group of R_∞ over $R_{\overline{V}}$. Note that Δ_∞ is isomorphic to $\mathbb{Z}_p(1)^d$. The natural map $R_\infty \to \overline{R}$ defines a homomorphism $\delta : \Delta \to \Delta_\infty$. Let Σ denote its kernel.

LEMMA 3.7. *The morphism $\delta : \Delta \to \Delta_\infty$ is surjective.*

PROOF. Let $z \in \operatorname{Spec}(R_{\overline{V}})$ be a point in the closed fiber, let R' denote the Zariski local ring of z, and let Δ' denote the fundamental group of $\operatorname{Spec}(R') \times_{\operatorname{Spec}(R_{\overline{V}})} \operatorname{Spec}(R_{\overline{K}}^o)$. Then there is a canonical map $\Delta' \to \Delta$ and it suffices to prove that $\Delta' \to \Delta_\infty$ is surjective. This is equivalent to showing that for any collections of nonnegative integers $\underline{a} = (a_1, \ldots, a_d)$ the ring

(3.7.1) $$R_{\underline{a}} := R'[x_1, \ldots, x_r]/(x_1^{p^{a_1}} - t_1, \ldots, x_r^{p^{a_d}} - t_d)$$

is an integral domain, where $t_i \in R'$ denotes the image of T_i in R' under 2.16.1. Since R' is a local ring, for this it suffices to show that the closed fiber $\operatorname{Spec}(R_{\underline{a}}/pR_{\underline{a}})$

is connected. This is clear because $\mathrm{Spec}(R_{\underline{a}}/pR_{\underline{a}}) \to \mathrm{Spec}(R'/pR')$ is a homeomorphism, and $(R'/pR')_{\mathrm{red}}$ is an integral domain being a localization of the ring $(R_{\overline{V}}/pR_{\overline{V}})_{\mathrm{red}}$ which is an integral domain by assumption. $\square$

LEMMA 3.8. *For any $\overline{R}$–module M with continuous action of Δ and $i \geq 0$, the natural map $H^i(\Delta_\infty, M^\Sigma) \to H^i(\Delta, M)$ is an almost isomorphism.*

PROOF. Consider the spectral sequence

$$(3.8.1) \qquad E_1^{pq} = H^p(\Delta_\infty, H^q(\Sigma, M)) \implies H^{p+q}(\Delta, M)$$

arising from writing the functor $M \mapsto M^\Delta$ as the composite of functors

$$(3.8.2) \qquad \mathrm{Rep}_{\mathrm{cts}}(\Delta) \xrightarrow{M \mapsto M^\Sigma} \mathrm{Rep}_{\mathrm{cts}}(\Delta_\infty) \xrightarrow{N \mapsto N^{\Delta_\infty}} \text{(abelian groups).}$$

The map in the Lemma is then the natural map $E_1^{i0} \to H^i(\Delta, M)$. To prove the lemma it therefore suffices to show that for $q > 0$ the group $H^q(\Sigma, M)$ is almost zero. For this note that

$$(3.8.3) \qquad H^q(\Sigma, M) = \varinjlim_H H^q(\Sigma/H, M^H),$$

where the limit is taken over normal subgroups $H \subset \Sigma$ of finite index, so it suffices to prove that each $H^q(\Sigma/H, M^H)$ is almost zero. Let $R_\infty \to S_H$ be the finite extension corresponding to H. Then by 2.17 the ring S_H is almost étale and Galois over R_∞. The result therefore follows from 3.4. $\square$

THEOREM 3.9. *Let $r > 0$ be an integer. Then the ring $H^*(\Delta, \overline{R}/p^r\overline{R})$ is almost isomorphic to the exterior algebra $R_{\overline{V}} \otimes_{\overline{V}} \bigwedge^\bullet (\overline{V}/p^r\overline{V})^d(-1)$.*

PROOF. By 3.8 the natural map

$$(3.9.1) \qquad H^*(\Delta_\infty, R_\infty/p^r R_\infty) \to H^*(\Delta, \overline{R}/p^r\overline{R})$$

is an almost isomorphism. It therefore suffices to compute $H^*(\Delta_\infty, R_\infty/p^r R_\infty)$.

LEMMA 3.10. *let $S \to S'$ be a flat ring homomorphism, and let M be an S–module with continuous action of a profinite group G over S. Then the natural map*

$$(3.10.1) \qquad H^*(G, M) \otimes_S S' \to H^*(G, M \otimes_S S')$$

is an isomorphism.

PROOF. Since $S \to S'$ is flat, we have

$$(3.10.2) \qquad (M \otimes_S S')^H = M^H \otimes_S S'$$

for any subgroup $H \subset G$. By the definition of continuous group cohomology, we have
$(3.10.3)$
$$H^i(G, M) = \varinjlim_{H \subset G} H^i(G/H, M^H), \quad H^i(G, M \otimes_S S') = \varinjlim_{H \subset G} (G/H, M^H \otimes_S S')$$

where the limit is taken over normal subgroups $H \subset G$ of finite index. This therefore reduces the proof to the case when G is finite.

Let $P_\bullet \to \mathbb{Z}$ denote the canonical resolution of $\mathbb{Z}$ defined in [**43**, VII, §3]. Recall that P_j is isomorphic to $\mathbb{Z}^{G^{j+1}}$. Then by loc. cit. the cohomology groups

$H^i(G, M)$ are equal to the the cohomology groups of $\mathrm{Hom}(P_\bullet, M)$ and similarly for $H^i(G, M \otimes_S S')$. The lemma then follows by observing that the natural map

$$(3.10.4) \qquad \mathrm{Hom}(P_\bullet, M) \otimes_S S' \to \mathrm{Hom}(P_\bullet, M \otimes_S S')$$

is an isomorphism since each P_j is a finitely generated free module. $\qquad\square$

It follows that $H^*(\Delta_\infty, R_\infty/p^r R_\infty)$ is isomorphic to $H^*(\Delta_\infty, S_\infty/p^r S_\infty) \otimes_{S_{\overline{V}}} R_{\overline{V}}$, where $S = V[T_1, \dots, T_d]$. Now the Galois cohomology of $S_\infty/p^r S_\infty$ can be computed directly as follows. For positive numbers $\underline{a} = (a_1, \dots, a_d)$ lying in $\mathbb{Z}[1/p]$ we have a well–defined element $T^{\underline{a}} = T^{a_1} \cdots T^{a_d} \in S_\infty$. Let $\mu_{\underline{a}} : \mathbb{Z}_p(1)^d \to \mu_{p^\infty}$ be the character giving the action on $T^{\underline{a}}$. If we consider an element $\varprojlim_n (\zeta_{1,n}, \dots, \zeta_{d,n}) \in \mathbb{Z}_p(1)^d$, where $\zeta_{i,n}$ is an n–th root of unity, and if we write $a_i = b_i/p^m$ for some m, then $\mu_{\underline{a}}$ sends $\varprojlim_n (\zeta_{1,n}, \dots, \zeta_{d,n})$ to $\zeta_{1,m}^{b_1} \cdots \zeta_{d,m}^{b_d}$. As in 3.5 let $L_{\mu_{\underline{a}}}$ denote the corresponding rank 1 free $(\overline{V}/p^r\overline{V})$–module with continuous Δ_∞–action. Then as an S–module with continuous Δ_∞–action, we have

$$(3.10.5) \qquad S_\infty = (\oplus_{\underline{a}, 0 \le a_i < 1 \text{ for all } i} L_{\mu_{\underline{a}}}) \otimes_{\overline{V}} S_{\overline{V}}.$$

Using 3.10 again we have

$$(3.10.6) \qquad H^*(\Delta_\infty, S_\infty/p^r S_\infty) \simeq H^*(\Delta_\infty, \oplus_{\underline{a}} L_{\mu_{\underline{a}}}) \otimes_{\overline{V}} S_{\overline{V}}.$$

By (3.5 (ii)) for $\underline{a} \ne (0, \dots, 0)$ the group $H^*(\Delta_\infty, L_{\mu_{\underline{a}}})$ is almost zero. Therefore, $H^*(\Delta_\infty, \oplus_{\underline{a}} L_{\mu_{\underline{a}}})$ is almost isomorphic to $H^*(\Delta_\infty, L_{\mu_{\underline{0}}})$ which by (3.5 (i)) is isomorphic to the exterior algebra $\bigwedge^\bullet (\overline{V}/p^r\overline{V})^d(-1)$. This completes the proof of 3.9. $\qquad\square$

3.11. Let $J \subset R$ be the ideal $(T_1 \cdots T_d)$, and let $\overline{J} \subset \overline{R}$ be the ideal of elements mapping to nilpotent elements in $\overline{R}/(T_1 \cdots T_d)$. Then we can also use the above techniques to (almost) compute $H^*(\Delta, \overline{J}/p^r\overline{J})$. By 3.8 and 3.10 the group $H^*(\Delta, \overline{J}/p^r\overline{J})$ is almost isomorphic to $H^*(\Delta_\infty, J_\infty/p^r J_\infty) \otimes_{S_{\overline{V}}} R_{\overline{V}}$, where $J_\infty \subset S_\infty$ is the ideal generated by elements $T_1^{a_1} \cdots T_d^{a_d}$ with $a_i \in \mathbb{Z}[1/p]$ and $a_i > 0$ for all i. Going through the last step in the above proof one obtains:

THEOREM 3.12. *All the morphisms in the following diagram*

$$(3.12.1) \qquad J \otimes_V \bigwedge^\bullet (\overline{V}/p^r\overline{V})^d(-1) \to J \otimes_{R_{\overline{V}}} H^*(\Delta, \overline{R}/p^r\overline{R}) \to H^*(\Delta, \overline{J}/p^r\overline{J})$$

are almost isomorphism.

3.13. In particular, there is a trace map $\mathrm{tr} : H^d(\Delta, \overline{J}) \to J \otimes_V \overline{V}/p^r\overline{V}(-d)$ in $\overline{R}^a - \mathrm{Mod}$ defined as the composition of the inverse of 3.12.1 and the canonical isomorphism

$$J \otimes_V \bigwedge^d (\overline{V}/p^r\overline{V})^d(-1) \simeq J \otimes_V (\overline{V}/p^r\overline{V})(-d).$$

If L is a representation of Δ on a finitely generated free $\mathbb{Z}/p^r$–module, we obtain a map

$$(3.13.1) \qquad R\Gamma_\Delta(L \otimes \overline{R}) \to R\mathrm{Hom}(R\Gamma_\Delta(L^* \otimes \overline{J}), J \otimes_V \overline{V}/p^r\overline{V}(-d)[-d])$$

defined in $\widetilde{D}(\overline{R})$. Here L^* denotes the dual $\mathrm{Hom}(L, \mathbb{Z}/p^r)$ and

$$(3.13.2) \qquad \Gamma_\Delta : \mathrm{Rep}^{\mathrm{cts}}(\Delta) \to (\mathrm{Groups})$$

is the functor taking Δ–invariants.

THEOREM 3.14. *The morphism 3.13.1 is an almost isomorphism.*

PROOF. By considering the distinguished triangles associated to the exact sequences

(3.14.1) $$0 \longrightarrow L/pL \xrightarrow{\times p} L \longrightarrow L/p^{r-1}L \longrightarrow 0$$

one reduces to the case when $r = 1$. Furthermore, using 3.8 and 3.10 it suffices to consider the case of $R = S$, $\overline{R} = S_\infty$, and $\Delta = \Delta_\infty$.

Let $\overline{k}$ denote the algebraic closure of the residue field k. For any continuous representation M of Δ_∞ on a finite dimensional $\overline{k}$–vector space, we then show that the map

(3.14.2) $R\Gamma_{\Delta_\infty}(M \otimes_{\overline{V}} S_\infty) \to R\mathrm{Hom}(R\Gamma_{\Delta_\infty}(M^* \otimes J_\infty), J_\infty \otimes_V \overline{V}/p\overline{V}(-d)[-d])$

is an isomorphism. By devissage as above it further suffices to consider the case when M is an irreducible representation.

LEMMA 3.15. *Any finite dimensional irreducible representation of Δ_∞ on $\overline{k}$–vector spaces is of rank* 1.

PROOF. A continuous representation of Δ_∞ on a finite dimensional $\overline{k}$–vector space W is given by d–commuting endomorphisms of W (after choosing a basis for Δ_∞). By standard linear algebra, if $\dim(W) > 1$ such a representation has a nontrivial invariant subspace. $\square$

It follows that it suffices to consider the case when $M = L_\mu$ for some character $\mu : \Delta_\infty \to \mu_{p^\infty}$ (notation as in 3.5). Choosing a compatible family of p^n–roots of unity ζ_n, the representation μ is given by a collection $(a_1, \ldots, a_d)$ with $a_i \in \mathbb{Z}[1/p]$ and $0 \le a_i < 1$. Using 3.12 the map 3.14.2 can be identified with the natural map

$$T_1^{1-a_1} \cdots T_d^{1-a_d} \cdot \bigwedge{}^{\bullet}(S_{\overline{V}}/(p))^d(-1)$$

(3.15.1)
$$\downarrow$$

$$\mathrm{Hom}(T_1^{a_1} \cdots T_d^{a_d} \cdot \bigwedge{}^{\bullet}(S_{\overline{V}}/(p))^d(-1), T_1 \cdots T_d \cdot S_{\overline{V}}/(p)(-d)[-d])$$

which is evidently an isomorphism. $\square$

3.16. More generally if $E \subset \{1, \ldots, n\}$ is a subset and $D_E \subset D$ the union of the divisors D_i with $i \in E$, then we define $\overline{J}_E \subset \overline{R}$ to be the ideal of elements mapping to nilpotent elements in the coordinate ring of $D_E \times_X \mathrm{Spec}(\overline{R})$. If $J_E \subset R$ denotes the ideal of D_E then the same argument used above shows that the canonical map

(3.16.1) $$J_E \otimes_V \bigwedge{}^{\bullet}(\overline{V}/p^r\overline{V})^d(-1) \to H^*(\Delta, \overline{J}_E/p^r\overline{J}_E)$$

is an almost isomorphism for all r.

If $F \subset \{1, \ldots, n\}$ denotes the complement of E, then there is a canonical map $\overline{J}_F \otimes \overline{J}_E \to \overline{J}$ which induces for any representation L of Δ a canonical map

(3.16.2) $$R\Gamma_\Delta(L \otimes \overline{J}_E) \otimes^{\mathbb{L}} R\Gamma_\Delta(L^* \otimes \overline{J}_F) \to R\Gamma_\Delta(\overline{J}).$$

Composing with the trace map we obtain a morphism

(3.16.3) $$R\Gamma_\Delta(L \otimes \overline{J}_E) \to R\mathrm{Hom}(R\Gamma_\Delta(L^* \otimes \overline{J}_F), J \otimes_V \overline{V}/p^r\overline{V}(-d)[-d]).$$

A local calculation as in the proof of 3.14 shows that this is an almost isomorphism.

4. Logarithmic geometry

4.1. In order to make the isomorphism in 3.9 canonical (i.e. independent of the choice of the morphism 2.16.1) it is useful to introduce a bit of log geometry in the sense of Fontaine, Illusie, and Kato [**29**].

4.2. Let S be a scheme which we view as a prelog scheme with the trivial log structure, and let $(X, M_X) \to S$ be a morphism of prelog schemes (we usually write $\alpha : M_X \to \mathscr{O}_X$ for the map in the definition of a prelog structure). Let E be a quasi–coherent $\mathscr{O}_X$–module.

DEFINITION 4.3 ([**38**, 1.1.2]). A *log derivation of $(\mathscr{O}_X, M_X)$ with values in E* is a pair $\partial = (D, \delta)$, where $\delta : \mathscr{O}_X \to E$ is a derivation and $D : M_X \to E$ is an additive map such that for every local section $m \in M_X$ we have $\alpha(m)D(m) = \delta(\alpha(m))$ (so $D(m)$ should be viewed as "$d\log(\alpha(m))$").

REMARK 4.4. Note that the map D and the equation $\alpha(m)D(m) = \delta(\alpha(m))$ determines the derivation δ since a derivation is determined by its value on the sections of $\mathscr{O}_X^*$.

4.5. If $\Omega^1_{(X,M_X)/S}$ denotes the logarithmic differentials [**29**, 1.7], then there is a universal log derivation (see for example [**38**, 1.1])

$$(4.5.1) \qquad d : \mathscr{O}_X \to \Omega^1_{(X,M_X)/S}, \quad D : M_X \to \Omega^1_{(X,M_X)/S}.$$

In other words, if E is a quasi–coherent $\mathscr{O}_X$–module then there is a canonical bijection
(4.5.2)
    (log derivations of $(\mathscr{O}_X, M_X)$ with values in E) $\simeq \operatorname{Hom}_{\mathscr{O}_X}(\Omega^1_{(X,M_X)/S}, E)$.

Note in particular that if X is affine and $P \to \Gamma(X, M_X)$ is a chart, then since $\Omega^1_{(X,P)/S} \simeq \Omega^1_{(X,M_X)/S}$ (this equality is explained in [**29**, 1.7]), to give a morphism $\Omega^1_{(X,M_X)/S} \to E$ is equivalent to giving an additive map $D : P \to \Gamma(X, E)$ and a derivation $\delta : \Gamma(X, \mathscr{O}_X) \to \Gamma(X, E)$ so that for all $p \in P$ we have $\alpha(p)D(p) = \delta(\alpha(p))$.

4.6. Now let the notation be as in 3.9, let $X = \operatorname{Spec}(R)$, let M_X be the log structure defined by the divisor $\operatorname{Spec}(R) - \operatorname{Spec}(R^o)$, let M_R denote $\Gamma(X, M_X)$, and let $\Omega^1_{(R,M_R)/V}$ denote the global sections of $\Omega^1_{(X,M_X)/\operatorname{Spec}(V)}$. Since there exists a morphism as in 2.16.1, the image of M_R under the map $d\log$ generates $\Omega^1_{(R,M_R)/V}$, and therefore for any R–module E, a homomorphism $\Omega^1_{(R,M_R)/V} \to E$ is determined by an additive map

$$(4.6.1) \qquad\qquad D : M_R \to E.$$

As in 3.9 let r be an integer. We now construct a morphism

$$(4.6.2) \qquad\qquad c : \Omega^1_{(R,M_R)/V} \to H^1(\Delta, \overline{R}/p^r\overline{R}(1))$$

depending only on $\operatorname{Spec}(R)$ with its log structure. For this we first construct an additive map

$$(4.6.3) \qquad\qquad D : M_R \to H^1(\Delta, \overline{R}/p^r\overline{R}(1)).$$

Note that the map $M_R \to R$ is an inclusion. For every $s \geq 0$ let $M_s \subset \overline{R}$ denote the set of elements $f \in \overline{R}$ with $f^{p^s} \in M_R$. There is then a natural exact sequence in the category of integral monoids

$$(4.6.4) \qquad 0 \longrightarrow \mu_{p^s} \longrightarrow M_s \xrightarrow{\times p^s} M_R \longrightarrow 0.$$

Raising to the p-th power induces a surjection $M_{s+1} \to M_s$. Let $E := \varprojlim M_s$ denote the projective limit, and note that the sequences 4.6.4 induce an exact sequence (again in the category of integral monoids)

$$(4.6.5) \qquad 0 \longrightarrow \mathbb{Z}_p(1) \longrightarrow E \longrightarrow M_R \longrightarrow 0.$$

The action of Δ on $\overline{R}$ induces an action of Δ on E which makes this an extension in the category of monoids with continuous Δ–action. The sequence 4.6.5 therefore induces an additive map

$$(4.6.6) \qquad\qquad M_R \to H^1(\Delta, \mathbb{Z}_p(1)).$$

Composing with the natural map $H^1(\Delta, \mathbb{Z}_p(1)) \to H^1(\Delta, \overline{R}/p^s\overline{R}(1))$ we obtain a map

$$(4.6.7) \qquad\qquad D : M_R \to H^1(\Delta, \overline{R}/p^s\overline{R}(1)).$$

THEOREM 4.7. *The map $M_R^{gp} \otimes_{\mathbb{Z}} R/p^s\overline{R} \to H^1(\Delta, R/p^s\overline{R}(1))$ induced by 4.6.7 descends to $\Omega^1_{(R,M_R)/V}$, and the resulting map*

$$(4.7.1) \qquad\qquad \Omega^1_{(R,M_R)/V} \otimes_R R_{\overline{V}}/p^s R_{\overline{V}}(-1) \to H^1(\Delta, \overline{R}/p^s\overline{R})$$

induces an almost isomorphism

$$(4.7.2) \qquad\qquad \overset{\bullet}{\bigwedge}(\Omega^1_{(R,M_R)/V} \otimes_R R_{\overline{V}}/p^s R_{\overline{V}}(-1)) \to H^*(\Delta, \overline{R}/p^s\overline{R}).$$

PROOF. For $f \in R^*$, let T_f denote the $\mathbb{Z}_p(1)$–torsor with Δ–action inducing $c(f) \in H^1(\Delta, \overline{R}/p^s\overline{R}(1))$. The torsor T_f is equal to the projective limit $T_f = \varprojlim T_{f,n}$, where $T_{f,n}$ is the set of elements $g \in \overline{R}^*$ such that $g^{p^n} = f$ and the map $T_{f,n+1} \to T_{f,n}$ sends g to g^p.

LEMMA 4.8. *The map $c|_{R^*} : R^* \to H^1(\Delta, \overline{R}/p^s\overline{R}(1))$ factors through the image of $d\log : R^* \to \Omega^1_{(R,M_R)/V}$.*

PROOF. It suffices to show that if $d\log(f) = d\log(g)$ then the torsors with Δ–action T_f and T_g are isomorphic, or equivalently that the torsor with Δ–action $T_{fg^{-1}} \simeq T_f \wedge T_g^{-1}$ is trivial. This is clear, for if $d\log(f) = d\log(g)$ then

$$(4.8.1) \qquad\qquad d\log(fg^{-1}) = d\log(f) - d\log(g) = 0$$

and hence $d(fg^{-1}) = 0$. Since $\mathrm{Spec}(R \otimes_V K) \to \mathrm{Spec}(K)$ is formally étale this implies that $fg^{-1} \in R \cap K = V$. This in turn implies that the action of Δ on $T_{fg^{-1}}$ is trivial which proves the lemma. $\square$

For the proof of 4.7, consider first the case when $R = S := V[T_1, \ldots, T_d]$. In this case $M_R = V^* \oplus \mathbb{N}^d$, where the i–th generator $e_i \in \mathbb{N}^d$ maps to the element t_i in S. The class $c(e_i) \in H^1(\Delta, \overline{S}/p^r\overline{S})$ is then given by the $\mathbb{Z}_p$–torsor of p–power roots of t_i in $\overline{S}$. Note that the action of Δ on this torsor factors through Δ_∞. It follows that in this case the map c factors through a map

$$(4.8.2) \qquad\qquad \tilde{c} : V^* \oplus \mathbb{N}^d \to H^1(\Delta_\infty, S_\infty/p^r S_\infty(1)).$$

Furthermore, $\tilde{c}(V^*) = 0$ which proves that $\tilde{c}$ factors through $\Omega^1_{(S,M_S)/V}$. Chasing through the construction of the isomorphism

$$(4.8.3) \qquad (S_{\overline{V}}/p^r S_{\overline{V}})^d \simeq H^1(\Delta_\infty, S_\infty/p^r S_\infty(1))$$

given in 3.9, one sees that the map $\tilde{c}$ induces an isomorphism

$$(4.8.4) \quad \Omega^1_{(S,M_S)/V} \otimes_S (S_{\overline{V}}/p^r S_{\overline{V}}) \simeq (S_{\overline{V}}/p^r S_{\overline{V}}) \otimes_{\mathbb{Z}} \mathbb{Z}^d \to H^1(\Delta_\infty, S_\infty/p^r S_\infty(1)).$$

This proves the theorem in the case when $R = S$.

For the general case choose a formally étale map $S \to R$ as in 2.16.1.

LEMMA 4.9. *The natural map $R^* \oplus_{S^*} M_S \to M_R$ is surjective.*

PROOF. Let $t_i \in R$ be the image of T_i. Then to prove the lemma it suffices to show that any element $m \in M_R \subset R$ can be written as $ut_1^{a_1} \cdots t_d^{a_d}$ where $u \in R^*$. This is clear, for if $m \in M_R$ let a_i denote the valuation of the image of m in the local ring of the generic point of the divisor of $\mathrm{Spec}(R)$ lying over $\mathrm{Spec}(S/(T_i)) \subset \mathrm{Spec}(S)$ (if the inverse image is empty then set $a_i = 1$). Since R is regular, the elements m and $t_1^{a_1} \cdots t_d^{a_d}$ must differ by a unit. $\square$

By the discussion at the beginning of the proof the composite

$$(4.9.1) \qquad M_S \oplus_{S^*} R^* \longrightarrow M_R \xrightarrow{\ c\ } H^1(\Delta, \overline{R}/p^r\overline{R}(1))$$

is equal to the composite

$$(4.9.2) \qquad M_S \oplus_{S^*} R^* \xrightarrow{\ a\ } \Omega^1_{(S,M_S)/V} \otimes_S R \longrightarrow H^1(\Delta_\infty, S_\infty/p^r S_\infty) \otimes_S R$$

$$H^1(\Delta, \overline{R}/p^r\overline{R}).$$

where the map a is induced by the map $d\log : R^* \to \Omega^1_{(R,M_R)/V} \simeq \Omega_{(S,M_S)/V} \otimes_S R$. By the surjectivity in 4.9 first statement in 4.7 follows. The second statement in 4.7 then also follows from the case $S = R$ since the map

$$(4.9.3) \qquad H^*(\Delta_\infty, S_\infty/p^r S_\infty) \otimes_{S_{\overline{V}}} R_{\overline{V}} \to H^*(\Delta, \overline{R}/p^r\overline{R})$$

is an almost isomorphism by the proof of 3.9. $\square$

COROLLARY 4.10. *There is a canonical (i.e. depending only on $\mathrm{Spec}(R)$ with its boundary divisor) almost isomorphism, called the trace map,*

$$(4.10.1) \qquad H^d(\Delta, \overline{J}/p^r\overline{J}) \to \Omega^d_{R/V} \otimes_V \overline{V}/p^r\overline{V}(-d).$$

PROOF. This follows from 3.12, 4.7, and the observation that there is a canonical isomorphism $J \otimes \Omega^1_{(R,M_R)/V} \simeq \Omega^1_{R/V}$. $\square$

Restating 3.14 we obtain:

COROLLARY 4.11. *The trace map induces a canonical isomorphism in $\widetilde{D}(R_{\overline{V}})$*

$$(4.11.1) \qquad R\Gamma_\Delta(L \otimes \overline{R}) \to R\mathrm{Hom}(R\Gamma_\Delta(L^* \otimes \overline{J}), \Omega^d_{R/V} \otimes_V \overline{V}/p^r\overline{V}(-d)[-d])$$

More generally, for a subset $E \subset \{1, \dots, n\}$ with complementary set F there is a canonical isomorphism in $\widetilde{D}(\overline{R})$

$$(4.11.2) \qquad R\Gamma_\Delta(L \otimes \overline{J}_E) \to R\mathrm{Hom}(R\Gamma_\Delta(L^* \otimes \overline{J}_F), \Omega^d_{R/V} \otimes_V \overline{V}/p^r\overline{V}(-d)[-d])$$

5. Coverings by $K(\pi, 1)$'s

5.1. The key to globalizing the above computations is the notion of a $K(\pi, 1)$. If X is a scheme, let $\mathrm{Fet}(X)$ denote the site whose underlying category is the category of finite étale morphisms $U \to X$, and whose coverings are surjective morphisms. We write X_{Fet} for the associated topos. The inclusion $\mathrm{Fet}(X) \subset \mathrm{Et}(X)$ induces a morphism of topoi

$$(5.1.1) \qquad \qquad \pi : X_{\mathrm{et}} \to X_{\mathrm{Fet}}.$$

If X is connected and $\bar{x} \to X$ is a geometric point, then X_{Fet} is equivalent to the category of sets $F_{\bar{x}}$ with continuous action of the fundamental group $\pi_1(X, \bar{x})$ so in particular for an abelian sheaf F the cohomology $H^*(X_{\mathrm{Fet}}, F)$ is isomorphic to group cohomology $H^*(\pi_1(X, \bar{x}), F_{\bar{x}})$.

In general, π^* identifies the category X_{Fet} with the category of sheaves $F \in X_{\mathrm{et}}$ for which F is equal to the union of its locally constant subsheaves.

In what follows let Λ denote $\mathbb{Z}/(n)$ for some integer n invertible on X.

REMARK 5.2. It is always true that for a locally constant sheaf L of Λ–modules on X the natural map $H^1(X_{\mathrm{Fet}}, L) \to H^1(X_{\mathrm{et}}, \pi^*L)$ is an isomorphism. This follows from interpreting these groups as classifying extensions of Λ by L and noting that any such extension is also locally constant.

DEFINITION 5.3. A scheme X is a $K(\pi, 1)$ if for integer n invertible on X and every locally constant sheaf F of $\mathbb{Z}/(n)$–modules the natural map $F \to R\pi_*\pi^*F$ is an isomorphism.

THEOREM 5.4. *Let V be a discrete valuation ring with field of fractions K of characteristic 0 and perfect residue field k. Let X/V be a smooth scheme. Then for every point $x \in X$ in the closed fiber of X there exists a Zariski open neighborhood $U \subset X$ of x such that the geometric generic fiber $U_{\overline{K}}$ is a $K(\pi, 1)$. In particular, if X/V is also proper then every point $x \in X$ is contained in an open subset $U \subset X$ such that $U_{\overline{K}}$ is a $K(\pi, 1)$.*

PROOF. By induction on the relative dimension d of X/V. We can without loss of generality assume that X is affine.

The case $d = 1$ can be seen as follows. Choose an embedding $X \hookrightarrow \mathbb{P}_V$ into some projective space and let $\overline{X}$ be the normalization of the closure. The generic fiber $\overline{X}_K$ is then a normal proper 1–dimensional K–scheme whence a smooth proper curve. Since $\overline{X}_K - X_K$ is not empty, the cohomological dimension of $X_{\overline{K}}$ is 1. This case therefore follows from 5.2.

The case $d = 1$ can also be relativized. As in [**1**, XI.3.1] define a morphism $f : Z \to W$ of K–schemes to be an *elementary fibration* if it extends to a proper smooth morphism $\bar{f} : \overline{Z} \to W$ such that the map $\overline{Z} - Z \to W$ (where $\overline{Z} - Z$ is given the reduced structure) is finite étale and surjective and for every geometric point $\bar{w} \to W$ the geometric fiber $\overline{Z}_{\bar{w}}$ is a smooth connected (and automatically proper) curve. Then $Z_{\bar{w}}$ is a smooth affine curve. We extend this notion slightly by defining a morphism $f : X \to Y$ of V–schemes to be an *elementary fibration* if the generic fiber $X_K \to Y_K$ is an elementary fibration.

LEMMA 5.5. *Let $f : Z \to W$ be an elementary fibration of smooth K–schemes. If W is a $K(\pi, 1)$ then Z is also a $K(\pi, 1)$.*

PROOF. We have to show that for any locally constant constructible sheaf L on Z and $i > 0$ the sheaf $R^i\pi_*L$ on Z_{Fet} is zero. For this we may clearly replace W by a finite étale cover, and hence making the base change $(\overline{Z} - Z) \to W$ we can assume that the complement $\overline{Z} - Z$ is given by a nonempty collection of sections $s : W \to \overline{Z}$. If $U \to Z$ is finite étale and surjective and $\overline{U}$ denotes the normalization of $\overline{Z}$ in U, then it follows from Abhyankar's lemma and the assumption $\text{char}(K) = 0$ that $\overline{U} \to W$ is also smooth and proper and that $\overline{U} - U \to W$ is finite étale and surjective. Replacing Z by a finite étale covering where L is trivial we may therefore assume that $L = \Lambda$. We may also base change to $\overline{K}$ and hence may assume that K is algebraically closed. Furthermore, it suffices to show that for any cohomology class $\alpha \in H^i(Z_{\text{et}}, \Lambda)$ there exists a finite étale surjective morphism $Z' \to Z$ such that the pullback of α to Z' is trivial.

Next consider the Leray spectral sequence

(5.5.1) $$E_2^{ij} = H^i(W, R^j f_*\Lambda) \implies H^{i+j}(Z, \Lambda).$$

Let $j : Z \hookrightarrow \overline{Z}$ and $i : (\overline{Z} - Z) \hookrightarrow \overline{Z}$ be the inclusions. Since $(\overline{Z} - Z) \subset \overline{Z}$ is a smooth divisor, there is by cohomological purity [**1**, XVI.3.9] a distinguished triangle on $\overline{Z}$

(5.5.2) $$\Lambda \to Rj_*j^*\Lambda \to i_*\Lambda(-1)[-1] \to \Lambda[1].$$

This induces a long exact sequence

(5.5.3) $$\cdots \to R^s\bar{f}_*\Lambda \to R^s f_*\Lambda \to R^{s-1}\bar{f}_*(i_*\Lambda(-1)) \to \cdots.$$

Now the sheaves $R^s\bar{f}_*\Lambda$ and $R^{s-1}(i_*\Lambda(-1))$ are locally constant constructible and their formation commutes with arbitrary base change [**1**, XVI.2.2]. In particular, the boundary map

$$R^0\bar{f}_*i_*\Lambda(-1) \to R^2\bar{f}_*\Lambda$$

is an isomorphism as this can be checked fiber by fiber. Also since $\bar{f} \circ i$ is étale we have $R^i\bar{f}_*(i_*\Lambda(-1)) = 0$ for $i > 0$. This implies that

$$R^0 f_*\Lambda \simeq R^0\bar{f}_*\Lambda, \quad R^1 f_*\Lambda \simeq R^1\bar{f}_*\Lambda, \quad R^s f_*\Lambda = 0 \text{ for } s > 1.$$

In particular, the sheaves $R^s f_*\Lambda$ are locally constant constructible. Since W is a $K(\pi, 1)$ any cohomology class in $H^i(W, R^j f_*\Lambda)$ for $i > 0$ is killed by a finite étale extension of W. We may therefore assume that α is given by a class in $H^0(W, R^j f_*\Lambda)$. Furthermore, since $R^j f_*\Lambda$ is zero for $j \geq 2$ it suffices to consider the case of $j = 1$ which follows from 5.2. $\qquad\square$

To prove 5.4 it therefore suffices by induction on d to show that for any point $x \in X$ of the closed fiber there exists a neighborhood U of x and an elementary fibration $f : U \to W$ with W smooth. For if $W' \subset W$ is an open neighborhood of $f(x)$ which is a $K(\pi, 1)$ then 5.5 implies that $(U \times_W W')_{\overline{K}}$ is also a $K(\pi, 1)$. For this we may as well replace V by its maximal unramified extension and hence may assume that the residue field k is algebraically closed. Furthermore it suffices to consider the case of a k–rational point x of X.

We can without loss of generality assume that X is affine. Choose an immersion $X \hookrightarrow \mathbb{P}^r$ for some r, and let $\overline{X}$ be the normalization of the closure of X. Choose a very ample sheaf M on $\overline{X}$ and let $\overline{X} \hookrightarrow \mathbb{P}_V^N$ be the embedding into projective space provided by $M^{\otimes r}$ for some $r \geq 2$. Let $Y = \overline{X} - X$ with the reduced subscheme structure, and let $Y^o \subset Y$ denote the open subset where Y is smooth over V. Note that the generic fiber of Y^o is dense in the generic fiber Y_K.

LEMMA 5.6. *There exist hyperplanes $H_1, \ldots, H_{d-1}$ in $\mathbb{P}_V^N$ containing x such that the intersection $L = H_1 \cap \cdots \cap H_{d-1}$ has relative dimension $N-d+1$ over V and L meets X transversally. We can furthermore choose the hyperplanes $H_1, \ldots, H_{d-1}$ such that the intersection $L_K \cap \overline{X}_K$ is contained in the smooth locus of $\overline{X}_K$, $L_K \cap Y_K$ is contained in Y_K^o, and L_K meets $\overline{X}_K$ and Y_K^o transversally. In fact, if $\mathbb{P}$ denotes the space classifying $(d-1)$–hyperplanes in $\mathbb{P}_V^N$ containing x (see the proof for a precise description of this space), then there is an open subset $\mathscr{U} \subset \mathbb{P}$ dense in every fiber such that for any morphism $\mathrm{Spec}(V) \to \mathscr{U}$ the corresponding hyperplanes $H_1, \ldots, H_{d-1}$ satisfy the above conditions.*

PROOF. Let $x : \mathrm{Spec}(V) \to X$ be a morphism sending the closed point to the previously denoted point x. The functor classifying hyperplanes in $\mathbb{P}^N$ through x is easily seen to be represented by a projective smooth scheme over V (and in particular with equidimensional fibers). It follows that the functor classifying $(d-1)$ hyperplanes in $\mathbb{P}^N$ containing x is also represented by a projective smooth V–scheme $\mathbb{P}$.

There is an open subset $\mathscr{U}_1 \subset \mathbb{P}$ classifying the condition that the intersection L has dimension $N-d+1$ in each fiber, and the open subset $\mathscr{U}_1$ is dense in every fiber. The condition that L meets X transversally is equivalent to the condition that $X \cap L$ is smooth over V. Consider the universal intersection $\mathscr{Z} = (X \times \mathbb{P}) \cap \mathscr{L}$ over $\mathbb{P}$ where $\mathscr{L}$ denotes the intersection of the universal hyperplanes over $\mathbb{P}$. Let $\mathscr{R} \subset \mathscr{Z}$ be the closed subset where $\mathscr{Z}$ is not smooth over $\mathbb{P}$ and let $\mathscr{W} \subset \mathbb{P}$ be its image. The subset $\mathscr{W}$ is a constructible subset of $\mathbb{P}$ whose complement is dense in both the generic and closed fiber by [**1**, XI.2.1]. Therefore the complement $\mathscr{U}_2 \subset \mathbb{P}$ of the closure of $\mathscr{W}$ is also dense in every fibre whence $\mathscr{U}_1 \cap \mathscr{U}_2$ is also nonempty and dense in every fiber. Furthermore as in the proof of [**1**, XI.3.1] there exists a dense open subset $\mathscr{V} \subset \mathbb{P}_K$ classifying hyperplanes such that $L_K \cap \overline{X}_K$ is contained in the smooth locus of $\overline{X}_K$ and $L_K \cap Y_K$ is contained in Y_K^o. Let $Q \subset \mathbb{P}$ be the closure of the complement of $\mathscr{V}$ (with the reduced structure). The scheme Q is flat over V and hence the dimension of the components of the closed fiber of Q is strictly smaller than the dimension of $\mathbb{P}_k$. In particular the intersection $Q^c \cap \mathscr{U}_1 \cap \mathscr{U}_2$ is dense in every fiber. This implies the lemma. $\square$

By a similar reasoning we can also choose a hyperplane H_0 not containing x meeting $X \cap L$ transversally and such that the intersection $Y \cap (H_0 \cap L)_K = \emptyset$. If $\mathbb{P}'$ denotes the space classifying d hyperplanes $H_0, \ldots, H_{d-1}$, then as in the proof of 5.6 one sees that there exists an open subset $\mathscr{U}' \subset \mathbb{P}'$ dense in every fiber such that $H_1, \ldots, H_{d-1}$ satisfy the conditions of the lemma, H_0 does not contain x, H_0 meets $X \cap L$ transversally, and the intersection $Y \cap (H_0 \cap L)_K$ is empty.

On $Y^o \subset Y$ the map $\Omega^1_{\mathbb{P}^N/V}|_{Y^o} \to \Omega^1_{Y^o/V}$ is surjective. Let $\mathrm{Gr}_{\mathbb{P}^N}(\Omega^1_{\mathbb{P}^N/V}, d-1)$ denote the scheme over $\mathbb{P}^N$ classifying rank $d-1$ quotients of $\Omega^1_{\mathbb{P}^N/V}$. We then obtain a morphism $Y^o \to \mathrm{Gr}_{\mathbb{P}^N}(\Omega^1_{\mathbb{P}^N/V}, d-1)$ over $\mathbb{P}^N$. Let $Z \subset \mathrm{Gr}_{\mathbb{P}^N}(\Omega^1_{\mathbb{P}^N/V}, d-1)$ be the closure of Y^o and let $\pi : Z \to Y$ be the projection. Then on Z there is a quotient $\Omega^1_{\mathbb{P}^N/V}|_Z \to \mathscr{E}$.

LEMMA 5.7. *As in 5.6, let $\mathbb{P}$ denote the V–scheme parametrizing $(d-1)$–hyperplanes in $\mathbb{P}^N$ through x, and for a (scheme–valued) point of $\mathbb{P}$ corresponding to hyperplanes $H_1, \ldots, H_{d-1}$ let L denote the intersection $H_1 \cap \cdots \cap H_{d-1}$. Then there exists an open subset $\mathscr{V} \subset \mathbb{P}$ dense in every fiber such that if $H_1, \ldots, H_{d-1}$*

define a point of $\mathscr{V}$ and $z \in Z$ with image $y \in Y$ then the natural map $\Omega^1_{\mathbb{P}^N_V/V}(y) \to \mathscr{E}(z) \oplus \Omega^1_{L/V}(y)$ is an isomorphism.

Similarly, if $\mathbb{P}''$ denotes the space of $(d-1)$ hyperplanes in $\mathbb{P}^N_V$ (not necessarily through x) then there exists an open subset $\mathscr{V} \subset \mathbb{P}''$ dense in every fiber such that if $H_1, \ldots, H_{d-1}$ define a point of $\mathscr{V}$ and $z \in Z$ with image $y \in Y$ then the natural map $\Omega^1_{\mathbb{P}^N_V/V}(y) \to \mathscr{E}(z) \oplus \Omega^1_{L/V}(y)$ is an isomorphism.

PROOF. We give the proof of the first statement leaving to the reader the task of proving the second statement using the same argument.

Let $\mathscr{V} \subset \mathbb{P}$ be the subfunctor associating to any V–scheme T the subset of $\mathbb{P}(T)$ classifying hyperplanes $H_1, \ldots, H_{d-1}$ such that the map $\Omega^1_{\mathbb{P}^N_V/V}|_{Z_T} \to \mathscr{E}|_{Z_T} \oplus \Omega^1_{L/T}|_{Z_T}$ is an isomorphism. The fact that Z is proper over V implies that in fact $\mathscr{V}$ is represented by an open subset of $\mathbb{P}$. We claim that $\mathscr{V}$ is dense in every fiber.

To verify this it suffices to consider the base change of Z by a map $V \to \Omega$, where Ω is an algebraically closed field. We claim that for any nonempty affine open subset $U \subset Z_\Omega$ there exists a dense open subset $U' \subset U$ classifying classifying hyperplanes $H_1, \ldots, H_{d-1}$ such that the map $\Omega^1_{\mathbb{P}^N_V/V}(y) \to \mathscr{E}(z) \oplus \Omega^1_{L/V}(y)$ is an isomorphism for every $z \in U'$ mapping to $y \in Y_\Omega$. Let $X_0, \ldots, X_N$ denote the coordinates of $\mathbb{P}^N$. After shrinking on U and possibly changing coordinates we can assume that U maps to the open subset $\mathbb{A}^N_\Omega \subset \mathbb{P}^N_\Omega$ given by $X_N \neq 0$ and that $\mathscr{E}|_U$ is isomorphic to $\mathscr{O}^{d-1}_U$. There is also a dense open subset $A \subset \mathbb{P}$ classifying hyperplanes $H_1, \ldots, H_{d-1}$ none of which are contained in $\{X_N = 0\}$. A point of A is given by the coefficients of linear equations

$$(5.7.1) \qquad \sum_{i=0}^{N-1} a_{i\nu} X_i = c_\nu,$$

where $\nu = 1, \ldots, d-1$ and $c_\nu \in \Omega$ (in the case when we require the hyperplanes to pass through x the constants c_ν will be fixed). Let R be the coordinate ring of U. The basis dX_i induces an isomorphism $\Omega^1_{\mathbb{P}^N_\Omega/\Omega}|_U \simeq R^N$ and the quotient $\Omega^1_{\mathbb{P}^N_\Omega/\Omega}|_U \to \Omega^1_{L/\Omega}|_U$ has kernel the submodule generated by the vectors $v_\nu = (a_{0\nu}, \ldots, a_{N-1\nu})$. Let $F \subset R^N$ be the kernel of the surjection $R^N \to \mathscr{E}|_U$. The intersection $I := F \cap \Omega^N \subset R^N$ must have dimension less than or equal to $N - d + 1$ since the map $I \otimes_\Omega R \to F$ is injective (because $I \otimes_\Omega R \to R^N$ is injective) and F has rank equal to $N - d + 1$. From this it is clear that there exists a dense open subset of A such that

$$I \cap \mathrm{Span}(v_\nu)^{d-1}_{\nu=1} = \{0\}.$$

This gives the desired dense open subset. $\qquad\square$

We can therefore choose hyperplanes $H_0, \ldots, H_{d-1}$ such that the conditions in 5.6 are satisfied, such that for every $z \in Z$ with image $y \in Y$ the natural map $\Omega^1_{\mathbb{P}^N_V/V}(y) \to \mathscr{E}(z) \oplus \Omega^1_{L/V}(y)$ is an isomorphism, and such that H_0 does not contain x, H_0 meets $X \cap L$ transversally and such that the intersection $Y \cap (H_0 \cap L)_K$ is empty. After changing coordinates, we can further assume that H_0 is the hyperplane $\{X_0 = 0\}$. Let C denote the intersection $H_0 \cap \cdots \cap H_{d-1}$. Choosing coordinates

write

$$(5.7.2) \qquad\qquad H_i : \sum_{\nu=0}^{N} a_{i\nu} X_\nu = 0,$$

and consider the map

$$(5.7.3) \qquad\qquad \mathbb{P}^N - C \to \mathbb{P}^{d-1}, \quad y_i = \sum_\nu a_{i\nu} X_\nu.$$

Let $U \subset \overline{X}$ be the neighborhood of x which is the complement of $H_0 \cap \overline{X}$. We then obtain a morphism $\epsilon : U \to \mathbb{P}^{d-1}$. The fiber $\epsilon^{-1}\epsilon(x)$ is isomorphic to $U \cap L$. In fact, let $\mathbb{A}^{d-1} \subset \mathbb{P}^{d-1}$ denote the open subset where $y_0 \neq 0$. By associating to a point $(y_1, \ldots, y_{d-1})$ the hyperplanes $y_i - a_{i0} = \sum_{\nu \geq 1} a_{i\nu} X_\nu$ we obtain a morphism $\mathbb{A}^{d-1} \to \mathbb{P}''$ over V. For $s \in \mathbb{A}^{d-1}$ let L_s denote the intersection of the $(d-1)$–hyperplanes corresponding to the image of s. Then the fiber of ϵ over s is equal to $U \cap L_s$. By 5.7, for general choice of $H_0, \ldots, H_{d-1}$, there exists an open subset $W \subset \mathbb{A}^{d-1}$ containing $\epsilon(x)$ such that for every $s \in W$ the conditions of 5.7 are satisfied for the image of s in $\mathbb{P}''$. It follows that for a point $s \in W_K$ of the generic fiber the intersection $\overline{X} \cap L_s$ is smooth. Using the argument given in [**1**, XI, page 7 last paragraph] one shows that the map $X_K \to W_K$ is an elementary fibration. $\square$

COROLLARY 5.8. *Let X/V be a smooth proper scheme. Then for any closed point $x \in X$ the scheme $\mathrm{Spec}(\mathcal{O}_{X,\bar{x}})_{\overline{K}}$ is a $K(\pi, 1)$.*

PROOF. The scheme $\mathrm{Spec}(\mathcal{O}_{X,\bar{x}})$ is a projective limit of $K(\pi, 1)$'s and a standard limit argument shows that such a projective limit is also a $K(\pi, 1)$. $\square$

EXAMPLE 5.9. Let $\overline{K}$ be an algebraically closed field of characteristic 0, and $X/\overline{K}$ a smooth $\overline{K}$–scheme with a divisor $D \subset X$ with normal crossings. Let $x \in X(\overline{K})$ be a point and let Y denote $\mathrm{Spec}(\mathcal{O}_{X,\bar{x}})$. Then if Y^o denotes $Y \times_X (X-D)$ then Y^o is a $K(\pi, 1)$.

For this note first that we are immediately reduced to the case when $X = \mathbb{A}^d$, x is the origin, and D is the standard hyperplanes. By Abhyankar's lemma the fundamental group Δ of Y^o is isomorphic to $\widehat{\mathbb{Z}}(1)^r$ where r is the number of components of D passing through x. Let L be a locally constant sheaf of Λ–modules on Y^o. By devissage we may assume that $\Lambda = \mathbb{Z}/(p)$ and that L corresponds to a simple representation of Δ. In this case L is isomorphic to a tensor product $L_{\mu_1} \otimes \cdots \otimes L_{\mu_r}$ where L_{μ_j} is a representation of the j–th factor of $\widehat{\mathbb{Z}}(1)$ in Δ. By the Künneth formula this further reduces the proof to the case when $X = \mathbb{A}^1$ and D is the origin. In this case Y^o is a projective limit of $K(\pi, 1)$'s and hence is also a $K(\pi, 1)$.

This argument also proves the following. Assume given an étale morphism

$$(5.9.1) \qquad\qquad \pi : X \to \mathrm{Spec}(\overline{K}[X_1, \ldots, X_d])$$

such that D is equal to the inverse image of the standard hyperplanes. Let $j : X^o := (X - D) \hookrightarrow X$ be the inclusion and let j be an integer. For $\underline{n} = (n_1, \ldots, n_d)$ let $X_n \to X$ denote the scheme obtained by taking the n_i–th roots of X_i, and let $j_n : X_n^o \hookrightarrow X_n$ be the inclusion. Then for any locally constant constructible sheaf L and cohomology class $\alpha \in R^i j_* L$ $(i > 0)$ there exists integers $\underline{n}$ such that the image of α in $R^i j_{n*} L$ is zero, $j_{n*} L$ is locally constant constructible, and the natural map $j_n^* j_{n*} L \to L$ is an isomorphism. Indeed this can be verified after passing to

the strict henselization of X at a point in which case it follows from the above computations.

COROLLARY 5.10. *Let X be a smooth proper V–scheme and $D \subset X$ a divisor with normal crossings relative to V. Let X^o denote the complement of D. Then for any closed point $x \in X$ the scheme $(\mathrm{Spec}(\mathcal{O}_{X,\bar{x}})_{\overline{K}})^o := X_{\overline{K}}^o \times_X \mathrm{Spec}(\mathcal{O}_{X,\bar{x}})$ is a $K(\pi, 1)$.*

PROOF. Let $X^\dagger$ denote $\mathrm{Spec}(\mathcal{O}_{X,\bar{x}})$ and let L be a locally constant constructible sheaf on $X_{\overline{K}}^{\dagger o}$. We have to show that for any cohomology class $\alpha \in H^i(X_{\overline{K}}^{\dagger o}, L)$ with $i > 0$ there exists a finite étale surjection $V \to X_{\overline{K}}^{\dagger o}$ over which α becomes zero. Choose a formally étale morphism

(5.10.1) $$\pi : X^\dagger \to \mathrm{Spec}(V[X_1, \ldots, X_d])$$

such that D is equal to the inverse image of $\{X_1 \cdots X_d = 0\}$. Let $j : X^{\dagger o} \hookrightarrow X^\dagger$ be the inclusion. By 5.9 and consideration of the Leray spectral sequence

(5.10.2) $$E_2^{pq} = H^p(X_{\overline{K}}^\dagger, R^q j_* L) \implies H^{p+q}(X_{\overline{K}}^{\dagger o}, L),$$

there exists integers $\underline{n} = (n_1, \ldots, n_d)$ such that if $X_{\underline{n}}^\dagger$ denotes the scheme

(5.10.3) $$X \times_{\mathrm{Spec}(V[X_1, \ldots, X_d])} \mathrm{Spec}(V[X_1^{1/n_1}, \ldots, X_d^{1/n_d}])$$

with inclusion $j_n : X_{\underline{n}}^{\dagger o} \hookrightarrow X_{\underline{n}}^\dagger$ then α is given by a class in $H^i(X_{\underline{n}, \overline{K}}^\dagger, j_{n*}L)$, $j_{n*}L$ is locally constant constructible, and $j_n^* j_{n*}L \to L$ is an isomorphism. Replacing $X^\dagger$ by $X_{\underline{n}}^\dagger$ we may therefore assume that L extends to a sheaf on $X_{\overline{K}}^\dagger$ which we again denote by L and that α is obtained from a cohomology class in $H^i(X_{\overline{K}}^\dagger, L)$. Since $X_{\overline{K}}^\dagger$ is a $K(\pi, 1)$ the result follows. $\square$

In fact 5.4 can be generalized as follows to also take into account a boundary.

THEOREM 5.11. *With notation as in 5.4, let $D \subset X$ be a divisor with simple normal crossings relative to V, set $X^o := X - D$. Let $D = D_1 \cup \cdots \cup D_n$ with each D_i an irreducible divisor. For a subset $A \subset \{1, \ldots, n\}$ (possibly empty) let D_A denote the intersection $\cap_{i \in A} D_i$, and let D_A^o denote $D_A - (\cup_{j \notin A} D_j)$. Then for any point $x \in X$ in the closed fiber there exists a Zariski open neighborhood U of x such that each $D_{A, \overline{K}}^o \cap U_{\overline{K}}$ is a $K(\pi, 1)$.*

PROOF. Note first that after possibly replacing the polarization M giving the embedding $X \hookrightarrow \mathbb{P}^N$ by $M^{\otimes r}$ for some r, we may assume that the closure $\overline{D}_i \subset \overline{X}$ of each irreducible component D_i is normal. Let $\epsilon : X \to W$ be the elementary fibration constructed in the proof of 5.4 by projection from the hyperplanes H_i. Then an examination of the proof and standard Bertini type arguments shows that we can furthermore choose the hyperplanes $H_1, \ldots, H_{d-1}$ so that if $L = H_1 \cap \cdots \cap H_{d-1}$ then the intersection $L_{\overline{K}} \cap \overline{D}_{i, \overline{K}}$ is transverse for all i so that this intersection is an étale $\overline{K}$–scheme, and furthermore can choose the H_i's such that the projection of the double intersections $D_i \cap D_j$ have image a divisor with simple normal crossings in W. Furthermore, we can choose the hyperplane H_0 so that it does not meet the intersections $L_{\overline{K}} \cap \overline{D}_{i, \overline{K}}$. Then it follows from the proof that after some shrinking on W the schemes $\overline{D}_{i, \overline{K}}$ are finite and étale over $W_{\overline{K}}$. This in turn implies that the

map $X_{\overline{K}}^o \to W_{\overline{K}}$ is also an elementary fibration whence $X_{\overline{K}}^o$ is a $K(\pi, 1)$, and the statement about the higher intersections follows by induction. $\qquad\square$

6. The topos $\mathscr{X}_{\overline{K}}^o$

6.1. Let X/V be a smooth V–scheme, and let $D \subset X$ be a divisor with normal crossings relative to V. For an X–scheme $\pi : U \to X$ let U^o denote $U - \pi^{-1}(D)$.

Define a site $\mathscr{S}$ as follows. The objects of $\mathscr{S}$ are pairs (U, N), where U is an étale separated X–scheme of finite type and $N \to U_{\overline{K}}^o$ is a finite and étale morphism. A morphism $(U', N') \to (U, N)$ is given by a commutative diagram of X–morphisms

$$(6.1.1) \qquad \begin{array}{ccc} N' & \longrightarrow & N \\ \downarrow & & \downarrow \\ U' & \longrightarrow & U. \end{array}$$

A collection of morphisms $\{(U_i, N_i) \to (U, N)\}$ is defined to be a covering if $\{U_i \to U\}$ is a covering in $\mathrm{Et}(X)$ and $\{N_i \to N\}$ is a covering in $\mathrm{Et}(X_{\overline{K}}^o)$. We write $\mathscr{X}_{\overline{K}}^o$ for the resulting topos.

LEMMA 6.2. *The category $\mathscr{S}$ has finite projective limits.*

PROOF. By [**1**, I.2.3], it suffices to show that finite products and equalizers are representable in $\mathscr{S}$. If $\{(U_i, N_i)\}_{i=1}^n$ is a finite set of objects in $\mathscr{S}$, then their product is given by the pair

$$(6.2.1) \qquad (U_1 \times_X U_2 \times \cdots \times_X U_n, N_1 \times_{X_{\overline{K}}^o} N_2 \times \cdots \times_{X_{\overline{K}}^o} N_n).$$

For the existence of equalizers, consider two morphisms $f, g : (U_1, N_1) \to (U_2, N_2)$ in $\mathscr{S}$, and let $N^\dagger$ (resp. $U^\dagger$) denote the equalizer of the morphisms $N_1 \rightrightarrows N_2$ (resp. $U_1 \rightrightarrows U_2$). Then $(U^\dagger, N^\dagger)$ represents the equalizer of f and g. The only point that needs verification is that $N^\dagger$ is finite over $U_{\overline{K}}^{\dagger o}$. For this note that $N^\dagger$ is equal to the fiber product of the diagram

$$\begin{array}{ccc} & & N_2 \\ & & \downarrow{\scriptstyle \Delta} \\ N_1 & \xrightarrow{a \times b} & N_2 \times_{X_{\overline{K}}^o} N_2, \end{array}$$

where we write $a, b : N_1 \to N_2$ for the two morphisms. Since $N_2 \to X_{\overline{K}}^o$ is separated this implies that $N^\dagger$ is a closed subscheme of N_1. By a similar argument $U^\dagger$ is a closed subscheme of U_1. We therefore have a commutative diagram

$$\begin{array}{ccc} N^\dagger & \hookrightarrow & N_1 \\ \downarrow & & \downarrow{\scriptstyle c} \\ U_{\overline{K}}^{\dagger o} & \hookrightarrow & U_{1,\overline{K}}^o, \end{array}$$

where the horizontal arrows are closed immersions, and c is finite. It follows that $N^\dagger \to U_{\overline{K}}^{\dagger o}$ is also finite. $\qquad\square$

REMARK 6.3. A similar argument to the one in the proof of 6.2 shows that a finite projective limit of finite étale morphisms is finite and étale. Indeed to verify this it suffices to consider finite products (immediate) and equalizers (proof of 6.2).

6.4. There is a continuous morphism of sites $\mathscr{S} \to \mathrm{Et}(X_{\overline{K}}^o)$ sending (U, N) to the composite $N \to U_{\overline{K}}^o \to X_{\overline{K}}^o$. This functor preserves finite inverse limits since it preserves products and equalizers, and therefore by [**1**, IV.4.9.2] induces a morphism of topoi

$$(6.4.1) \qquad u_X : X_{\overline{K},\mathrm{et}}^o \to \mathscr{X}_{\overline{K}}^o.$$

There is also a continuous morphism of sites $\mathrm{Et}(X) \to \mathscr{S}$ sending $U \to X$ to $(U, \mathrm{id} : U_{\overline{K}}^o \to U_{\overline{K}}^o)$. This functor also preserves finite projective limits, and induces a morphism of topoi

$$(6.4.2) \qquad \nu_X : \mathscr{X}_{\overline{K}}^o \to X_{\mathrm{et}}.$$

Note that the composite $\nu_X \circ u_X : X_{\overline{K},\mathrm{et}}^o \to X_{\mathrm{et}}$ is just the morphism of topoi induced by the morphism of schemes $X_{\overline{K}}^o \to X$.

Finally there is a morphism of sites $\mathrm{Fet}(X_{\overline{K}}^o) \to \mathscr{S}$ sending $N \to X_{\overline{K}}^o$ to the object $(X, N) \in \mathscr{S}$. This morphism induces a morphism of topoi

$$(6.4.3) \qquad \epsilon_X : \mathscr{X}_{\overline{K}}^o \to X_{\overline{K},\mathrm{Fet}}^o.$$

Observe that $\epsilon_X \circ u_X$ is the natural projection $X_{\overline{K},\mathrm{et}}^o \to X_{\overline{K},\mathrm{Fet}}^o$.

LEMMA 6.5. *Let L be a locally constant constructible abelian sheaf on $X_{\overline{K}}^o$. Then $R^i u_{X*} L = 0$ for $i > 0$ and $u_{X*} L$ is the sheaf $(U, N) \mapsto L(N)$.*

PROOF. Fix an integer $i > 0$. By 5.11 any object of $\mathscr{S}$ can be covered by objects (U, N) with U^o a $K(\pi, 1)$. To prove that $R^i u_{X*} L = 0$, it therefore suffices to show that for any such $(U, N) \in \mathscr{S}$ and class $o \in H^i(X_{\overline{K},\mathrm{et}}^o|_{u_X^{-1}(U,N)}, L)$, there exists a covering $(U', N') \to (U, N)$ such that the pullback of o to $H^i(X_{\overline{K},\mathrm{et}}^o|_{u_X^{-1}(U',N')}, L)$ is zero.

For this note that for any $(U, N) \in \mathscr{S}$ there is a natural isomorphism

$$(6.5.1) \qquad H^i(X_{\overline{K},\mathrm{et}}^o|_{u_X^{-1}(U,N)}, L) \simeq H^i(N_{\mathrm{et}}, L).$$

Now if U^o is a $K(\pi, 1)$, then N is also a $K(\pi, 1)$. This implies that there exists a finite étale covering $N' \to N$ such that o maps to zero in $H^i(N'_{\mathrm{et}}, L)$. The resulting covering $(U, N') \to (U, N)$ in $\mathscr{S}$ then kills o.

The description of $u_{X*} L$ is immediate from the definitions. $\qquad \square$

LEMMA 6.6. *Assume that X^o is a $K(\pi, 1)$. Then for any abelian sheaf F in $X_{\overline{K},\mathrm{Fet}}^o$, the natural map $F \to R\epsilon_{X*} \epsilon_X^* F$ is an isomorphism.*

PROOF. Note that by the preceding lemma, $R\epsilon_{X*} \epsilon_X^* F \simeq R(\epsilon_X \circ u_X)_*(F|_{X_{\overline{K},\mathrm{et}}^o})$. The lemma therefore follows from the fact that X^o is a $K(\pi, 1)$ which implies that the adjunction map $F \to R(\epsilon_X \circ u_X)_*(F|_{X_{\overline{K},\mathrm{et}}^o})$ is an isomorphism. $\qquad \square$

6.7. An annoying technical difficulty about the topos $\mathscr{X}_{\overline{K}}^o$ is that most of the sheaves we will consider will be obtained from presheaves by sheafification. This means that if $U \in \mathrm{Et}(X)$ then it is rather difficult to describe the restriction of the sheaf to $U_{\overline{K},\mathrm{Fet}}^o$.

We can overcome this as follows. Recall from [**1**, V, §7] (in particular [**1**, V.7.4.1 (4)]) the following way of computing cohomology using hypercovers. Let S be a site with representable finite products and fiber products and let $S^{\sim}$ be the associated topos. Let HR_S (or just HR if the reference to the site S is clear) denote the *homotopy category of hypercovers* in S. This is the category whose objects are hypercovers by objects of S of the initial object of the topos $S^{\sim}$ and whose morphisms are homotopy classes of morphisms of hypercovers (see [**1**, V.7.3.1.6] for the notion of a homotopy between two morphisms of hypercovers). By [**1**, V.7.3.2 (1)] the opposite category HR^o is filtering (while the category of hypercovers before passing to the homotopy category is not filtering). Let G be a presheaf of abelian groups on S with associated sheaf G^a. For any hypercover $U.$ of the initial object in $S^{\sim}$, let $G(U.)$ denote the complex obtained by evaluating G on each of the U_n to get a cosimplicial abelian group and then taking the total complex. Two homotopic morphisms $f, g : U'. \to U.$ induce the same map $H^*(G(U.)) \to H^*(G(U'.))$ and hence one can form the limit

$$(6.7.1) \qquad\qquad \varinjlim_{U. \in HR} H^*(G(U.)).$$

There is a natural morphism

$$(6.7.2) \qquad\qquad \varinjlim_{U. \in HR} H^*(G(U.)) \to H^*(S, G^a)$$

which by [**1**, 7.4.1 (4)] is an isomorphism, where sq_k (resp. cosq_k) denotes the k-th skeleton (resp. coskeleton) functor. For a fixed q, one can do slightly better. Namely, recall that for an integer k an object $U. \in HR$ is *k-truncated* if the adjunction map

$$(6.7.3) \qquad\qquad U. \to \mathrm{cosq}_k \mathrm{sq}_k U.$$

is an isomorphism. We denote by $HR_k \subset HR$ the full subcategory of k-truncated objects. By [**1**, V.7.3.2 (1)] the category HR_k^o is also filtering, and by [**1**, 7.4.1 (3)] the natural map

$$(6.7.4) \qquad\qquad \varinjlim_{U. \in HR_k} H^q(G(U.)) \to H^q(S, F)$$

is an isomorphism for any $k > q$.

6.8. Now let $\mathscr{S}$ be as in 6.1. Let F be a presheaf on $\mathscr{S}$ such that for every $U \in \mathrm{Et}(X)$ the restriction F_U of F to $\mathrm{Fet}(U_{\overline{K}}^o)$ is a sheaf. Let $\nu_X : \mathscr{X}_{\overline{K}}^o \to X_{\mathrm{et}}$ be the morphism of topoi defined in 6.4. For any $U \in \mathrm{Et}(X)$ let $\mathscr{S}|_U$ denote the site of objects in $\mathscr{S}$ with a morphism to the object $(U, U_{\overline{K}}^o)$. The topos associated to $\mathscr{S}|_U$ is the localized topos $\mathscr{X}_{\overline{K}}^o|_U$. We also have the morphism of topoi

$$(6.8.1) \qquad\qquad \epsilon_U : \mathscr{X}_{\overline{K}}^o|_U \to U_{\overline{K}, \mathrm{Fet}}^o.$$

The map of presheaves $F \to F^a$ induces a morphism $F_U \to R\epsilon_{U*} F^a$. Applying the functor $H^q(U_{\overline{K}, \mathrm{Fet}}^o, -)$ we obtain a morphism

$$(6.8.2) \qquad\qquad H^q(U_{\overline{K}, \mathrm{Fet}}^o, F_U) \to H^q(\mathscr{X}_{\overline{K}}^o|_U, F^a).$$

This map is functorial in U, so if $\mathscr{H}^q(F)$ denotes the sheaf on $\mathrm{Et}(X)$ associated to the presheaf

$$(6.8.3) \qquad\qquad U \mapsto H^q(U_{\overline{K}, \mathrm{Fet}}^o, F_U)$$

we obtain a morphism of sheaves on $\mathrm{Et}(X)$

$$(6.8.4) \qquad \mathscr{H}^q(F) \to R^q \nu_{X*} F^a.$$

PROPOSITION 6.9. *The morphism 6.8.4 is an isomorphism.*

PROOF. The key point is the following lemma:

LEMMA 6.10. *Let $U \in \mathrm{Et}(X)$ and let $(U'_\cdot, N'_\cdot) \to (U, U^o_{\overline{K}})$ be a k-truncated hypercover in $\mathscr{S}$. Then after replacing U by an étale cover $W \to U$ and $(U'_\cdot, N'_\cdot) \to (U, U^o_{\overline{K}})$ by $(U'_\cdot \times_U W, N'_\cdot \times_U W) \to (W, W^o_{\overline{K}})$ there exists a morphism of k-truncated hypercovers $(U''_\cdot, N''_\cdot) \to (U'_\cdot, N'_\cdot)$ over $(U, U^o_{\overline{K}})$ such that each N''_n is finite and étale over $U^o_{\overline{K}}$.*

PROOF. By a standard limit argument, it suffices to consider the case when U is the spectrum of a strictly henselian local ring (indeed the assumption that the hypercovers are k-truncated ensures that they are given by finitely many schemes and maps, and hence if we prove the lemma over the strict henselization of a geometric point $\bar{x} \to U$ we can "spread out" to get the lemma in some étale neighborhood of $\bar{x}$).

In this case any étale U–scheme $Z \to U$ decomposes canonically as $Z^g \coprod Z^f$, where $Z^f \to U$ is finite étale and the inclusion $Z^f \hookrightarrow Z$ reduces to an isomorphism over the closed point of U. This decomposition is functorial in Z. Let $U'^f_\cdot$ be the simplicial U–scheme $[n] \mapsto U'^f_n$. The simplicial scheme $U'^f_\cdot$ is again a hypercover of U. Indeed for any integer $n \geq 0$ to verify that the morphism of finite étale U–schemes

$$(6.10.1) \qquad U'^f_{n+1} \to (\mathrm{cosq}_n \mathrm{sq}_n U'^f_\cdot)_{n+1}$$

is surjective, it suffices to check that it induces a surjection over the closed point of U which follows from the fact that $U'_\cdot \to U$ is a hypercover.

We claim that in this local case we can find a morphism $(U''_\cdot, N''_\cdot) \to (U'_\cdot, N'_\cdot)$ as in the lemma with $U''_\cdot = U'^f_\cdot$. We prove this stronger statement by induction on k.

For the case $k = 0$, we have a commutative diagram

$$(6.10.2) \qquad \begin{array}{ccc} N'_0 & \xrightarrow{\ \alpha\ } & U^o_{\overline{K}} \\ t \downarrow & & \downarrow \\ U'_0 & \xrightarrow{\ \beta\ } & U, \end{array}$$

where α and β are étale and surjective, and $(U'_\cdot, N'_\cdot) \to (U, U^o_{\overline{K}})$ is obtained by taking the 0-coskeletons of the horizontal arrows. Set $N'^f_0 := N'_0 \times_{U'_0} U'^f_0$. We then obtain a commutative diagram mapping to 6.10.2

$$(6.10.3) \qquad \begin{array}{ccc} N'^f_0 & \xrightarrow{\ \alpha^f\ } & U^o_{\overline{K}} \\ t^f \downarrow & & \downarrow \\ U'^f_0 & \xrightarrow{\ \beta^f\ } & U, \end{array}$$

where the map $N'^f_0 \to U'^{fo}_{0,\overline{K}}$ induced by t^f is finite and étale, and α^f and β^f are étale surjections. Taking 0-coskeletons of the horizontal arrows in 6.10.3 we obtain the desired hypercover in $\mathscr{S}$ mapping to $(U'_\cdot, N'_\cdot) \to (U, U^o_{\overline{K}})$.

For the inductive step we assume the lemma holds for k and prove it for $k+1$. By induction we can find a hypercover

$$(6.10.4) \qquad (\mathrm{cosq}_k \mathrm{sq}_k U'^f_\cdot, Z_\cdot) \to (U, U^o_{\overline{K}})$$

and a commutative diagram over U

$$(6.10.5) \qquad \begin{array}{ccc} Z_\cdot & \longrightarrow & \mathrm{cosq}_k \mathrm{sq}_k N'_\cdot \\ \downarrow & & \downarrow \\ \mathrm{cosq}_k \mathrm{sq}_k U'^f_\cdot & \longrightarrow & \mathrm{cosq}_k \mathrm{sq}_k U'_\cdot. \end{array}$$

Set

$$(6.10.6) \qquad W := N_{k+1} \times_{(\mathrm{cosq}_k \mathrm{sq}_k N_\cdot)_{k+1}} (\mathrm{cosq}_k \mathrm{sq}_k Z_\cdot)_{k+1}$$

so that there is a commutative diagram (whose top square is cartesian)

$$(6.10.7) \qquad \begin{array}{ccc} W & \xrightarrow{\ \delta\ } & N_{k+1} \\ {\scriptstyle \tilde{\beta}}\downarrow & & \downarrow{\scriptstyle \beta} \\ (\mathrm{cosq}_k \mathrm{sq}_k Z_\cdot)_{k+1} & \xrightarrow{\ \alpha\ } & (\mathrm{cosq}_k \mathrm{sq}_k N_\cdot)_{k+1} \\ {\scriptstyle \sigma}\downarrow & & \downarrow \\ (\mathrm{cosq}_k \mathrm{sq}_k U'^f_\cdot)_{k+1} & \xrightarrow{\ \gamma\ } & (\mathrm{cosq}_k \mathrm{sq}_k U_\cdot)_{k+1}. \end{array}$$

The morphism

$$(6.10.8) \qquad \sigma' : (\mathrm{cosq}_k \mathrm{sq}_k Z_\cdot)_{k+1} \to (\mathrm{cosq}_k \mathrm{sq}_k U'^{f_o}_{\cdot,\overline{K}})_{k+1}$$

induced by σ is finite and étale being an inverse limit of finite étale morphisms 6.3. It follows that $(\mathrm{cosq}_k \mathrm{sq}_k Z_\cdot)_{k+1}$ is proper over $U^o_{\overline{K}}$, and therefore the morphism α is also finite and étale. Since the top square is cartesian we conclude that δ is also finite étale. The morphism $\tilde{\beta}$ is surjective since β is surjective (since $N_\cdot$ is a hypercover).

Next set

$$(6.10.9) \qquad \widetilde{W} := U'^f_{k+1} \times_{(\mathrm{cosq}_k \mathrm{sq}_k U'^f_\cdot)_{k+1}} W.$$

Since $U'^f_\cdot$ is a hypercover of U, the morphism $U'^f_{k+1} \to (\mathrm{cosq}_k \mathrm{sq}_k U'^f_\cdot)_{k+1}$ is surjective and finite. This implies that the map $\widetilde{W} \to W$ is also surjective and that $\widetilde{W}$ is finite over $U^o_{\overline{K}}$.

Let $\underline{\Delta}_{k+1} \subset \underline{\Delta}$ denote the full subcategory whose objects are $\{[s]| s \le k+1\}$. Define

$$(6.10.10) \qquad R : \underline{\Delta}^o_{k+1} \to (U^o_{\overline{K}}\text{-schemes})$$

as follows. For $s < k+1$ set $R_s := Z_s$, and set $R_{k+1} := \widetilde{W}$. For a morphism $\delta : [s] \to [k+1]$ with $s < k+1$ define $\delta^* : R_{k+1} \to R_s$ to be the composite

$$(6.10.11) \qquad \widetilde{W} \longrightarrow (\mathrm{cosq}_k \mathrm{sq}_k Z_\cdot)_{k+1} \longrightarrow Z_s,$$

where the second morphism is the one given by the simplicial structure on $\mathrm{cosq}_k \mathrm{sq}_k Z_\cdot$. For a morphism $\delta : [k+1] \to [s]$ let $\delta^* : R_s \to R_{k+1} = \widetilde{W}$ be the morphism defined

as follows. First the commutative diagram

$$(6.10.12) \quad \begin{array}{ccc} Z_s & \longrightarrow & N_s \\ \| & & \downarrow{\scriptstyle\delta} \\ Z_s & & N_{k+1} \\ \downarrow & & \downarrow \\ (\mathrm{cosq}_k\mathrm{sq}_k Z.)_{k+1} & \longrightarrow & (\mathrm{cosq}_k\mathrm{sq}_k N.)_{k+1} \end{array}$$

defines a morphism $h : Z_s \to W$. This morphisms sits in a commutative diagram

$$(6.10.13) \quad \begin{array}{ccc} Z_s & \longrightarrow & W \\ \downarrow & & \| \\ U_s'^f & & W \\ {\scriptstyle\delta}\downarrow & & \downarrow \\ U_{k+1}'^f & \longrightarrow & (\mathrm{cosq}_k\mathrm{sq}_k U.'^f)_{k+1} \end{array}$$

and hence factors through a morphism $\delta^* : R_s = Z_s \to \widetilde{W} = R_{k+1}$. We leave to the reader the verification that these definitions are compatible with compositions of morphisms in $\underline{\Delta}_{k+1}$. Since all the morphisms in the diagram

$$(6.10.14) \qquad \widetilde{W} \longrightarrow W \longrightarrow (\mathrm{cosq}_k\mathrm{sq}_k Z.)_{k+1}$$

are surjections the truncated simplicial scheme R is a hypercover of $U_{\overline{K}}^o$. Furthermore we have a commutative diagram of truncated simplicial schemes

$$(6.10.15) \quad \begin{array}{ccc} R & \longrightarrow & N. \\ \downarrow & & \downarrow \\ \mathrm{sq}_{k+1}U.'^f & \longrightarrow & U.. \end{array}$$

Taking $(k+1)$-coskeletons we obtain the desired morphism of hypercovers in $\mathscr{S}$. $\qquad\square$

To deduce 6.9 from this, let $s \in R^q\epsilon_{X*}F^a$ be a section over some $U \in \mathrm{Et}(X)$. After shrinking on U we can find a hypercover $(U.', N.') \to (U, U_{\overline{K}}^o)$ and a section $\tilde{s} \in F_{U_q'}(N_q')$ representing the class s. By 6.10, after further shrinking on U, we can assume that each U_n' is finite étale over U. In this case $N.$ is a hypercover of $U_{\overline{K}}^o$ in $\mathrm{Fet}(U_{\overline{K}}^o)$ and therefore $\tilde{s} \in F_{U_q'}(N_q') = F_U(N_q')$ defines a class in $H^q(U_{\overline{K},\mathrm{Fet}}^o, F_U)$ mapping to s. This shows that 6.8.4 is surjective.

For the injectivity of 6.8.4, consider $U \in \mathrm{Et}(X)$ and an element

$$(6.10.16) \qquad \sigma \in H^q(U_{\overline{K},\mathrm{Fet}}^o, F_U)$$

mapping to zero in $R^q\nu_{X*}F^a$. After shrinking on U, we can find an étale hypercover $N. \to U_{\overline{K}}^o$ in $\mathrm{Fet}(U_{\overline{K}}^o)$, a section $\tilde{\sigma} \in F_U(N_q)$ defining σ, and a morphism of hypercovers in $\mathscr{S}$

$$(6.10.17) \qquad (U.', N.') \to (U, N.)$$

such that $\tilde{\sigma}$ maps to a boundary in $F_{U'}(N.')$. By 6.10, after shrinking on U we can assume each U_n' is finite and étale over U. In this case $N.' \to N.$ is a morphism of hypercovers in $U_{\overline{K},\mathrm{Fet}}^o$ such that the image of $\tilde{\sigma}$ under the map $F_U(N.) \to F_U(N.')$ is

a boundary. Therefore $\tilde{\sigma}$ defines the zero section of $\mathscr{H}^q$. This completes the proof of 6.9.   □

COROLLARY 6.11. *With notation and assumptions as in 6.8, the natural map*
(6.11.1)
$$\varinjlim_{U. \in HR_{\mathrm{Et}(X)}} H^*(U^o_{\cdot,\overline{K},\mathrm{Fet}}, F_{U.}) \to \varinjlim_{U. \in HR_{\mathrm{Et}(X)}} H^*(U^o_{\cdot,\overline{K},\mathrm{Fet}}, R\epsilon_{U.*}F^a) \simeq H^*(\mathscr{X}^o_{\overline{K}}, F^a).$$
is an isomorphism.

PROOF. For any $U. \in HR_{\mathrm{Et}(X)}$ there is a canonical spectral sequence

(6.11.2) $E_1^{pq} = H^q(U^o_{p,\overline{K},\mathrm{Fet}}, F_{U_p}) \implies H^{p+q}(U^o_{\cdot,\overline{K},\mathrm{Fet}}, F_{U.})$

and similarly

(6.11.3) $E_1^{pq} = H^q(U^o_{p,\overline{K},\mathrm{Fet}}, R\epsilon_{U_p*}F^a) \implies H^{p+q}(U^o_{\cdot,\overline{K},\mathrm{Fet}}, R\epsilon_{U.*}F^a),$

and it follows from the construction that the morphism 6.11.1 is induced by a morphism of spectral sequences from 6.11.2 to 6.11.3. The E_2^{pq}-term of 6.11.2 computes the p–th Cech cohomology of the presheaf

(6.11.4) $U \mapsto H^q(U^o_{\overline{K},\mathrm{Fet}}, F_U),$

and the E_2^{pq}-term of 6.11.3 computes the p–th Cech cohomology of the presheaf

(6.11.5) $U \mapsto H^q(\mathscr{X}^o_{\overline{K}}|_U, F^a).$

Proposition 6.9 therefore shows that the map from 6.11.2 to 6.11.3 induces an isomorphism on E_2-terms when passing to the limit over all $U. \in HR_{\mathrm{Et}(X)}$.   □

6.12. Let $\mathscr{O}$ denote the sheaf on $\mathscr{S}$ which to any (U,N) associates $\Gamma(U, \mathscr{O}_U)$. Define $\mathscr{O}_{\mathscr{X}^o_{\overline{K}}}$ to be the sheaf associated to the presheaf $\widetilde{\mathscr{O}}_{\mathscr{X}^o_{\overline{K}}}$ sending (U,N) to the global sections of the normalization of U in N. Also define $\mathscr{J}_X \subset \mathscr{O}_{\mathscr{X}^o_{\overline{K}}}$ to be the sheaf of ideals associated to the presheaf $\widetilde{\mathscr{J}}_X$ which to any (U,N) associates the ideal of the inverse image (with the reduced structure) of D in the normalization $\overline{N}$ of $U_{\overline{K}}$ in N.

If $U = \mathrm{Spec}(R)$ is an affine étale X-scheme admitting a morphism as in 2.16.1 and if Δ_U denotes the fundamental group of $U^o_{\overline{K}}$ with respect to a generic base point, then the sheaf $\widetilde{\mathscr{O}}_{\mathscr{X}^o_{\overline{K}},U}$ (resp. $\widetilde{\mathscr{J}}_{X,U}$) is the sheaf on $U^o_{\overline{K},\mathrm{Fet}}$ corresponding to the Δ_U-representation $\overline{R}$ (resp. $\overline{J}_X$) defined in 3.6. From this, 4.7, and 3.12 it follows that for any integer $r \geq 1$ there are almost isomorphisms

$$\mathscr{H}^q(\widetilde{\mathscr{O}}_{\mathscr{X}^o_{\overline{K}}}/p^r) \simeq \Omega^q_{(X,M_X)/V} \otimes_V (\overline{V}/p^r)(-q),$$

$$\mathscr{H}^q(\widetilde{\mathscr{J}}_X/p^r) \simeq J_X \otimes \Omega^q_{(X,M_X)/V} \otimes_V (\overline{V}/p^r)(-q),$$

where $J_X \subset \mathscr{O}_X$ is the ideal defining $X - X^o$. From this and 6.9 we conclude:

COROLLARY 6.13. *For any $r \geq 1$, the sheaves*

$$R^i\nu_{X*}\mathscr{O}_{\mathscr{X}^o_{\overline{K}}}/p^r\mathscr{O}_{\mathscr{X}^o_{\overline{K}}}, \quad and \quad R^i\nu_{X*}\mathscr{J}_X/p^r\mathscr{J}_X$$

are almost isomorphic to quasi–coherent sheaves on X_{et}, are almost zero for $i > d$, and there is a canonical trace isomorphism

(6.13.1) $\mathrm{tr} : R^d\nu_{X*}\mathscr{J}_X/p^r\mathscr{J}_X \to \Omega^d_{X/V} \otimes_V (\overline{V}/p^r)(-d)$

defined in $\mathscr{O}^a_X - \mathrm{Mod}$.

COROLLARY 6.14. *Assume that X is smooth and proper over* $\mathrm{Spec}(V)$. *Then* $H^i(\mathscr{X}^o_{\overline{K}}, \mathscr{J}_X/p^r\,\mathscr{J}_X)$ *is almost zero for* $i > 2d$ *and there is a canonical trace isomorphism defined in* $\overline{V}^a - \mathrm{Mod}$

(6.14.1) $$\mathrm{tr}: H^{2d}(\mathscr{X}^o_{\overline{K}}, \mathscr{J}_X/p^r\,\mathscr{J}_X) \to \overline{V}/p^r\overline{V}(-d).$$

PROOF. By the preceding corollary and standard facts about cohomology of quasi–coherent sheaves, we obtain that $H^i(\mathscr{X}^o_{\overline{K}}, \mathscr{J}_X/p^r\,\mathscr{J}_X)$ is almost zero for $i > 2d$ and that for $i = 2d$ this group is almost isomorphic to $H^d(X, \Omega^d \otimes \overline{V}/p^r\overline{V})(-d)$. By Serre duality there is a canonical isomorphism

(6.14.2) $$H^d(X, \Omega^d \otimes \overline{V}/p^r\overline{V})(-d) \simeq H^0(X \otimes (V/p^r), \mathscr{O}_{X \otimes V/p^r}) \otimes \overline{V}(-d)$$

which implies the result. □

By the same argument one obtains the following:

COROLLARY 6.15. *Assume that X is smooth and proper over* $\mathrm{Spec}(V)$. *Let L be a locally constant sheaf of free $\mathbb{Z}/p^r\mathbb{Z}$–modules on* $X^o_{\overline{K}}$, *let $L^* := \mathscr{H}om(L, \mathbb{Z}/p^r)$ denote its dual, and set $\mathscr{L} := u_{X*}L$ and $\mathscr{L}^* := u_{X*}L^*$.*

(i) The sheaves $R^i\nu_{X*}(\mathscr{L} \otimes \mathscr{O}_{\mathscr{X}^o_{\overline{K}}})$ and $R^i\nu_{X*}(\mathscr{L} \otimes \mathscr{J}_X)$ are almost isomorphic to quasi–coherent sheaves on X_{et}, and are almost zero for $i > d$.

(ii) The map in $\widetilde{D}(\mathscr{O}_{X_{\overline{V}}})$

$$R\nu_{X*}(\mathscr{L} \otimes \mathscr{O}_{\mathscr{X}^o_{\overline{K}}}) \to R\mathscr{H}om(R\nu_{X*}(\mathscr{L}^* \otimes \mathscr{J}_X), \Omega^d_{X/V} \otimes \overline{V}/p^r(-d))$$

induced by the trace map 6.13.1, is an isomorphism.

(iii) The trace map induces a canonical isomorphism in $\widetilde{D}(\overline{V})$

(6.15.1) $$R\Gamma(\mathscr{X}^o_{\overline{K}}, \mathscr{L} \otimes \mathscr{O}_{\mathscr{X}^o_{\overline{K}}}) \to R\mathrm{Hom}(R\Gamma(\mathscr{X}^o_{\overline{K}}, \mathscr{L}^* \otimes \mathscr{J}_X), \overline{V}/p^r\overline{V}(-d)[-2d]).$$

PROOF. Statement (i) follows from the same argument giving 6.13, and this argument combined with 4.11 also gives (ii). For (iii), note that by Serre duality and the fact that the complex $R\nu_{X*}(\mathscr{L}^* \otimes \mathscr{J}_X)$ is almost isomorphic to a bounded complex with coherent cohomology sheaves, the canonical pairing

$$R\mathrm{Hom}(R\nu_{X*}(\mathscr{L}^* \otimes \mathscr{J}_X), \Omega^d_{X/V} \otimes \overline{V}/p^r) \otimes R\Gamma R\nu_{X*}(\mathscr{L}^* \otimes \mathscr{J}_X) \to \overline{V}/p^r\overline{V}$$

is perfect. Now by (ii) we have

$$R\mathrm{Hom}(R\nu_{X*}(\mathscr{L}^* \otimes \mathscr{J}_X), \Omega^d_{X/V} \otimes \overline{V}/p^r) \simeq R\Gamma(\mathscr{L} \otimes \mathscr{O}_{\mathscr{X}^o_{\overline{K}}})$$

which implies (iii). □

COROLLARY 6.16. *Assume $r = 1$. Then for any integer i, there is a perfect pairing in $\overline{V}^a - Mod$*

(6.16.1) $$H^i(\mathscr{X}^o_{\overline{K}}, \mathscr{L} \otimes \mathscr{O}_{\mathscr{X}^o_{\overline{K}}}) \times H^{2d-i}(\mathscr{X}^o_{\overline{K}}, \mathscr{L}^* \otimes \mathscr{J}_X) \to \overline{V}/p^r\overline{V}(-d).$$

By 6.5, the natural map

(6.16.2) $$H^*(\mathscr{X}^o_{\overline{K}}, \mathscr{L}) \to H^*(X^o_{\overline{K},\mathrm{et}}, L)$$

is an isomorphism. The map of sheaves $\mathscr{L} \to \mathscr{L} \otimes \mathscr{O}_{\mathscr{X}^o_{\overline{K}}}$ therefore induces a map on cohomology

(6.16.3) $$H^*(X^o_{\overline{K},\mathrm{et}}, L) \otimes \overline{V} \to H^*(\mathscr{X}^o_{\overline{K}}, \mathscr{L} \otimes \mathscr{O}_{\mathscr{X}^o_{\overline{K}}}).$$

The following theorem will be proven in section 8:

THEOREM 6.17. *The morphism 6.16.3 is an isomorphism. In addition there is a canonical almost isomorphism*

$$(6.17.1) \qquad H^*_c(X^o_{\overline{K},\mathrm{et}}, L) \otimes \overline{V} \simeq H^*(\mathscr{X}^o_{\overline{K}}, \mathscr{L} \otimes \mathscr{J}_X),$$

and these isomorphisms are compatible with the pairing 6.15.1 and the usual Poincaré duality for étale cohomology.

6.18. The isomorphism 6.15.1 can also be generalized to a statement with partial compact supports (this generalization is in fact crucial for the proof). Suppose given a decomposition $D = E \cup F$, where E and F are divisors on X with no common irreducible component. Define $\mathscr{J}_E \subset \mathscr{O}_{\mathscr{X}^o_{\overline{K}}}$ to be the sheaf of ideals which to any $(U, N) \in \mathscr{S}$ associates the ideal of $(E \times_X \overline{N})_{\mathrm{red}} \subset \overline{N}$, where $\overline{N}$ denotes the normalization of $U_{\overline{K}}$ in N. Note that multiplication induces a map $\mathscr{J}_E \otimes \mathscr{J}_F \to \mathscr{J}_X$ and hence using Poincaré duality we obtain for any locally constant sheaf of $\mathbb{Z}/p^r$-modules L a map

$$(6.18.1) \quad R\Gamma(\mathscr{X}^o_{\overline{K}}, \mathscr{L} \otimes \mathscr{J}_E) \to R\mathrm{Hom}(R\Gamma(\mathscr{X}^o_{\overline{K}}, \mathscr{L}^* \otimes \mathscr{J}_F), \overline{V}/p^r\overline{V}(-d)[-2d]).$$

By the same argument used to prove 6.15 (iii) this map is an isomorphism.

7. Computing compactly supported cohomology using Galois cohomology

Before proving 6.17, we make in this section some general comments about how to compute compactly supported cohomology using Galois cohomology. These observations will only be used in the proof of 6.17 and can be omitted by the reader who skips this proof.

Throughout this section Λ denotes the ring $\mathbb{Z}/p^n$ for some n, and we consider Λ-modules unless otherwise stated.

7.1. Let Y be a smooth $\overline{K}$–scheme, $D = D_1 \cup \cdots \cup D_n$ a divisor with simple normal crossings on Y, let $A \subset \{1, \ldots, n\}$ be a subset, and let $D_A \subset Y$ denote the intersection $\cap_{a \in A} D_A$. Let D^o_A denote the complement in D_A of $\cup_{i \notin A}(D_i \cap D_A)$, and let $j_A : D^o_A \hookrightarrow Y$ and $j : Y^o \hookrightarrow Y$ be the inclusions. Assume that $Y = \mathrm{Spec}(R)$ is affine, and that each D^o_A is a $K(\pi, 1)$. We will further assume that the exists a formally étale morphism

$$(7.1.1) \qquad Y \to \mathrm{Spec}(\overline{K}[X_1, \ldots, X_d])$$

for some $d \geq 0$ such that D is equal to the inverse image of $X_1 \cdots X_n = 0$ for some $n \leq d$.

7.2. We will also sometimes consider the following additional assumption (but will always state explicitly when we assume it):

(R): For every nonempty $A \subset \{1, \ldots, n\}$ there exists a retraction of the inclusion $D_A \hookrightarrow Y$

$$(7.2.1) \qquad r_A : Y \to D_A$$

such that the inverse image of D^o_A contains Y^o.

EXAMPLE 7.3. The relevance of condition (R) for our purposes is the following. Let X/V be a smooth proper scheme with $D \subset X$ a divisor with simple normal crossings relative to V, and let $\bar{x} \to X$ be a geometric point. Then $Y := \mathrm{Spec}(\mathscr{O}_{X,\bar{x}} \otimes_V \overline{K})$ with the pullback of the divisor D satisfies condition (R).

7.4. For a finite étale morphism $Z^o \to Y^o$, let $Z \to Y$ be the normalization of Y in Z^o, and let $Z_A^o \subset Z$ denote the maximal reduced closed subscheme of $D_A^o \times_Y Z$. We then have inclusions

$$(7.4.1) \qquad Z_A^o \xrightarrow{\;j_A^Z\;} Z \xleftarrow{\;j^Z\;} Z^o.$$

LEMMA 7.5. *The morphism $Z_A^o \to D_A^o$ is finite and étale.*

PROOF. The assertion is étale local on Y so it suffices to consider the case when Y is the spectrum of a strictly henselian local ring. In this case the result follows from Abhyankar's lemma [**25**, Appendice 1, 5.2]. $\qquad\square$

LEMMA 7.6. *For any locally constant constructible sheaf of Λ–modules L on Y^o and $\nu \geq 0$ the sheaf $j_A^* R^\nu j_* L$ is locally constant constructible sheaf on D_A^o.*

PROOF. If $f : Y'^o \to Y^o$ is a finite étale morphism, then $f_* \Lambda$ is a locally constant constructible sheaf of Λ–modules on Y^o. Furthermore, for any locally constant constructible sheaf L on Y^o there exists a finite étale covering $f : Y'^o \to Y^o$ such that $f^* L$ is trivial. If we fix an isomorphism $f^* L \simeq \Lambda^r$ for some r then the adjunction map $L \to f_* \Lambda^r$ is an inclusion. From this one deduces that for any locally constant constructible sheaf of Λ–modules L on Y^o, there exists a resolution $L \to K^\bullet$ in the category of locally constant constructible sheaves of Λ–modules on Y^o such that each K^n is a direct sum of sheaves of the form $f_* \Lambda$ for some finite étale morphisms $f : Y'^o \to Y^o$. The spectral sequence corresponding to the "stupid filtration" on $K^\bullet$ yields

$$(7.6.1) \qquad E_1^{pq} = R^q j_* K^p \implies R^{p+q} j_* L.$$

From this it follows that it suffices to prove the lemma for $L = f_* \Lambda$ for some finite étale surjection $f : Y'^o \to Y^o$.

In this case, let $\bar{f} : Y' \to Y$ be the normalization of Y in Y'^o, and let $D_A'^o$ (resp. D_A') denote the maximal reduced subscheme of $Y' \times_Y D_A^o$ (resp. $Y' \times_Y D_A$) so that we have

$$(7.6.2) \qquad
\begin{array}{ccccc}
Y'^o & \xrightarrow{\;j'\;} & Y' & \xleftarrow{\;j_A'\;} & D_A'^o \\[4pt]
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} & & \downarrow{\scriptstyle g} \\[4pt]
Y^o & \xrightarrow{\;j\;} & Y & \xleftarrow{\;j_A\;} & D_A^o.
\end{array}$$

As in 7.5, the subscheme $D_A' \subset Y'$ is a divisor with normal crossings. We then have

$$
\begin{aligned}
j_A^* R^\nu j_* L &= j_A^* R^\nu j_* (f_* \Lambda) \\
&\simeq j_A^* R^\nu j_* (R f_* \Lambda) \\
&\simeq j_A^* R^\nu \bar{f}_* (R j_*' \Lambda) \\
&\simeq g_* j_A'^* R^\nu j_*' \Lambda,
\end{aligned}
$$

where the last isomorphism is by the proper base change theorem (note that the right square in 7.6.2 is not cartesian but that $D_A' \to Y' \times_Y D_A$ is a closed immersion defined by a nilpotent ideal so that étale cohomology of a sheaf on $Y' \times_Y D_A$ is equal to the étale cohomology of its pullback to D_A'). It follows that it suffices to prove that $j_A'^* R^\nu j_*' \Lambda$ is locally constant constructible on $D_A'^o$. This further reduces the proof to the case when $L = \Lambda$.

So let us finally prove the result for $L = \Lambda$. By replacing Y by $Y - \cup_{i \notin A} D_i$ we may assume that in fact $A = \{1, \ldots, n\}$. Furthermore, the assertion is evidently étale local on Y so we may assume that there is a smooth map

$$(7.6.3) \qquad \pi : Y \to \mathrm{Spec}(\overline{K}[T_1, \ldots, T_n])$$

such that D_i is the inverse image of the zero locus of T_i so that $D_A = \pi^{-1}(0)$. This morphism π defines a morphism

$$(7.6.4) \qquad H^\nu(\mathbb{G}_m^n, \Lambda) \to j_A^* R^\nu j_* \Lambda$$

which we claim is an isomorphism. To prove this we may replace Y by $\mathrm{Spec}(\mathscr{O}_{Y,\bar{y}})$ for a geometric point $\bar{y} \to D_A$. In this case Y^o is a $K(\pi, 1)$ by 5.10 with fundamental group $\mathbb{Z}(1)^n$ so $(j_A^* R^\nu j_* \Lambda)_{\bar{y}}$ is isomorphic to the group cohomology $H^\nu(\mathbb{Z}(1)^n, L)$. On the other hand, using Künneth and the fact that $\mathbb{G}_m$ is a $K(\pi, 1)$ one sees that $H^\nu(\mathbb{G}_m^n, \Lambda)$ is also isomorphic to $H^\nu(\mathbb{Z}(1)^n, \Lambda)$ and under these identifications the map 7.6.4 is the identity map. $\qquad \square$

7.7. Fix a geometric generic point $\mathrm{Spec}(\Omega) \to Y$, and let $\overline{R} \subset \Omega$ be the integral closure of R in the compositum of all field extensions $k(Y) \subset L \subset \Omega$ for which the normalization of R in L is étale over Y^o. Let Δ denote the étale fundamental group of Y^o with respect to the chosen base point so that Δ acts on $\mathrm{Spec}(\overline{R})$.

Let $A \subset \{1, \ldots, n\}$ be a subset and let $\bar{\eta} \to D_A^o$ be a geometric generic point. For any choice of lifting $\bar{\eta} \to \mathrm{Spec}(\overline{R})$ we obtain a *decomposition group* $H_A \subset \Delta$ defined to be the subgroup of elements fixing the image of $\bar{\eta}$. The subgroup $H_A \subset \Delta$ is closed. Let $\overline{R}_A$ be the quotient of $\overline{R}$ by the prime ideal corresponding to the image of $\bar{\eta}$. Then $\mathrm{Spec}(\overline{R}_A) \to D_A^o$ is an inverse limit of finite étale morphisms, and H_A acts on the inverse limit. Let $I_A \subset H_A$ denote the subgroup of elements which act trivially on $\overline{R}_A$ (the *inertia group*). If Δ_A denotes the étale fundamental group of D_A^o with respect to the base point $\bar{\eta}$, then the action of H_A/I_A on $\overline{R}_A$ defines a surjection

$$(7.7.1) \qquad \Delta_A \to H_A/I_A.$$

LEMMA 7.8. *If condition (R) holds, then the homomorphism 7.7.1 is an isomorphism.*

PROOF. It suffices to show that any finite set with continuous Δ_A-action is obtained by pullback from a set with continous H_A/I_A-action. This is clear for if S is a set with continuous Δ_A–action corresponding to an étale morphism $V \to D_A^o$, then $r_A^* V \to Y^o$ is an étale morphism, and the corresponding H_A/I_A-set pulls back to S. $\qquad \square$

7.9. In terms of Galois representations, the sheaf $j_A^* j_* L$ can be described as follows. Let M be the Δ–representation corresponding to L. Then $j_A^* j_* L$ is the sheaf corresponding to the representation of Δ_A acting on M^{I_A} via the surjection $\Delta_A \to H_A/I_A$. This implies in particular the following:

COROLLARY 7.10. *Let $A \subset B \subset \{1, \ldots, n\}$ so that $D_B \subset D_A$. Then for any locally constant constructible sheaf of Λ–modules L on Y^o the natural map $j_B^* j_* L \to j_B^* j_{A*} j_A^* j_* L$ is an isomorphism.*

7.11. Let L be a locally constant sheaf of abelian groups on Y_{Fet}^o. For $A \subset \{1, \ldots, n\}$ define a presheaf $\psi_A(L)$ on $\mathrm{Fet}(Y^o)$ by

$$(7.11.1) \qquad (Z^o \to Y^o) \mapsto \Gamma(Z_A^o, j_A^{Z*} j_*^Z(L|_{Z^o})).$$

In terms of Galois representations the presheaf $\psi_A(L)$ can be described as follows. Let M be the Δ–representation corresponding to L, and let $H_A \subset \Delta$ be the decomposition group of D_A obtained by making suitable choices as in 7.7. Then we claim that $\psi_A(L)$ is the sheaf corresponding to the Δ–representation

$$(7.11.2) \qquad \operatorname{Ind}_{H_A}^{\Delta}(M|_{H_A}) := \operatorname{Hom}_{H_A}^{\mathrm{cts}}(\Delta, M|_{H_A}).$$

Here $\operatorname{Ind}_{H_A}^{\Delta}(-)$ is the right adjoint to the restriction functor from Δ-modules to H_A-modules.

To see this let $Z^o \to Y^o$ be a finite étale morphism corresponding to a Δ–set S. The restriction of S to H_A decomposes into H_A-orbits

$$(7.11.3) \qquad S|_{H_A} = \coprod_i S_i.$$

For each i let $H_i \subset H_A$ denote the kernel of the map $H_A \to \operatorname{Aut}(S_i)$. We then have

$$
\begin{aligned}
\operatorname{Hom}_{\Delta}(S, \operatorname{Ind}_{H_A}^{\Delta}(M|_{H_A})) &\simeq \operatorname{Hom}_{H_A}(S, M) \\
&\simeq \prod_i \operatorname{Hom}_{H_A}(S_i, M) \\
&\simeq \prod_i M^{H_i} \\
&\simeq \Gamma(Z_A^o, j_A^{Z*} j_*^Z L).
\end{aligned}
$$

COROLLARY 7.12. *The presheaf $\psi_A(L)$ is a sheaf on Y_{Fet}^o.*

COROLLARY 7.13. *The functor $\psi_A(-)$ is an exact functor on the category of sheaves on Y_{Fet}^o.*

PROOF. This is because $\operatorname{Ind}_{H_A}^{\Delta}(-)$ is an exact functor (this follows immediately from the definition). $\square$

LEMMA 7.14. *Let Δ be a profinite group and $D \subset \Delta$ a closed subgroup.*
(i) Any injective Δ–representation I is a direct summand of an induced module $\operatorname{Ind}_D^{\Delta}(N)$ with N an injective D–representation.
(ii) For any injective Δ–representation I and $i > 0$ the group $H^i(D, I|_D)$ is zero.

PROOF. Let $\widetilde{I}$ denote the restriction of I to D and choose an inclusion $\widetilde{I} \hookrightarrow N$ with N an injective D-representation. The adjunction map $I \to \operatorname{Ind}_D^{\Delta}(\widetilde{I}) \to \operatorname{Ind}_D^{\Delta}(N)$ is then injective. Since I is injective this inclusion $I \hookrightarrow \operatorname{Ind}_D^{\Delta}(N)$ is split which proves (i).

For (ii) note first that by (i) applied to the inclusion of the trivial group into Δ, it suffices to consider the case of $I = \operatorname{Ind}^{\Delta}(N)$ (induction for the trivial subgroup $\{e\} \hookrightarrow \Delta$) for some injective abelian group N. Let $K_n \subset \Delta$ be a decreasing system of closed subgroups of finite index defining a base for the topology so that $\Delta = \varprojlim \Delta/K_n$ and $D = \varprojlim_n D/(D \cap K_n)$ (note since D is closed in Δ such a system of subgroups K_n exists). We then have

$$
\begin{aligned}
H^i(D, I|_D) &= \varinjlim_n H^i(D/(D \cap K_n), \operatorname{Hom}^{\mathrm{cts}}(\Delta, N)^{D \cap K_n}) \\
&= \varinjlim_n \varinjlim_m H^i(D/(D \cap K_n), \operatorname{Hom}(\Delta/K_m, N)^{D \cap K_n}).
\end{aligned}
$$

We can write this double limit as the single limit

$$(7.14.1) \qquad \varinjlim_{n} H^i(D/(D \cap K_n), \mathrm{Hom}(\Delta/K_n, N)).$$

The D–module $\mathrm{Hom}(\Delta/K_n, N)$ is a direct sum of modules $\mathrm{Hom}(D/(D \cap K_n), N)$ which is injective in the category of $D/(D \cap K_n)$–modules. From this it follows that the higher cohomology groups are zero. $\square$

COROLLARY 7.15. *Let M be an injective object in the category of continuous representations of Δ on Λ-modules. Then the underlying Λ-module of M is flat.*

PROOF. Taking $D = \{1\}$ in 7.14 we see that M is a direct summand of $\mathrm{Ind}_{\{1\}}^{\Delta}(N)$ for some injective Λ-module N. Therefore it suffices to show that any injective Λ-module is flat over Λ. This is a standard consequence of the fact that Λ is Gorenstein of dimension 0, which implies that Λ is injective as a Λ-module (see for example [**12**, §21.2]). For the convenience of the reader let us recall the argument. It suffices to show that for any nonzero element $m \in M$ there exists an inclusion $i : \Lambda \hookrightarrow M$ such that $i(\Lambda)$ contains m. For since Λ is injective such an inclusion i is split so this will show that M is isomorphic to filtered colimit of finitely generated free Λ-modules, and hence flat. Since M is a Λ-module there exists a minimal integer s such that $p^s m = 0$, and this gives an inclusion $j : \mathbb{Z}/(p^s) \hookrightarrow M$ sending 1 to m. Since M is injective there exists a dotted arrow g filling in the diagram

$$
\begin{array}{ccccc}
0 & \longrightarrow & \mathbb{Z}/(p^s) & \xrightarrow{1 \mapsto p^{n-s}} & \Lambda \\
 & & \downarrow{\scriptstyle j} & \overset{g}{\swarrow} & \\
 & & M. & &
\end{array}
$$

Because j is an inclusion the map g is also an inclusion, since by construction $g(1)$ has exact order p^n. $\square$

LEMMA 7.16. *With notation as in 7.7, let M be an injective representation of Δ. Then*
  (i)

$$(7.16.1) \qquad\qquad H^i(H_A, M|_{H_A}) = 0$$

for any $i > 0$, and $H^0(H_A, M|_{H_A}) = H^0(\Delta_A, M^{I_A})$.
  (ii) If in addition condition (R) holds then for any $i > 0$ we have

$$(7.16.2) \qquad\qquad H^i(\Delta_A, M^{I_A}) = 0.$$

PROOF. Since $H_A \subset \Delta$ is closed the vanishing of $H^i(H_A, M|_{H_A})$ follows from 7.14. The statement about H^0 is immediate.

For the vanishing of $H^i(\Delta_A, M^{I_A})$ in the case when (R) holds, note that the short exact sequence

$$(7.16.3) \qquad\qquad 1 \to I_A \to H_A \to \Delta_A \to 1$$

induces a spectral sequence

$$(7.16.4) \qquad E_2^{pq} = H^p(\Delta_A, H^q(I_A, M)) \implies H^{p+q}(H_A, M).$$

Since $I_A \subset \Delta$ is also closed we have $H^q(I_A, M) = 0$ for $q > 0$ by 7.14, and therefore there is a canonical isomorphism

$$(7.16.5) \qquad\qquad H^p(\Delta_A, M^{I_A}) \simeq H^p(H_A, M).$$

From this (ii) follows. $\qquad\square$

We can in fact sharpen the above lemma as follows.

LEMMA 7.17. *Let Δ be a profinite group. A filtering direct limit of injective continuous Δ–representations is an injective continuous Δ–representation.*

PROOF. Let $I = \varinjlim I_\lambda$ with each I_λ an injective Δ-representation. Consider an inclusion of Δ–representations $N \hookrightarrow M$ and a map $\rho : N \to I$. We need to show that ρ extends to a morphism $M \to I$. For this write $M = \varinjlim M_k$ with each M_k a finitely presented abelian group with continuous Δ–action (this is possible since the action is continuous). Let N_k denote $N \cap M_k$. We then inductively extend the maps $\rho_k : N_k \to I$ obtained from ρ.

For this fix some k_0. The map $N_{k_0} \to I$ factors through some I_λ since N_{k_0} is finitely presented and therefore there exists an extension $\tilde{\rho}_{k_0} : M_{k_0} \to I$.

Now assume that $\tilde{\rho}_k : M_k \to I$ has been constructed. By the definition of N_k the map

$$(7.17.1) \qquad\qquad N_{k+1} \oplus_{N_k} M_k \to M_{k+1}$$

is injective. Using the above argument applied to the map

$$(7.17.2) \qquad\qquad \rho_{k+1} \oplus \tilde{\rho}_k : N_{k+1} \oplus_{N_k} M_k \to I$$

we then obtain the extension $\tilde{\rho}_{k+1}$. $\qquad\square$

LEMMA 7.18. *Let Δ be a profinite group and let $D \subset \Delta$ be a closed subgroup. If M is an injective Δ–representation then $M|_D$ is an injective D–representation.*

PROOF. Write $\Delta = \varprojlim \Delta/K_n$ and set $S_n = \Delta/(D, K_n)$. Then the set of cosets $S := \Delta/D$ is equal to $\varprojlim S_n$ (and in particular has the profinite topology). Choose a continuous section σ of the projection $\Delta \to \Delta/D$. Then for any continuous D–representation N we have an isomorphism of D–representations

$$(7.18.1) \qquad \mathrm{Hom}_D^{\mathrm{cts}}(\Delta, N)|_D \to \mathrm{Hom}^{\mathrm{cts}}(S, N), \quad (f : \Delta \to N) \mapsto (s \mapsto f(\sigma(s))).$$

Since

$$(7.18.2) \qquad\qquad \mathrm{Hom}^{\mathrm{cts}}(S, N) = \varinjlim_n \mathrm{Hom}(S_n, N)$$

this, together with 7.17, proves the result in the case when $M = \mathrm{Ind}_D^\Delta(N)$ for some injective D–representation N. For the general case, note that by 7.14 any injective M is a direct summand of a representation $\mathrm{Ind}_D^\Delta(N)$ with N an injective D–representation. $\qquad\square$

LEMMA 7.19. *Let Δ be a profinite group and let $D \subset \Delta$ be a closed subgroup. Assume that $D \subset \Delta$ is normal, and let G be the quotient. If M is an injective Δ–representation, then M^D is an injective G–representation.*

PROOF. Immediate since the functor $M \mapsto M^D$ takes injectives to injectives. $\qquad\square$

COROLLARY 7.20. *Let Δ be a profinite group, let $D_1, D_2 \subset \Delta$ be two closed subgroups, and let M be an injective Δ–representation. Then for any $i > 0$ we have*

$$(7.20.1) \qquad\qquad H^i(D_2, \mathrm{Ind}_{D_1}^{\Delta}(M|_{D_1})|_{D_2}) = 0.$$

7.21. Let L be a locally constant sheaf of abelian groups on Y_{et}^o. Define a complex of sheaves $\overline{\Psi}(L)$ on $\mathrm{Et}(Y)$ as follows. For $r \geq 0$ set

$$(7.21.1) \qquad\qquad \overline{\Psi}(L)^r := \oplus_{A \subset \{1,\ldots,n\}, |A|=r} j_{A*} j_A^* j_* L.$$

For $1 \leq j \leq r$ and $A = \{i_1 < i_2 < \cdots < i_r\} \subset \{1,\ldots,n\}$ let A_j denote the set $\{i_1, i_2, \ldots, \hat{i}_j, i_{j+1}, \ldots, i_r\}$ so that $A_j \subset A$. Using 7.10 we then have a map

$$(7.21.2) \qquad \partial_j : j_{A_j*} j_{A_j}^* j_* L \to j_{A*} j_A^* j_{A_j*} j_{A_j}^* j_* L \simeq j_{A*} j_A^* j_* L.$$

Taking the sum of these maps we get a morphism

$$(7.21.3) \qquad\qquad \partial_j : \overline{\Psi}(L)^{r-1} \to \overline{\Psi}(L)^r.$$

Define

$$(7.21.4) \qquad\qquad \partial : \overline{\Psi}(L)^{r-1} \to \overline{\Psi}(L)^r$$

to be the alternating sum $\sum_{j=1}^{r+1} (-1)^j \partial_j$. One verifies immediately that $\partial^2 = 0$ so we get a complex $\overline{\Psi}(L)$ of sheaves on $\mathrm{Et}(Y)$.

We have $\overline{\Psi}(L)^0 = j_* L$ and the composite

$$(7.21.5) \qquad\qquad j_! L \longrightarrow j_* L = \overline{\Psi}(L)^0 \xrightarrow{\ \partial\ } \overline{\Psi}(L)^1$$

is zero.

LEMMA 7.22. *The map of complexes of sheaves*

$$(7.22.1) \qquad\qquad j_! L \to \overline{\Psi}(L)$$

is a quasi-isomorphism.

PROOF. It suffices to show that for any geometric point $\bar{y} \to Y$ the map of complexes of abelian groups

$$(7.22.2) \qquad\qquad (j_! L)_{\bar{y}} \to \overline{\Psi}(L)_{\bar{y}}$$

is a quasi–isomorphism. If $\bar{y}$ maps to Y^o this is immediate. If the image of $\bar{y}$ is in the boundary, the ring $\mathscr{O}_{Y,\bar{y}}$ is isomorphic to the strict henselization of the ring $K[X_1, \ldots, X_d]$ at the origin and D is the zero locus of $X_1 \cdots X_r$ for some $r \leq d$. In this case the fundamental group Δ of $\mathrm{Spec}(\mathscr{O}_{Y,\bar{y}})^o$ is isomorphic to $\mathbb{Z}(1)^r$. If M denotes the representation of Δ corresponding to L, then it follows from the construction that $\overline{\Psi}(L)_{\bar{y}}$ is isomorphic to the complex

$$(7.22.3) \qquad M^{\Delta} \otimes (\mathbb{Z} \to \oplus_{1 \leq i \leq r} \mathbb{Z} \to \oplus_{1 \leq i_1 < i_2 \leq r} \mathbb{Z} \to \cdots),$$

where

$$\oplus_{1 \leq i \leq r} \mathbb{Z} \to \oplus_{1 \leq i_1 < i_2 \leq r} \mathbb{Z} \to \cdots$$

is the acyclic complex computing the Cech cohomology of $\mathrm{Spec}(\mathbb{Z})$ with respect to the covering

$$\coprod_{i=1}^r \mathrm{Spec}(\mathbb{Z}) \to \mathrm{Spec}(\mathbb{Z}).$$

$\square$

LEMMA 7.23. *Let L be an injective Λ–module in Y_{Fet}^o. For any subset $A \subset \{1,\ldots,n\}$ the natural map $j_{A*}j_A^*j_*L \to Rj_{A*}j_A^*j_*L$ is an isomorphism.*

PROOF. Let $\bar{y} \to Y$ be a geometric point, and let $\widetilde{Y}$ denote $\mathrm{Spec}(\mathscr{O}_{Y,\bar{y}})$. Then it suffices to show that for any $\nu > 0$ we have

$$(7.23.1) \qquad\qquad H^\nu(\widetilde{Y}_A^o, j_A^*j_*L) = 0.$$

The existence of the map 7.1.1 ensures that the morphism of fundamental groups $\pi_1(\widetilde{Y}^o) \to \pi_1(Y^o)$ (obtained by choosing suitable base points) is injective. Using 7.18 this implies that it suffices to prove the lemma with Y replaced by $\widetilde{Y}$. In this case $j_A^*j_*L$ is given by an injective representation of $\pi_1(Y_A^o)$ which implies the lemma. $\qquad\square$

LEMMA 7.24. *Let L be an injective Λ-module in Y_{Fet}^o and assume condition (R) holds. Then for any $i > 0$ and any integer r the group $H^i(Y_{\mathrm{et}}, \overline{\Psi}(L)^r)$ is zero.*

PROOF. For any $A \subset \{1,\ldots,n\}$ of size r we have by 7.23

$$(7.24.1) \qquad\qquad H^i(Y_{\mathrm{et}}, j_{A*}j_A^*j_*L) = H^i(D_{A,\mathrm{et}}^o, j_A^*j_*L),$$

which since D_A^o is a $K(\pi,1)$ and $j_A^*j_*L$ is locally constant is isomorphic to

$$(7.24.2) \qquad\qquad H^i(D_{A,\mathrm{Fet}}^o, j_A^*j_*L).$$

In the notation of 7.7, if M is the Δ-representation corresponding to L we then want

$$(7.24.3) \qquad\qquad H^i(\Delta_A, M^{I_A}) = 0$$

for $i > 0$. This follows from 7.16. $\qquad\square$

7.25. It follows that for a locally constant constructible sheaf L of Λ–modules on Y^o, we can compute $H^*(Y_{\mathrm{et}}, j_!L)$ as follows. Choose an injective resolution $L \to I^\bullet$ in the category of Λ–modules in Y_{Fet}^o, and let $\overline{\Psi}(I^\bullet)$ be the complex on Y_{et} obtained by taking the total complex of the double complex obtained by applying $\overline{\Psi}(-)$ to each I^p. We then have a commutative diagram

$$(7.25.1) \qquad\qquad \begin{array}{ccc} j_!L & \xrightarrow{\;a\;} & j_!I^\bullet \\ {\scriptstyle b}\big\downarrow & & \big\downarrow{\scriptstyle c} \\ \overline{\Psi}(L) & \xrightarrow{\;d\;} & \overline{\Psi}(I^\bullet). \end{array}$$

Since $j_!$ is an exact functor the morphism a is a quasi–isomorphism, and b and c are quasi-isomorphisms by 7.22. It follows that d is also a quasi-isomorphism. We then have a map

$$(7.25.2) \qquad\qquad \Gamma(Y_{\mathrm{et}}, \overline{\Psi}(I^\bullet)) \to R\Gamma(Y_{\mathrm{et}}, j_!L)$$

which by 7.24 is an isomorphism if (R) holds.

7.26. For a sheaf of Λ–modules L in Y_{Fet}^o we can also define a complex $\Psi(L)$ in Y_{Fet}^o as follows. We define

$$(7.26.1) \qquad\qquad \Psi(L)^r := \oplus_{A\subset\{1,\ldots,n\},|A|=r}\psi_A(L),$$

where $\psi_A(L)$ is defined as in 7.11.

If $A \subset B \subset \{1,\ldots,n\}$, then there is a natural map

$$(7.26.2) \qquad\qquad \psi_A(L) \to \psi_B(L).$$

Indeed for any $Z^o \to Y^o$ with inclusions

$$(7.26.3) \qquad\qquad j^Z : Z^o \hookrightarrow Z, \;\; j^Z_A : Z^o_A \hookrightarrow Z \;\; \text{etc.}$$

we have a map
(7.26.4)
$$\psi_A(L)(Z^o) = j^Z_{A*} j^{Z*}_A j^Z_* L(Z) \to j^{*}_B j^Z_{A*} j^{Z*}_A j^Z_* L(Z^o_B) = j^{Z*}_B j^Z_* L(Z^o_B) = \psi_B(L)(Z^o),$$

where the second to last isomorphism is by 7.10. Taking the alternating sum as in 7.21 we obtain a map

$$(7.26.5) \qquad\qquad \Psi(L)^{r-1} \to \Psi(L)^r,$$

and we write $\Psi(L)^\bullet$ for the resulting complex of sheaves on Y^o_{Fet}.

We can compute $R\Gamma(Y^o_{\mathrm{Fet}}, \Psi(L)^\bullet)$ as follows. Let $L \to I^\bullet$ be an injective resolution in the category of Λ–modules in Y^o_{Fet}, and let $\Psi(I^\bullet)$ be the total complex of the double complex obtained by applying $\Psi(-)$ to each I^p. By 7.13 the functor $\Psi(-)^r$ is exact, so we have a quasi-isomorphism

$$(7.26.6) \qquad\qquad \Psi(L)^\bullet \to \Psi(I^\bullet).$$

On the other hand, it follows from 7.16 that for any integer $i > 0$ we have $H^i(Y^o_{\mathrm{Fet}}, \Psi(I^\bullet)^r) = 0$. Therefore we obtain an isomorphism

$$(7.26.7) \qquad\qquad R\Gamma(Y^o_{\mathrm{Fet}}, \Psi(I^\bullet)) \simeq \Gamma(Y^o_{\mathrm{Fet}}, \Psi(I^\bullet)),$$

and hence also an isomorphism

$$(7.26.8) \qquad\qquad R\Gamma(Y^o_{\mathrm{Fet}}, \Psi(L)) \simeq \Gamma(Y^o_{\mathrm{Fet}}, \Psi(I^\bullet)).$$

Now observe that by the definition of the sheaves $\psi_A(L)$ we have

$$(7.26.9) \qquad\qquad \Gamma(Y^o_{\mathrm{Fet}}, \psi_A(I^p)) = \Gamma(D^o_A, j^*_A j_* I^p).$$

It follows that

$$(7.26.10) \qquad\qquad \Gamma(Y^o_{\mathrm{Fet}}, \Psi(I^\bullet)) = \Gamma(Y_{\mathrm{et}}, \overline{\Psi}(I^\bullet)).$$

Combining 7.25.2 and 7.26.8 we obtain the following:

THEOREM 7.27. *For any locally constant constructible sheaf L in Y^o_{Fet}, there is a canonical morphism*

$$(7.27.1) \qquad\qquad R\Gamma(Y^o_{\mathrm{Fet}}, \Psi(L)) \to R\Gamma(Y_{\mathrm{et}}, j_! L),$$

which is an isomorphism if condition (R) holds.

REMARK 7.28. In the above we have chosen an injective resolution of L. One verifies immediately that the morphism in 7.27 does not depend on this choice.

REMARK 7.29. The preceding constructions are compatible with finite étale base change $Y' \to Y$. In particular, if $a : Y^o_{\mathrm{Fet}} \to Y_{\mathrm{Fet}}$ and $b : Y_{\mathrm{et}} \to Y_{\mathrm{Fet}}$ are the natural morphisms of topoi, then $\Psi(I^\bullet)$ (resp. $\overline{\Psi}(I^\bullet)$) is a complex of sheaves acyclic for a_* (resp. b_*) and there is a canonical isomorphism

$$(7.29.1) \qquad\qquad a_* \Psi(I^\bullet) \simeq b_* \overline{\Psi}(I^\bullet)$$

which induces the morphism 7.27.1 by applying $R\Gamma$.

7.30. Let $i : Z \subset Y$ be a smooth divisor meeting D transversally, and let Z^o denote $Z \times_Y Y^o$. Assume that Z^o is a $K(\pi, 1)$ and that there is a retraction $r : Y^o \to Z^o$ of the inclusion $Z^o \hookrightarrow Y^o$. Then for any geometric point $\bar{y} \to Z^o$ the identity map on $\pi_1(Z^o, \bar{y})$ factors through the map $\pi_1(Z^o, \bar{y}) \to \pi_1(Y^o, \bar{y})$. In particular, the map $\pi_1(Z^o, \bar{y}) \to \pi_1(Y^o, \bar{y})$ is injective and identifies $\pi_1(Z^o, \bar{y})$ with a closed subgroup of $\pi_1(Y^o, \bar{y})$.

COROLLARY 7.31. *Let L be an injective sheaf on Y^o_{Fet} and assume condition (R) holds. Then for any $A \subset \{1, \ldots, n\}$ the sheaf $i^* \Psi_A(L)$ on Z^o_{Fet} is acyclic for the global section functor.*

PROOF. If Δ denotes the fundamental group of Y^o and Δ_Z the fundamental group of Z^o then we have an inclusion $\Delta_Z \subset \Delta$ (well–defined up to conjugation). We also have the decomposition group $H_A \subset \Delta$ (also well–defined up to conjugation) and if M is the representation of Δ corresponding to L then $\psi_A(L)$ corresponds to the representation $\mathrm{Ind}^\Delta_{H_A}(M)$. The corollary therefore follows from 7.20. $\square$

COROLLARY 7.32. *Let L be an injective sheaf on Y^o_{Fet} and assume condition (R) holds. Then for any subset $A \subset \{1, \ldots, n\}$ the sheaf $j_A^* j_* L$ corresponds to a injective representation of $\pi_1(D_A^o)$.*

PROOF. Let Δ denote the fundamental group of Y^o, let $H_A \subset \Delta$ denote the decomposition group, and let $I_A \subset H_A$ be the inertia group. We then have H_A/I_A isomorphic to the fundamental group of D_A^o. If M is the Δ-representation corresponding to L, then $j_A^* j_* L$ corresponds to the representation $(M|_{H_A})^{I_A}$ of H_A/I_A. By 7.18 the H_A–representation $M|_{H_A}$ is injective, and therefore the corollary follows from 7.19. $\square$

8. Proof of 6.17

Let Λ denote the ring $\mathbb{Z}/p^n$ for some n.

8.1. Note first the if Y is another smooth proper V–scheme with a divisor $D_Y \subset Y$ with relative normal crossings, and if $f : Y \to X$ is a proper morphism such that $f^{-1}(D_X) \subset D_Y$, then there is an induced morphism of topoi

(8.1.1) $$f : \mathscr{Y}^o_{\overline{K}} \to \mathscr{X}^o_{\overline{K}}$$

such that the diagram

(8.1.2)
$$
\begin{array}{ccc}
Y^o_{\overline{K},\mathrm{et}} & \xrightarrow{u_Y} & \mathscr{Y}^o_{\overline{K}} \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\
X^o_{\overline{K},\mathrm{et}} & \xrightarrow{u_X} & \mathscr{X}^o_{\overline{K}}
\end{array}
$$

commutes.

8.2. We use this to reduce the proof of 6.17 to the case when D has simple normal crossings as follows. By [**33**, 4.2.12] there exists a blow-up $f : Y \to X$ with support in D such that Y/V is smooth, $D_Y := f^{-1}(D)$ is a divisor with simple normal crossings on Y and the induced morphism of log schemes $f : (Y, M_Y) \to (X, M_X)$ is log étale. This last observation implies that $f^* \Omega^1_{(X,M_X)/V} \simeq \Omega^1_{(Y,M_Y)/V}$. The morphism 8.1.1 induces morphisms

(8.2.1) $$H^*(\mathscr{X}^o_{\overline{K}}, \mathscr{L} \otimes \mathscr{O}_{\mathscr{X}^o_{\overline{K}}}) \to H^*(\mathscr{Y}^o_{\overline{K}}, f^* \mathscr{L} \otimes \mathscr{O}_{\mathscr{Y}^o_{\overline{K}}})$$

and

$$(8.2.2) \qquad H^*(\mathscr{X}_{\overline{K}}^o, \mathscr{L}^* \otimes \mathscr{J}_X) \to H^*(\mathscr{Y}_{\overline{K}}^o, f^*\mathscr{L}^* \otimes \mathscr{J}_Y).$$

We claim that 8.2.1 and 8.2.2 are monomorphisms in $\overline{V}^a - \mathrm{Mod}$. Indeed by a standard devissage to prove this it suffices to consider the case when L is a sheaf $\mathbb{Z}/p$-modules, in which case it follows from the fact that 8.2.1 and 8.2.2 are compatible with the perfect pairings 6.16 (this follows from the construction). In particular, to prove 6.17 it suffices to show that the composite morphism

$$(8.2.3) \qquad H^*(X_{\overline{K}}^o, L) \to H^*(\mathscr{X}_{\overline{K}}^o, \mathscr{L} \otimes \mathscr{O}_{\mathscr{X}_{\overline{K}}^o}) \to H^*(\mathscr{Y}_{\overline{K}}^o, f^*\mathscr{L} \otimes \mathscr{O}_{\mathscr{Y}_{\overline{K}}^o})$$

is an isomorphism. For then all the maps are isomorphisms. Using Poincaré duality this then also implies that the map 8.2.2 is an almost isomorphism.

We may therefore assume that the divisor D has simple normal crossings. Write $D = D_1 \cup \cdots \cup D_n$ with each $D_i \subset X$ a smooth irreducible divisor. For a subset $A \subset \{1, \ldots, n\}$ we write D_A for the intersection $\cap_{a \in A} D_a$. If $A = \emptyset$ then D_A denotes X.

8.3. Let $\mathscr{S}$ denote the site defining $\mathscr{X}_{\overline{K}}^o$. For an abelian sheaf $\mathscr{L}$ in $\mathscr{X}_{\overline{K}}^o$ define a complex of presheaves $\Psi(\mathscr{L})$ on $\mathscr{S}$ by associating to any $U \in \mathrm{Et}(X)$ the complex of sheaves $\Psi(\mathscr{L}_U)$ on $U_{\overline{K}, \mathrm{Fet}}^o$ obtained by applying the construction in 7.26 to the sheaf $\mathscr{L}_U$ in $U_{\overline{K}, \mathrm{Fet}}^o$. We denote by $\Psi^a(\mathscr{L})$ the associated complex of sheaves in $\mathscr{X}_{\overline{K}}^o$.

More generally if $\mathscr{L}^\bullet$ is a bounded below complex of abelian sheaves in $\mathscr{X}_{\overline{K}}^o$ we define $\Psi(\mathscr{L}^\bullet)$ to be the total complex of the bicomplex obtained by applying $\Psi(-)$ to each component of $\mathscr{L}^\bullet$. Passing to the associated sheaves we also define $\Psi^a(\mathscr{L}^\bullet)$. It follows from 7.13 that if $\mathscr{L}^\bullet \to \mathscr{M}^\bullet$ is a quasi-isomorphism, then the induced morphism

$$(8.3.1) \qquad \Psi^a(\mathscr{L}^\bullet) \to \Psi^a(\mathscr{M}^\bullet)$$

is also a quasi-isomorphism.

8.4. Let L be a locally constant constructible sheaf of Λ–modules in $X_{\overline{K}, \mathrm{et}}^o$, and let $\mathscr{L} := u_{X*}L$ denote the induced sheaf in $\mathscr{X}_{\overline{K}}^o$. Let $HR(X)' \subset HR(X)$ be the full subcategory of hypercoverings $U. \to X$ such that for every n the scheme $U_{n, \overline{K}}$ satisfies the assumptions of 7.1. It follows from 5.11 that the subcategory $HR(X)'$ is cofinal in $HR(X)$.

Let $\mathscr{L} \to I^\bullet$ be an injective resolution in $\mathscr{X}_{\overline{K}}^o$. For $U. \in HR(X)'$ let

$$(8.4.1) \qquad \epsilon_{U.} : \mathscr{X}_{\overline{K}}^o \to U_{.,\overline{K}, \mathrm{Fet}}^o$$

be the projection.

LEMMA 8.5. *For every n the map*

$$(8.5.1) \qquad L|_{U_{n,\overline{K}, \mathrm{Fet}}^o} = \epsilon_{U_n*}\mathscr{L} \to \epsilon_{U_n*}I^\bullet$$

is a quasi–isomorphism. Consequently, $L|_{U_{.,\overline{K}, \mathrm{Fet}}^o} \to \epsilon_{U.*}I^\bullet$ is an injective resolution in the category of Λ–modules in the simplicial topos $U_{.,\overline{K}, \mathrm{Fet}}^o$.

PROOF. By 6.5 the natural map $\mathscr{L} \to Ru_{X*}L$ is an isomorphism. It follows that if $\pi : U_{n,\overline{K}, \mathrm{et}}^o \to U_{n,\overline{K}, \mathrm{Fet}}^o$ is the projection then the right side of 8.5.1 computes $R\pi_*L$. The result then follows from the fact that $U_{n,\overline{K}}^o$ is a $K(\pi, 1)$. □

8.6. We can also apply the $\overline{\Psi}(-)$–construction in 7.21 to the simplicial sheaves $L|_{U^o_{\cdot,\overline{K},\mathrm{Fet}}}$ and $\epsilon_{U\cdot*}I^\bullet$. This gives complexes of sheaves $\overline{\Psi}_{U\cdot}(L)$ and $\overline{\Psi}(\epsilon_{U*}I^\bullet)$ on $U_{\overline{K},\mathrm{et}}$ and a commutative diagram of quasi-isomorphisms

(8.6.1)
$$
\begin{array}{ccc}
j_!L|_{U\cdot} & \longrightarrow & j_!(\epsilon_{U*}I^\bullet) \\
\downarrow & & \downarrow \\
\overline{\Psi}_{U\cdot}(L) & \longrightarrow & \overline{\Psi}(\epsilon_{U*}I^\bullet).
\end{array}
$$

In particular we have

(8.6.2)
$$
R\Gamma(U_{\cdot,\overline{K},\mathrm{et}}, j_!L|_{U_{\cdot,\overline{K}}}) \simeq R\Gamma(U_{\cdot,\overline{K},\mathrm{et}}, \overline{\Psi}(\epsilon_{U*}I^\bullet)).
$$

There is a canonical map

(8.6.3)
$$
\Gamma(U_{\cdot,\overline{K},\mathrm{et}}, \overline{\Psi}(\epsilon_{U*}I^\bullet)) \to R\Gamma(U_{\cdot,\overline{K},\mathrm{et}}, \overline{\Psi}(\epsilon_{U*}I^\bullet))
$$

LEMMA 8.7. *The map*
(8.7.1)
$$
\mathrm{hocolim}_{U\cdot \in HR(X)'}\Gamma(U_{\cdot,\overline{K},\mathrm{et}}, \overline{\Psi}(\epsilon_{U*}I^\bullet)) \to \mathrm{hocolim}_{U\cdot \in HR(X)'}R\Gamma(U_{\cdot,\overline{K},\mathrm{et}}, \overline{\Psi}(\epsilon_{U*}I^\bullet))
$$

induced by 8.6.3 is an isomorphism.

PROOF. Write just $\overline{\Psi}(I^\bullet)$ for $\overline{\Psi}(\epsilon_{U*}I^\bullet)$. The map on cohomology obtained from 8.7.1 extends to a map of spectral sequences from

(8.7.2)
$$
E_1^{pq} = \varinjlim_{U\cdot} H^q(\Gamma(U_{p,\overline{K}}, \overline{\Psi}(I^\bullet))) \implies \varinjlim_{U\cdot} H^{p+q}(\Gamma(U_{\cdot,\overline{K}}, \overline{\Psi}(I^\bullet)))
$$

to

(8.7.3)
$$
E_1^{pq} = \varinjlim_{U\cdot} H^q(U_{p,\overline{K}}, \overline{\Psi}(I^\bullet)) \implies \varinjlim_{U\cdot} H^{p+q}(U_{\cdot,\overline{K}}, \overline{\Psi}(I^\bullet)).
$$

It follows that it suffices to show that for every (p,q) the map

(8.7.4)
$$
\varinjlim_{U\cdot} H^q(\Gamma(U_{p,\overline{K}}, \overline{\Psi}(I^\bullet))) \to \varinjlim_{U\cdot} H^q(U_{p,\overline{K}}, \overline{\Psi}(I^\bullet))
$$

is an isomorphism. For this in turn it suffices to show that for every (p,q) with $q > 0$ and $j, r \in \mathbb{Z}$ we have

(8.7.5)
$$
\varinjlim_{U\cdot} H^q(U_{p,\overline{K}}, \overline{\Psi}(I^j)^r) = 0.
$$

To verify this it suffices by a standard limit argument to consider the case of the strict henselization of X at a point. In this case condition (R) in 7.2 holds so the result follows from 7.24.   □

8.8. By 7.26.10 we also have

(8.8.1)
$$
\Gamma(U_{\cdot,\overline{K},\mathrm{et}}, \overline{\Psi}(\epsilon_{U*}I^\bullet)) = \Gamma(U^o_{\cdot,\overline{K},\mathrm{Fet}}, \Psi(\epsilon_{U*}I^\bullet))
$$

We then have

$$
\begin{aligned}
R\Gamma(\mathscr{X}^o_{\overline{K}}, \Psi^a(\mathscr{L})) &= R\Gamma(\mathscr{X}^o_{\overline{K}}, \Psi^a(I^\bullet)) \text{ (by exactness of } \Psi^a) \\
&= \operatorname{hocolim}_{U_\cdot \in HR(X)'} R\Gamma(U^o_{\cdot, \overline{K}, \mathrm{Fet}}, \Psi(\epsilon_{U_\cdot *} I^\bullet)) \text{ (by 6.11)} \\
&= \operatorname{hocolim}_{U_\cdot \in HR(X)'} \Gamma(U^o_{\cdot, \overline{K}, \mathrm{Fet}}, \Psi(\epsilon_{U_\cdot *} I^\bullet)) \text{ (by 7.26.7)} \\
&= \operatorname{hocolim}_{U_\cdot \in HR(X)'} \Gamma(U_{\cdot, \overline{K}, \mathrm{et}}, \overline{\Psi}(\epsilon_{U_\cdot *} I^\bullet)) \text{ (by 8.8.1)} \\
&= \operatorname{hocolim}_{U_\cdot \in HR(X)'} R\Gamma(U_{\cdot, \overline{K}, \mathrm{et}}, j_! L|_{U_{\overline{K}}}) \text{ (by 8.7 and 8.6.2)} \\
&= R\Gamma(X_{\overline{K}}, j_! L).
\end{aligned}
$$

Summarizing:

THEOREM 8.9. *There is a canonical isomorphism*

$$
H^*_c(X^o_{\overline{K}}, L) \simeq H^*(\mathscr{X}^o_{\overline{K}}, \Psi^a(\mathscr{L}))
$$

functorial in L.

REMARK 8.10. In the above we have chosen the injective resolution $\mathscr{L} \to I^\bullet$. A standard verification shows that the isomorphism in 8.9 is independent of this choice.

8.11. In fact this isomorphism is even compatible with cup products. Let $\mathscr{L}$ be the sheaf on $\mathscr{X}^o_{\overline{K}}$ associated (by u_{X*}) to a locally constant constructible sheaf L of projective Λ–modules on $X^o_{\overline{K}}$ and let L^* be the dual of L with corresponding sheaf $\mathscr{L}^* := u_{X*} L^*$. Then it follows from the construction that there are canonical maps

$$
\tag{8.11.1} \mathscr{L} \otimes \Psi(\mathscr{L}^*) \to \Psi(\mathbb{Z}/p^r)
$$

and

$$
\tag{8.11.2} j_* L \otimes \overline{\Psi}(L^*) \to \overline{\Psi}(\mathbb{Z}/p^r)
$$

on $\mathscr{X}^o_{\overline{K}}$ and $X_{\overline{K}}$ respectively, which induce cup products

$$
\tag{8.11.3} \cup : H^*(\mathscr{X}^o_{\overline{K}}, \mathscr{L}) \times H^*(\mathscr{X}^o_{\overline{K}}, \Psi^a(\mathscr{L}^*)) \to H^*(\mathscr{X}^o_{\overline{K}}, \Psi^a(\mathbb{Z}/p^r)),
$$

and

$$
\tag{8.11.4} \cup : H^*(X^o_{\overline{K}}, L) \times H^*_c(X^o_{\overline{K}}, L^*) \to H^*_c(X^o_{\overline{K}}, \mathbb{Z}/p^r)
$$

and the isomorphisms obtained from 6.5 and 8.9 are compatible with these pairings.

8.12. There is also a finer variant of the above construction. Let $D = E \cup F$ be a decomposition of D into two divisors, and let $\Psi_E(\mathscr{L}) \subset \Psi(\mathscr{L})$ denote the subcomplex obtained by taking in 8.3 only the sum over divisors lying in E and intersections of such divisors. Then by the same argument used to prove 8.9, one gets a canonical isomorphism in the derived category

$$
\tag{8.12.1} R\Gamma(\mathscr{X}^o_{\overline{K}}, \Psi^a_E(\mathscr{L})) \simeq R\Gamma((X - F)_{\overline{K}}, j_! L).
$$

In what follows we denote the right side of this equation by $R\Gamma_{E,F}(X^o_{\overline{K}}, L)$ and by $H^*_{E,F}(X^o_{\overline{K}}, L)$ the corresponding cohomology groups (though of course these groups depend on the compactification X of X^o).

These cohomology groups with partial compact support also satisfy Poincaré duality:

PROPOSITION 8.13. *With notation as above, there is a canonical isomorphism*

(8.13.1) $$R\Gamma_{E,F}(X_{\overline{K}}^o, L) \to R\mathrm{Hom}(R\Gamma_{F,E}(X_{\overline{K}}^o, L^*), \Lambda(-d)[-2d]).$$

PROOF. Consider the commutative diagram

(8.13.2)
$$\begin{array}{ccc} X - F & \xrightarrow{\ j_F\ } & X \\ {\scriptstyle j_E}\big\uparrow & & \big\uparrow{\scriptstyle j_E} \\ X - D & \xrightarrow{\ j_F\ } & X - E. \end{array}$$

LEMMA 8.14. *For any locally constant constructible sheaf H of Λ–modules on $(X - D)_{\overline{K}}$ there is a canonical isomorphism*

(8.14.1) $$j_{E!}Rj_{F*}(H) \to Rj_{F*}j_{E!}(H).$$

PROOF. Note first there there is a canonical map $j_{E!}j_{F*} \to j_{F*}j_{E!}$, induced by adjunction from the natural isomorphism $j_E^* j_{F*} j_{E!} \simeq j_{F*}$. By the universal property of derived functors we therefore obtain a morphism of functors

(8.14.2) $$j_{E!}Rj_{F*} \to Rj_{F*}j_{E!}$$

This defines the morphism 8.14.1. To verify that it is an isomorphism for H locally constant constructible, we may pass to the strict henselization of $X_{\overline{K}}$ at a geometric point. Let d_E (resp. d_F) denote the number of components of E (resp. F). Then the fundamental group Δ_E of $X - E$ is isomorphic to $\widehat{\mathbb{Z}}(1)^{d_E}$ and the fundamental group Δ_F of $X-F$ is isomorphic to $\widehat{\mathbb{Z}}(1)^{d_F}$, and the fundamental group Δ of $X-D$ is isomorphic to $\Delta_E \times \Delta_F$. By filtering H first by subsheaves of the form nH ($n \in \mathbb{Z}$), and then further by irreducible subrepresentations the proof is then further reduced to the case when H is of the form $H_E \otimes H_F$ for some representations H_E (resp. H_F) of Δ_E (resp. Δ_F). This reduces the proof the case when $X = \mathbb{A}^{d_E} \times \mathbb{A}^{d_F}$ for some integers d_E and d_F, $X - E = \mathbb{G}_m^{d_E} \times \mathbb{A}^{d_F}$, $X - F = \mathbb{A}^{d_E} \times \mathbb{G}_m^{d_F}$, and H is obtained from a sheaf H_E on $\mathbb{G}_m^{d_E}$ tensored with a sheaf H_F pulled back from $\mathbb{G}_m^{d_F}$. In this case the result follows from the Künneth formula [**24**, III, 1.6.4 and 1.7.1]. $\qquad\square$

For a locally constant constructible sheaf H of projective Λ–modules on $(X - D)_{\overline{K}}$ we have

(8.14.3) $$Rj_{F*}j_{E!}(H) \otimes^{\mathbb{L}} Rj_{E*}j_{F!}H^* \simeq j_!(H \otimes H^*).$$

The trace map therefore induces a morphism

(8.14.4) $$R\Gamma(X_{\overline{K}}, Rj_{F*}j_{E!}(H)) \otimes^{\mathbb{L}} R\Gamma(X_{\overline{K}}, Rj_{E*}j_{F!}(H^*)) \to \Lambda(-d)[-2d]$$

which defines the map 8.13.1. To see that it is an isomorphism, one can either proceed by a local group cohomology computation, or as follows using local duality. Let $f : X_{\overline{K}} \to \mathrm{Spec}(\overline{K})$ momentarily denote the structure morphism, and let $\omega_{X_{\overline{K}}} := f^!\Lambda \simeq \Lambda(-d)[-2d]$ denote the dualizing complex [**24**, I]. Write $D_X(-)$ for the functor $R\mathscr{H}om(-, \omega_{X_{\overline{K}}})$ and $D_{\overline{K}}(-)$ for $R\mathrm{Hom}(-, \Lambda)$.

By duality [**9**, Th. Finitude, 4.3] and 8.14 above we then have

$$\begin{aligned} Rj_{F*}j_{E!}(H) & \simeq & j_{E!}Rj_{F*}(H) \\ & \simeq & D_X Rj_{E*} D_{X-E} Rj_{F*} D_{X-D}(D_{X-D}(H)) \\ & \simeq & D_X Rj_{E*} j_{F!}(D_{X-D}(H)). \end{aligned}$$

Since f is proper so that $Rf_! = D_{\overline{K}} Rf_* D_X$ is equal to Rf_*, we get
(8.14.5)
$$D_{\overline{K}} Rf_* Rj_{F*} j_{E!}(H) \simeq Rf_* Rj_{E*} j_{F!}(D_{X-D}(H)) \simeq Rf_* Rj_{E*} j_{F!}(H^*)(-d)[-2d].$$
$\square$

8.15. Next we define another complex $\Phi(\mathcal{O}_{\mathscr{X}_{\overline{K}}^o})$ on $\mathscr{X}_{\overline{K}}^o$.

For $A \subset \{1, \ldots, n\}$ the V-scheme D_A is smooth, and the closed subscheme

(8.15.1)
$$\bigcup_{i \notin A} (D_i \cap D_A) \subset D_A$$

is a divisor with normal crossings in D_A. We can therefore apply the discussion in section 6 to D_A with respect to the divisor 8.15.1 to get a topos $\mathscr{D}_{A,\overline{K}}^o$. We write $\mathscr{S}_A$ for the site defining $\mathscr{D}_{A,\overline{K}}^o$.

For an object (U, N) of $\mathscr{S}$ (the site defining $\mathscr{X}_{\overline{K}}^o$), let $\overline{N} \to U_{\overline{K}}$ be the normalization of $U_{\overline{K}}$ in N, and let $\overline{N}_A^o$ denote the maximal reduced subscheme of the pullback of $\overline{N}$ to $D_A^o \times_X \overline{N}$. Then $\overline{N}_A^o \to D_{A,\overline{K}}^o$ is finite and étale by 7.5, and therefore $(U_A, \overline{N}_A^o)$ is an object of the site $\mathscr{S}_A$.

Define $\Phi(\mathcal{O}_{\mathscr{X}_{\overline{K}}^o})^r$ to be the presheaf

(8.15.2)
$$(U, N) \mapsto \oplus_{A \subset \{1,\ldots,n\}, |A|=r} \Gamma((U_A, \overline{N}_A^o), \mathcal{O}_{\mathscr{D}_{A,\overline{K}}^o}),$$

and let $\Phi^a(\mathcal{O}_{\mathscr{X}_{\overline{K}}^o})^r$ be the associated sheaf. We define a map

(8.15.3)
$$d : \Phi^a(\mathcal{O}_{\mathscr{X}_{\overline{K}}^o})^r \to \Phi^a(\mathcal{O}_{\mathscr{X}_{\overline{K}}^o})^{r+1}$$

as follows. Say $A = \{i_1, \ldots, i_{r+1}\}$ and for $1 \leq j \leq r+1$ let $A_j = \{i_1, \ldots, \hat{i}_j, \ldots, i_{r+1}\}$.

Let $(U, N) \in \mathscr{S}$ be an object with $U = \mathrm{Spec}(R)$ affine, and let S denote the coordinate ring of the normalization of U in N. If $I_A \subset R$ (resp. $I_{A_j} \subset R$) denotes the ideal of D_A (resp. D_{A_j}), then $\Gamma((U_A, \overline{N}_A^o), \widetilde{\mathcal{O}}_{\mathscr{D}_A^o})$ is equal to the normalization of $(S/I_A S)_{\mathrm{red}}$, and $\Gamma((U_{A_j}, \overline{N}_{A_j}^o), \widetilde{\mathcal{O}}_{\mathscr{D}_{A_j}^o})$ is equal to the normalization of $(S/I_{A_j} S)_{\mathrm{red}}$ (here $\widetilde{\mathcal{O}}_{\mathscr{D}_A^o}$ and $\widetilde{\mathcal{O}}_{\mathscr{D}_{A_j}^o}$ are as in 6.12). Since $D_A \subset D_{A_j}$ we have $I_{A_j} \subset I_A$, and hence there is a canonical map
(8.15.4)
$$\partial_j : \Gamma((U_{A_j}, \overline{N}_{A_j}^o), \widetilde{\mathcal{O}}_{\mathscr{D}_{A_j,\overline{K}}^o}) = (S/I_{A_j}S)\widetilde{} \to (S/I_A S)\widetilde{} = \Gamma((U_A, \overline{N}_A^o), \widetilde{\mathcal{O}}_{\mathscr{D}_{A,\overline{K}}^o}).$$

Taking the alternating sums $\sum_{j=1}^{r+1}$ for various choices of A and sheafifying we obtain the map 8.15.3. It follows immediately from the construction that $d^2 = 0$ so we get a complex denoted $\Phi^a(\mathcal{O}_{\mathscr{X}_{\overline{K}}^o})$.

Note that $\Phi^a(\mathcal{O}_{\mathscr{X}_{\overline{K}}^o})^0 = \mathcal{O}_{\mathscr{X}_{\overline{K}}^o}$. In particular, there is a canonical map $\mathscr{J}_X \to \Phi^a(\mathcal{O}_{\mathscr{X}_{\overline{K}}^o})^0$ and it follows immediately from the above computations that $d(\mathscr{J}_X) = 0$. We therefore get a morphism of complexes

(8.15.5)
$$\mathscr{J}_X \to \Phi^a(\mathcal{O}_{\mathscr{X}_{\overline{K}}^o}).$$

Similarly, for any integer $r \geq 1$ we get a complex $\Phi^a(\mathcal{O}_{\mathscr{X}_{\overline{K}}^o}/p^r)$ by replacing $\mathcal{O}_{\mathscr{D}_{A,\overline{K}}^o}$ in the above with $\mathcal{O}_{\mathscr{D}_{A,\overline{K}}^o}/p^r$, and the map 8.15.5 induces a morphism

(8.15.6)
$$\mathscr{J}_X/p^r \to \Phi^a(\mathcal{O}_{\mathscr{X}_{\overline{K}}^o}/p^r).$$

PROPOSITION 8.16. *For every integer $r \geq 1$, the induced morphism*

$$R\nu_{X*}\mathscr{J}_X/p^r \to R\nu_{X*}\Phi^a(\mathscr{O}_{\mathscr{X}_{\overline{K}}^o}/p^r)$$

is an isomorphism in $\widetilde{D}(\mathscr{O}_{X_{\overline{V}}})$.

PROOF. By a standard reduction it suffices to consider the case when $X = \mathrm{Spec}(R)$ is the spectrum of a strictly henselian local ring, and to prove the corresponding result for global cohomology groups. In this case by our assumptions each D_A also has the same form and we can also apply the purity theorem 2.17 to D_A. Let $S = V[T_1, \ldots, T_d]$ and let S_∞ be as in 2.16. For any integer r and $1 \leq i_1 < i_2 < \cdots < i_r \leq d$ let $S_{i_1 \ldots i_r, \infty}$ denote $(S_\infty/(T_{i_1}, \ldots, T_{i_r}))_{\mathrm{red}} = (S/(T_{i_1}, \ldots, T_{i_r}))_\infty$. Then by 3.8 and 6.9 the complex $R\nu_{X*}\Phi^a(\mathscr{O}_{\mathscr{X}_{\overline{K}}^o}/p^r)$ is almost isomorphic to

$$(8.16.1) \qquad R\Gamma(\Delta_\infty, R/p^r \otimes_S (S_\infty \to \oplus_i S_{i,\infty} \to \oplus_{i_1 < i_2} S_{i_1 i_2, \infty} \to \cdots)),$$

and as before

$$R\nu_{X*}\mathscr{J}_X/p^r \simeq R\Gamma(\Delta_\infty, R/(p^r) \otimes_S J_\infty) \quad \text{(almost isomorphism)}.$$

It therefore suffices to show that the natural map

$$(8.16.2) \qquad J_\infty \to (S_\infty \to \oplus_i S_{i,\infty} \to \oplus_{i_1 < i_2} S_{i_1 i_2, \infty} \to \cdots)$$

is a quasi–isomorphism (since this is a complex of p-torsion free modules this will also imply that it is a quasi-isomorphism modulo p^r). For this note that the complex on the right side is equal to the tensor product of the complexes obtained from each $V[T_i]$ and J_∞ is equal to the tensor product of the complexes obtained from the ideal defining the origin in each $V[T_i]$. It therefore suffices to consider the case when $d = 1$ in which case the result is immediate. $\square$

8.17. More generally for a subset $E \subset \{1, \ldots, n\}$ there is a complex $\Phi_E^a(\mathscr{O}_{\mathscr{X}_{\overline{K}}^o})$ obtained by summing only over subsets $A \subset E$ in 8.15.2. The same argument used above shows that there is a canonical morphism

$$(8.17.1) \qquad \mathscr{J}_E \to \Phi_E^a(\mathscr{O}_{\mathscr{X}_{\overline{K}}^o}),$$

which induces an almost isomorphism after dividing by p^r and applying $R\nu_{X*}$.

8.18. Fix $A \subset \{1, \ldots, n\}$. The presheaf

$$(8.18.1) \qquad (U, N) \mapsto \Gamma(\overline{N}_A^o, \widetilde{\mathscr{O}}_{\mathscr{D}_A^o})$$

can be described Galois–theoretically as follows. Let $U \to X$ be an étale morphism with $U = \mathrm{Spec}(R)$ affine, and assume that $U_{\overline{K}}^o$ satisfies the assumptions in 7.1. Define groups Δ, H_A, I_A etc. as in 7.7. Let R_A denote the coordinate ring of D_A. Let $\overline{R}_A$ denote the integral closure of R_A in the maximal subextension of the chosen geometric generic point of $D_{A,\overline{K}}^o$ which is unramified over $D_{A,\overline{K}}^o$. Then $\overline{R}_A$ is a continuous representation of Δ_A and $\widetilde{\mathscr{O}}_{\mathscr{D}_A^o}$ is the corresponding presheaf on $\mathscr{S}_A$ (defined as in 8.15). The restriction of the presheaf 8.18.1 to $U_{\overline{K}, \mathrm{Fet}}^o$ is then the sheaf corresponding to the induced representation $\mathrm{Ind}_{H_A}^\Delta(\overline{R}_A)$. Note in particular that by the projection formula for any continuous Δ–representation M there is a canonical isomorphism

$$(8.18.2) \qquad M \otimes \mathrm{Ind}_{H_A}^\Delta(\overline{R}_A) \to \mathrm{Ind}_{H_A}^\Delta(M|_{\Delta_A} \otimes \overline{R}_A).$$

This implies in particular that for any sheaf $\mathscr{L}$ in $\mathscr{X}_{\overline{K}}^o$ there is a canonical map of complexes in $\mathscr{X}_{\overline{K}}^o$

$$(8.18.3) \qquad\qquad \Psi^a(\mathscr{L}) \to \mathscr{L} \otimes \Phi^a(\mathscr{O}_{\mathscr{X}_{\overline{K}}^o}),$$

and similarly for the complexes defining cohomology with partial compact support.

8.19. Fix now a decomposition $D = E \cup F$. Let L be a locally constant constructible sheaf of flat $\mathbb{Z}/p^r$–modules on $X_{\overline{K}}^o$ and set $\mathscr{L} := u_{X*}L$. We then get a morphism

$$(8.19.1) \qquad\qquad \mathscr{L} \otimes \mathscr{J}_E \to \mathscr{L} \otimes \Phi_E^a(\mathscr{O}_{\mathscr{X}_{\overline{K}}^o}),$$

which induces an almost quasi-isomorphism after applying $R\nu_{X*}$. Combining this with 8.18.3 we obtain a morphism in $\overline{V}^a - \mathrm{Mod}$

$$(8.19.2) \qquad H_{E,F}^*(X_{\overline{K}}^o, L) \simeq H^*(\mathscr{X}_{\overline{K}}^o, \Psi_E^a(\mathscr{L})) \to H^*(\mathscr{X}_{\overline{K}}^o, \mathscr{L} \otimes \mathscr{J}_E).$$

It follows from the construction that the resulting diagram

$$(8.19.3) \qquad
\begin{array}{ccc}
H_{E,F}^*(X_{\overline{K}}^o, L) \times H_{F,E}^*(X_{\overline{K}}^o, L^*) & \xrightarrow{\cup} & H_c^*(X_{\overline{K}}^o, \mathbb{Z}/p^r) \\
\downarrow & & \downarrow \\
H^*(\mathscr{X}_{\overline{K}}^o, \mathscr{L} \otimes \mathscr{J}_E) \times H^*(\mathscr{X}_{\overline{K}}^o, \mathscr{L}^* \otimes \mathscr{J}_F) & \xrightarrow{\cup} & H^*(\mathscr{X}_{\overline{K}}^o, \mathscr{J})
\end{array}$$

commutes.

This defines the map 6.17.1. To prove 6.17 we prove the stronger statement that 8.19.2 is an isomorphism and in addition we will show that under this isomorphism the usual trace map on étale cohomology agrees with the one defined in 6.14.

8.20. Let (Y, D_Y) be another smooth proper V–scheme with D_Y a divisor with simple normal crossings. Let $D_Y = E_Y \cup F_Y$ be a decomposition of the divisor into two divisors with no common irreducible component, and let $f : Y \to X$ be a morphism such that $f^{-1}(E) \subset E_Y$ and $f^{-1}(F) \subset F_Y$. It follows from the construction that then there is an induced morphism

$$(8.20.1) \qquad f^* : H^*(\mathscr{X}_{\overline{K}}^o, \mathscr{L} \otimes \mathscr{J}_E) \to H^*(\mathscr{Y}_{\overline{K}}^o, f^*\mathscr{L} \otimes \mathscr{J}_{E_Y}).$$

and this morphism is compatible with cup products and the trace maps. In particular, if d_X and d_Y denote the relative dimensions over V of X and Y respectively, then for any class $\alpha \in H^i(\mathscr{Y}_{\overline{K}}^o, f^*\mathscr{L} \otimes \mathscr{J}_{E_Y})$ we obtain a map

$$(8.20.2) \qquad
\begin{array}{ccc}
H^{2d_Y - i}(\mathscr{X}_{\overline{K}}^o, \mathscr{L}^* \otimes \mathscr{J}_F) & \xrightarrow{f^*} & H^{2d_Y - i}(\mathscr{Y}_{\overline{K}}^o, f^*\mathscr{L}^* \otimes \mathscr{J}_{F_Y}) \\
& \xrightarrow{\cup \alpha} & H^{2d_Y}(\mathscr{Y}_{\overline{K}}^o, \mathscr{J}_Y/p^r \mathscr{J}_Y) \\
& \xrightarrow{\mathrm{tr}_Y} & \overline{V}/p^r\overline{V}(-d_Y).
\end{array}$$

By Poincaré duality there exists a unique class

$$f_*(\alpha) \in H^{2d_X - 2d_Y + i}(\mathscr{X}_{\overline{K}}^o, \mathscr{L} \otimes \mathscr{J}_E)(d_X - d_Y)$$

such that this map is equal to

$$
\begin{array}{ccc}
H^{2d_Y - i}(\mathscr{X}_{\overline{K}}^o, \mathscr{L}^* \otimes \mathscr{J}_F) & \xrightarrow{\cup f_*(\alpha)} & H^{2d_X}(\mathscr{X}_{\overline{K}}^o, \mathscr{J}_X/p^r \mathscr{J}_X)(d_X - d_Y) \\
& \xrightarrow{\mathrm{tr}} & \overline{V}/p^r\overline{V}(-d_Y).
\end{array}$$

This defines a homomorphism

$$(8.20.3) \qquad f_* : H^i(\mathscr{Y}_{\overline{K}}^o, f^*\mathscr{L} \otimes \mathscr{J}_{E_Y}) \to H^{2d_X - 2d_Y + i}(\mathscr{X}_{\overline{K}}^o, \mathscr{L} \otimes \mathscr{J}_E)(d_X - d_Y)$$

in $\overline{V}^a - \mathrm{Mod}$.

Similarly Poincaré duality for étale cohomology with partial compact supports 8.13 defines a morphism

$$(8.20.4) \qquad f_*^{\mathrm{et}} : H^i_{E_Y, F_Y}(Y^o_{\overline{K}}, f^*L) \to H^{2d_X - 2d_Y + i}_{E, F}(X^o_{\overline{K}}, L)(d_X - d_Y).$$

THEOREM 8.21. *Assume f is a closed immersion, and that Y meets D transversally, and $E_Y = E \cap Y$ and $F_Y = F \cap Y$. Then the diagram*

$$(8.21.1) \qquad \begin{array}{ccc} H^i(\mathscr{Y}^o_{\overline{K}}, f^*\mathscr{L} \otimes \mathscr{J}_{E_Y}) & \xrightarrow{\ f_*\ } & H^{2d_X - 2d_Y + i}(\mathscr{X}^o_{\overline{K}}, \mathscr{L} \otimes \mathscr{J}_E)(d_X - d_Y) \\ \sigma_Y \uparrow & & \uparrow \sigma_X \\ H^i_{E_Y, F_Y}(Y^o_{\overline{K}}, f^*L) \otimes \overline{V} & \xrightarrow{\ f_*^{\mathrm{et}}\ } & H^{2d_X - 2d_Y + i}_{E, F}(X^o_{\overline{K}}, L) \otimes \overline{V}(d_X - d_Y) \end{array}$$

commutes in $\overline{V}^a - \mathrm{Mod}$, where the vertical arrows are the morphisms 8.19.2.

PROOF. Let us first consider the case when $Y \subset X$ is a smooth divisor meeting D transversally (the general case will then be deduced from this special case). Let Y^o denote $Y - (Y \cap D)$ and let $D_Y = Y \cap D = E_Y \cup F_Y$ be the decomposition of the divisor D_Y obtained from $D = E \cup F$. Write d for the relative dimension of X over V, and let $\widetilde{X}^o$ denote $X^o - Y^o$. Let $\widetilde{\mathscr{X}^o_{\overline{K}}}$ denote the topos associated to the pair $(X, Y \cup D)$. Let $\widetilde{E} = E$ and $\widetilde{F} = F \cup Y$ so that we obtain a decomposition $Y \cup D = \widetilde{E} \cup \widetilde{F}$. The topos $\widetilde{\mathscr{X}^o_{\overline{K}}}$ is the category of sheaves on the site $\widetilde{\mathscr{S}}$ whose objects are pairs (U, N) where $U \to X$ is étale and $N \to \widetilde{X}^o_{\overline{K}} \times_X U$ is finite étale and surjective. There is a natural functor

$$(8.21.2) \qquad \mathscr{S} \to \widetilde{\mathscr{S}}, \quad (U, N) \mapsto (U, N \times_{X^o} \widetilde{X}^o)$$

which induces a morphism of topoi $j : \widetilde{\mathscr{X}^o_{\overline{K}}} \to \mathscr{X}^o_{\overline{K}}$. There is also a morphism of topoi $i : \mathscr{Y}^o_{\overline{K}} \to \mathscr{X}^o_{\overline{K}}$. If $\mathscr{S}_Y$ denotes the site in the definition of $\mathscr{Y}^o_{\overline{K}}$ then the morphism i is obtained from the functor

$$(8.21.3) \qquad \mathscr{S} \to \mathscr{S}_Y, \quad (U, N) \mapsto (U \times_X Y, N \times_X Y).$$

PROPOSITION 8.22. *Let $\mathscr{L}$ be as in 8.21. There is a canonical distinguished triangle*

$$(8.22.1) \quad \mathscr{L} \otimes \mathscr{J}_E \to Rj_*(j^*\mathscr{L} \otimes \mathscr{J}_{\widetilde{E}}) \to i_*(i^*\mathscr{L} \otimes \mathscr{J}_{E_Y})(-1)[-1] \to \mathscr{L} \otimes \mathscr{J}_E[1]$$

in $\widetilde{D}(\mathscr{X}^o_{\overline{K}})$.

PROOF. By the projection formula it suffices to consider the case when $\mathscr{L} = \mathbb{Z}/p^r$ for some r. Let $(U, W) \in \mathscr{S}$ be an object. Then the localized topos $\widetilde{\mathscr{X}^o_{\overline{K}}}|_{j^{-1}(U, W)}$ is isomorphic to the topos of sheaves on the site whose objects are pairs (U', W'), where $U' \to U$ is étale and $W' \to W \times_{X^o} \widetilde{X}^o$ is a finite étale morphism. Let $\widetilde{W}$ denote $W \times_{X^o} \widetilde{X}^o$. Any finite étale surjective morphism $W' \to \widetilde{W}$ defines such a pair (namely (U, W')) and hence we obtain a morphism of topoi

$$(8.22.2) \qquad \epsilon : \widetilde{\mathscr{X}^o_{\overline{K}}}|_{j^{-1}(U, W)} \to \widetilde{W}_{\mathrm{Fet}}.$$

If U admits an étale morphism

$$(8.22.3) \qquad \pi : U \to \mathrm{Spec}(V[X_1, \ldots, X_{d+l}]) \times \mathrm{Spec}(V[y])$$

for some integers $d, l \geq 0$, such that Y is the inverse image of $\{y = 0\}$ and D is the inverse image of $\{X_1 \cdots X_d = 0\}$ (and say E is the inverse image of $\{X_1 \cdots X_s = 0\}$) and $W = \mathrm{Spec}(R)$ for some R, then using the almost purity theorem we can compute

$$(8.22.4) \qquad H^*(\widetilde{\mathscr{X}_{\overline{K}}^o}|_{j^{-1}(U,W)}, \mathscr{J}_E/p^r \, \mathscr{J}_E)$$

as follows. Namely, let $\widetilde{S}_\infty$ denote the ring

$$(8.22.5) \qquad \widetilde{S}_\infty := V[X_1^{1/p^\infty}, \dots, X_d^{1/p^\infty}, y^{1/p^\infty}],$$

and let

$$(8.22.6) \qquad S_\infty := V[X_1^{1/p^\infty}, \dots, X_d^{1/p^\infty}, y].$$

Then if $\overline{J}_{\widetilde{E}} \subset \widetilde{S}_\infty$ denotes the ideal $(X_1 \cdots X_s)$ we have as in the proof of 6.13 almost isomorphisms

$$\begin{aligned}
H^*(\widetilde{\mathscr{X}_{\overline{K}}^o}|_{j^{-1}(U,W)}, \mathscr{J}_{\widetilde{E}}/p^r \, \mathscr{J}_{\widetilde{E}}) &\simeq H^*(\widetilde{W}_{\mathrm{Fet}}, \epsilon_* \mathscr{J}_{\widetilde{E}}/p^r \, \mathscr{J}_{\widetilde{E}}) \\
&\simeq R \otimes_S H^*(\widetilde{\Delta}_\infty, \overline{J}_{\widetilde{E}}/p^r \overline{J}_{\widetilde{E}}),
\end{aligned}$$

where $\widetilde{\Delta}_\infty$ denotes the Galois group of $\widetilde{S}_\infty$ over S. Let Δ_∞ denote the Galois group of S_∞ over S so that there is a surjection $\widetilde{\Delta}_\infty \to \Delta_\infty$ with kernel Q isomorphic to $\mathbb{Z}_p(1)$. The kernel Q corresponds to the étale covers of $\widetilde{W}$ obtained by taking roots of y. We then have the Hochschild-Serre spectral sequence

$$(8.22.7) \qquad E_2^{ij} = H^i(\Delta_\infty, H^j(Q, \overline{J}_{\widetilde{E}}/p^r \overline{J}_{\widetilde{E}})) \implies H^{i+j}(\widetilde{\Delta}_\infty, \overline{J}_{\widetilde{E}}/p^r \overline{J}_{\widetilde{E}}).$$

Now for any cohomology class $c \in H^i(\Delta_\infty, H^j(Q, \overline{J}_{\widetilde{E}}/p^r \overline{J}_{\widetilde{E}}))$ with $i > 0$, there exists a finite index subgroup $\Delta'_\infty \subset \Delta_\infty$, corresponding to a covering $W' \to W$ obtained by extracting some roots of the X_i, such that c maps to zero in $H^i(\Delta'_\infty, H^j(Q, \overline{J}_{\widetilde{E}}/p^r \overline{J}_{\widetilde{E}}))$. It follows that the sheaf on W_{Fet} associated to the presheaf sending a finite étale morphism $W' \to W$ to

$$H^i(\widetilde{\mathscr{X}_{\overline{K}}^o}|_{j^{-1}(U,W')}, \mathscr{J}_{\widetilde{E}}/p^r \, \mathscr{J}_{\widetilde{E}})$$

is almost isomorphic to the sheaf corresponding to the Δ_∞-representation

$$H^i(Q, \overline{J}_{\widetilde{E}}/(p^r)) \otimes_S R.$$

We conclude that the sheaf on $\mathscr{S}$ associated to the presheaf

$$(8.22.8) \qquad (U, W) \mapsto H^i(\widetilde{\mathscr{X}_{\overline{K}}^o}|_{j^{-1}(U,W')}, \mathscr{J}_{\widetilde{E}}/p^r \, \mathscr{J}_{\widetilde{E}})$$

is almost isomorphic to the sheaf associated to the presheaf whose restriction to W_{Fet} for any object (U, W) as above is the sheaf corresponding to $H^i(Q, \overline{J}_{\widetilde{E}}/(p^r)) \otimes_S R$.

Note that as a Q–module, $\overline{J}_{\widetilde{E}}$ is isomorphic to $\overline{J}_E \otimes \overline{V}[y^{1/p^\infty}]$ Thus we have

$$(8.22.9) \qquad H^i(Q, \overline{J}_{\widetilde{E}}/p^r \overline{J}_{\widetilde{E}}) \otimes_S R \simeq H^i(Q, \overline{V}[y^{1/p^\infty}]/(p^r)) \otimes_{K[y]} J_E \otimes_S R.$$

This implies that $H^i(Q, \overline{V}[y^{1/p^\infty}]/(p^r))$ is zero for $i \geq 2$ since $\mathbb{Z}_p(1)$ has cohomological dimension 1, that $H^1(Q, \overline{V}[y^{1/p^\infty}]/(p^r)) \otimes_{V[y]} J_E \otimes_S R \simeq J_E \otimes_S R/(p^r) \cdot d\log(y)$, and that $H^0(Q, \overline{V}[y^{1/p^\infty}]/(p^r)) \otimes_{V[y]} J_E \simeq J_E \otimes_S R/(p^r)$. Here $d\log(y)$ should be interpreted as the class of the Q–torsor of all p^n–th roots of y in $\overline{V}[y^{1/p^\infty}]$ as in 3.5. With this interpretation all these almost isomorphisms are canonical.

Now we have to sheafify the presheaf 8.22.8 by also allowing étale localization in U and W. For this we may as well assume that $1 - y$ is invertible in R, and pass

to the étale extension of W obtained by taking p^n–th roots of $(1 - y)$. If $\zeta \in R$ is such a p^n–th root, then an elementary calculation shows that

$$(8.22.10) \qquad y d \log(y) = -p^n (1 - y) d \log \zeta.$$

This implies that $R^1 j_* j^* \mathscr{J}_{\widetilde{E}}/p^r \mathscr{J}_{\widetilde{E}}$ is annihilated by y and there is a canonical almost epimorphism

$$(8.22.11) \qquad i_*(\mathscr{J}_{E_Y}/p^r \mathscr{J}_{E_Y}) \to R^1 j_* j^* \mathscr{J}_{\widetilde{E}}/p^r \mathscr{J}_{\widetilde{E}}.$$

That this morphism is in fact an almost isomorphism follows by reduction to the case of $\mathbb{A}^1$ via the same argument used above in which case it is immediate. $\qquad \square$

Similarly on $(X - F)_{\overline{K}}$ there is a distinguished triangle of étale sheaves

$$(8.22.12) \qquad j_{E!} L \to R j_* j^* j_{E!} L \to i_* i^* j_{E!} L(-1)[-1] \to j_{E!} L[1],$$

where $j_E : X - D \hookrightarrow X - F$ and $j : X - Y \cup F \hookrightarrow X - F$ are the inclusions. Indeed using 8.14 it suffices to consider the higher direct images of $j^* L$ under $j : X - Y \cup D \to X - D$. Furthermore, by the projection formula it suffices to consider the case of $L = \mathbb{Z}/(p^r)$. By working locally on X one further reduces to computing $R^1 j_* \mathbb{Z}/(p^r)$ for the inclusion of $\mathbb{G}_m \hookrightarrow \mathbb{A}^1$ which follows from a group cohomology computation as in 3.5.

PROPOSITION 8.23. *The morphisms denoted momentarily by i'_* and i'^{et}_**

$$(8.23.1) \qquad i'_* : H^s(\mathscr{Y}^o_{\overline{K}}, i^* \mathscr{L} \otimes \mathscr{J}_{\widetilde{E}}) \to H^{s+2}(\mathscr{X}^o_{\overline{K}}, \mathscr{L} \otimes \mathscr{J}_E)(1)$$

and

$$(8.23.2) \qquad i'^{\text{et}}_* : H^s_{\widetilde{E}, \widetilde{F}}(Y^o_{\overline{K}}, i^* L) \to H^{s+2}_{E, F}(X^o_{\overline{K}}, L)(1)$$

obtained from the boundaries in the long exact sequences obtained from 8.22.1 and 8.22.12 agree with the maps i_* and i^{et}_* respectively.

PROOF. Note first of all that for any class $\alpha \in H^s(\mathscr{Y}^o_{\overline{K}}, i^* \mathscr{L} \otimes \mathscr{J}_{\widetilde{E}})$ and $\beta \in H^t(\mathscr{X}^o_{\overline{K}}, \mathscr{L}^* \otimes \mathscr{J}_F)$ we have

$$(8.23.3) \qquad i_*(\alpha \cup i^* \beta) = i_*(\alpha) \cup \beta, \quad i'_*(\alpha \cup i^* \beta) = i'_*(\alpha) \cup \beta$$

in $H^{s+t+2}(\mathscr{X}^o_{\overline{K}}, \mathscr{J}_X/p^r \mathscr{J}_X)$, and similarly for i^{et}_* and i'^{et}. Using Poincaré duality it follows that it suffices to consider the case when $L = \mathbb{Z}/(p^r)$, $E = D$ (so $F = \emptyset$), and $s = 2d - 2$. In other words it suffices to show that the boundary maps of 8.22.1 and 8.22.12 are compatible with the trace maps.

This is well–known in the case of étale cohomology. Indeed by the construction the induced map

$$(8.23.4) \qquad H^*_c(Y^o_{\overline{K}}, R i^! \mathbb{Z}/p^r) \simeq H^*_c(Y^o_{\overline{K}}, \mathbb{Z}/p^r(-1)[-2]) \to H^*_c(X^o_{\overline{K}}, \mathbb{Z}/p^r)$$

given by the boundary of 8.22.1 is equal to the map induced by the adjunction $R i_! R i^! \mathbb{Z}/p^r \to \mathbb{Z}/p^r$. This combined with construction of the trace map in [**9**, Cycle, 2.3] shows that the two trace maps in the étale theory coincide.

For the compatibility of i'_* with the trace maps it suffices to study the sequence

$$(8.23.5) \qquad \cdots R^d \nu_{X*} \mathscr{J}_X \to R j_* j^* R^d \nu_{X*} \mathscr{J}_X \to R^{d-1} \nu_{X*} i_* \mathscr{J}_Y \to 0.$$

obtained by applying $R \nu_{X*}$ to the sequence 8.22.1. Now a local calculation as in the proof of 4.7 shows that this sequence is almost isomorphic to the sequence

$$(8.23.6) \qquad \cdots \Omega^d_X \to \Omega^d_X(\log Y) \to \Omega^{d-1}_Y \to 0.$$

Here we have used the isomorphism between Ω_Y^{d-1} and

(8.23.7) $$\mathrm{Coker}(\Omega_X^d \to \Omega_X^d(\log Y)) \simeq i^*\Omega_X^d(\log Y)$$

obtained by choosing locally a local coordinate y for Y in X and defining the isomorphism

(8.23.8) $$\Omega_Y^{d-1} \to i^*\Omega_X^d(\log Y), \quad \omega \mapsto i^*(\tilde{\omega} \wedge d\log(y)),$$

where $\tilde{\omega} \in \Omega_X^{d-1}$ is a lifting of ω. If $u \in \mathscr{O}_X^*$ is a unit, then $d\log(uy)$ is equal to $u^{-1}du + d\log(y)$ from which one sees that the map 8.23.8 is independent of the choices. We thus get a global isomorphism. It follows that i'_* is induced from the boundary map

(8.23.9) $$H^{d-1}(Y, \Omega_Y^{d-1}) \to H^d(X, \Omega_X^d)$$

obtained from the short exact sequence

(8.23.10) $$0 \to \Omega_X^d \to \Omega_X^d(\log Y) \to i_*\Omega_Y^{d-1} \to 0.$$

We leave to the reader the task of verifying that this map is equal under Serre duality to the canonical isomorphism

(8.23.11) $$H^0(X, \mathscr{O}_X) \simeq H^0(Y, \mathscr{O}_Y)$$

thereby completing the proof.   □

8.24. We obtain a diagram

(8.24.1)
$$
\begin{array}{ccc}
R\Gamma_{E,F}(X_{\overline{K}}^o, L) & \xrightarrow{\quad a \quad} & R\Gamma(\mathscr{X}_{\overline{K}}^o, \mathscr{L} \otimes \mathscr{J}_E) \\
\downarrow & & \downarrow \\
R\Gamma_{\widetilde{E},\widetilde{F}}(\widetilde{X}_{\overline{K}}^o, j^*L) & \xrightarrow{\quad b \quad} & R\Gamma(\widetilde{\mathscr{X}_{\overline{K}}^o}, j^*\mathscr{L} \otimes \mathscr{J}_{\widetilde{E}}) \\
\downarrow & & \downarrow \\
R\Gamma_{E_Y,F_Y}(Y_{\overline{K}}^o, i^*L)(-1)[-1] & \xrightarrow{\quad c \quad} & R\Gamma(\mathscr{Y}_{\overline{K}}^o, i^*\mathscr{L} \otimes \mathscr{J}_{E_Y})(-1)[-1] \\
\downarrow & & \downarrow \\
R\Gamma_{E,F}(X_{\overline{K}}^o, L)[1] & \xrightarrow{\quad a[1] \quad} & R\Gamma(\mathscr{X}_{\overline{K}}^o, \mathscr{L} \otimes \mathscr{J}_E)[1],
\end{array}
$$

where the columns are the distinguished triangles obtained by applying the global section functor to 8.22.1 and 8.22.12, and the horizontal arrows are the morphisms 8.19. To prove 8.21 in the present special case it suffices to show that 8.24.1 commutes.

8.25. First let us make some general observations about the construction of the complexes $\Psi(-)$, $\overline{\Psi}(-)$, and $\Phi(-)$.

Let $\widetilde{\underline{\Delta}}^+$ denote the category whose objects are finite ordered sets (including the empty set!) and whose morphisms are order preserving monomorphisms. Let $\underline{\Delta}^+ \subset \widetilde{\underline{\Delta}}^+$ denote the full subcategory of nonempty ordered sets. For $n \geq 0$ let $[n]$ denote the ordered set $\{0, 1, \ldots, n\}$ and by convention we define $[-1]$ to be the empty set.

For a topos T let $T^{\widetilde{\underline{\Delta}}^+}$ denote the constant $\widetilde{\underline{\Delta}}^+$-topos associated to T (see for example $[\mathbf{1}, \mathrm{V}^{\mathrm{bis}}]$ for the notion of a D-topos for a category D). So a sheaf $F \in T^{\widetilde{\underline{\Delta}}^+}$

consists of a sheaf F_n for every $[n] \in \widetilde{\underline{\Delta}}^+$ and for every inclusion $[n] \hookrightarrow [m]$ a morphism $F_n \to F_m$. In other words, the restriction of F to $\underline{\Delta}^+$ is a strictly cosimplicial sheaf in T and so the category $T^{\widetilde{\underline{\Delta}}^+}$ is equivalent to the category of triples (F_{-1}, F_+, σ), where F_{-1} is in T, F_+ is a strictly cosimplicial sheaf in T, and $\sigma : F_{-1} \to F_+$ is an augmentation (where F_{-1} is viewed as a constant cosimplicial sheaf). If S is a site defining T, then the topos $T^{\widetilde{\underline{\Delta}}^+}$ is isomorphic to the topos associated to the site $S^{\widetilde{\underline{\Delta}}^+}$ whose objects are pairs $(U, [n])$, where $U \in S$ and $[n] \in \widetilde{\underline{\Delta}}^+$, and whose morphisms $(U', [n']) \to (U, [n])$ are pairs $f : U' \to U$ and $\delta : [n] \hookrightarrow [n']$, where f is a morphism in S and δ is a morphism in $\widetilde{\underline{\Delta}}^+$. Such a morphism is a covering if f is a covering in S and $[n] = [n']$.

8.26. If Λ is a ring in T, then $T^{\widetilde{\underline{\Delta}}^+}$ is also ringed by the sheaf of rings which in degree n is equal to Λ. Let $D^+(T^{\widetilde{\underline{\Delta}}^+}, \Lambda)$ denote the bounded below derived category of complexes of Λ–modules in $T^{\widetilde{\underline{\Delta}}^+}$. There is a *total complex functor*

$$(8.26.1) \qquad \int : D^+(T^{\widetilde{\underline{\Delta}}^+}, \Lambda) \to D^+(T, \Lambda)$$

defined as follows. A Λ–module $K \in T^{\widetilde{\underline{\Delta}}^+}$ defines a complex in T whose degree r-term is $K([r-1])$ and whose transition morphism $K([r-1]) \to K([r])$ is given by the sum

$$(8.26.2) \qquad \Sigma_{i=0}^r (-1)^i \partial_i : K([r-1]) \to K([r]),$$

where $\partial_i : [r-1] \to [r]$ is the unique injective map whose image does not contain i (if $r = 0$ we take the map induced by the unique map $[-1] \to [0]$). This construction is functorial in K and therefore if $K^\bullet$ is a complex of Λ–modules in $T^{\widetilde{\underline{\Delta}}^+}$ we obtain a bicomplex by applying the above construction in each degree. We define $\int K^\bullet$ to be the associated single complex. We also sometimes refer to this complex as the *normalized complex of $K^\bullet$*.

8.27. A complex $K^\bullet$ in $T^{\widetilde{\underline{\Delta}}^+}$ is equivalent to a complex $K_{-1}^\bullet$ in T, a strictly cosimplicial complex $K_+^\bullet$ in T, and an augmentation $K_{-1}^\bullet \to K_+^\bullet$. If $\mathrm{Tot}(K_+^\bullet)$ denotes the usual total complex of the cosimplicial complex $K_+^\bullet$, then the augmentation $K_{-1}^\bullet \to K_+^\bullet$ induces a morphism of complexes $K_{-1}^\bullet \to \mathrm{Tot}(K_+^\bullet)$ and $\int K_\bullet$ is the cone of this morphism of complexes. It follows that $\int$ takes quasi–isomorphisms to quasi-isomorphisms and hence passes to the derived category. Moreover this implies that $\int$ takes distinguished triangles to distinguished triangles. This defines the triangulated functor 8.26.1.

For a morphism $f : T' \to T$ of topoi there is an induced morphism of topoi $T'^{\widetilde{\underline{\Delta}}^+} \to T^{\widetilde{\underline{\Delta}}^+}$ which we usually denote by the same letter f. If T' and T are ringed by Λ' and Λ respectively, and f is a morphism of ringed topoi then the diagram

$$(8.27.1) \qquad \begin{array}{ccc} D^+(T'^{\widetilde{\underline{\Delta}}^+}, \Lambda') & \xrightarrow{Rf_*} & D^+(T^{\widetilde{\underline{\Delta}}^+}, \Lambda) \\ {\scriptstyle \int} \downarrow & & \downarrow {\scriptstyle \int} \\ D^+(T', \Lambda^+) & \xrightarrow{Rf_*} & D^+(T, \Lambda) \end{array}$$

commutes.

We use this to prove that 8.24.1 commutes as follows. To ease the notation, we give the proof in the case when $E = D$ so that $F = \emptyset$. The general case is obtained by exactly the same argument replacing X in the following by $X - F$. Let $s : X^o \hookrightarrow X$ be the inclusion, and write also s for the inclusions $Y^o \hookrightarrow Y$, $\widetilde{X}^o \hookrightarrow \widetilde{X}$, $X_{\overline{K}}^o \hookrightarrow X_{\overline{K}}$ etc.

LEMMA 8.28. *Let $U \to X$ be an étale morphism and let L be a sheaf of Λ–modules on $U_{\overline{K},\mathrm{Fet}}^o$. Set $Y_{U,\overline{K}} := U_{\overline{K}} \times_X Y$ so that we have a cartesian diagram*

(8.28.1)
$$
\begin{array}{ccc}
Y_{U,\overline{K}} & \xrightarrow{\ i\ } & U_{\overline{K}} \\
{\scriptstyle s}\uparrow & & \uparrow{\scriptstyle s} \\
Y_{U,\overline{K}}^o & \xrightarrow{\ i\ } & U_{\overline{K}}^o.
\end{array}
$$

Then the natural map in the derived category of Λ–modules in $Y_{U,\overline{K},\mathrm{et}}$

(8.28.2)
$$
i^* s_* L \to s_* i^* L
$$

is an isomorphism.

PROOF. Let $\bar{y} \to Y_{U,\overline{K}}$ be a geometric point. Base changing the diagram 8.28.1 to $\mathrm{Spec}(\mathscr{O}_{U_{\overline{K}},\bar{y}})$ we reduce the proof to the analogous statement for $\mathrm{Spec}(\mathscr{O}_{U_{\overline{K}},\bar{y}})$. In this case the result is immediate because $U_{\overline{K}}^o$ and $Y_{U,\overline{K}}^o$ are $K(\pi,1)$'s and the map $Y_{U,\overline{K}}^o \to U_{\overline{K}}^o$ induces an isomorphism on fundamental groups. $\qquad\square$

8.29. Let $U. \in HR(X)'$ be a hypercover of X, where $HR(X)'$ is the category defined in 9.4. Let $\widetilde{U}_{.,\overline{K}}$ denote $U_{.,\overline{K}} \times_X \widetilde{X}$, and let $Y_{U,.,\overline{K}}$ denote $U_{.,\overline{K}} \times_X Y$. We then have a commutative diagram of simplicial topoi

(8.29.1)
$$
\widetilde{U}_{.,\overline{K},\mathrm{et}} \xrightarrow{\ j\ } U_{.,\overline{K},\mathrm{et}} \xleftarrow{\ i\ } Y_{U,.,\overline{K},\mathrm{et}}.
$$

For a sheaf $F.$ in $U_{.,\overline{K},\mathrm{Fet}}^o$ we then obtain as in 7.21 complexes $\overline{\Psi}(j^*F)$, $\overline{\Psi}(F)$, and $\overline{\Psi}(i^*F)$ on $\widetilde{U}_{.,\overline{K},\mathrm{et}}$, $U_{.,\overline{K},\mathrm{et}}$, and $Y_{U,.,\overline{K},\mathrm{et}}$ respectively. It follows immediately from the construction of these complexes that $\overline{\Psi}(j^*F)$ is canonically isomorphic to $j^*\overline{\Psi}(F)$. Also the maps 8.28.2 define a morphism

(8.29.2)
$$
i^*\overline{\Psi}(F) \to \overline{\Psi}(i^*F)
$$

which by 8.28 is an isomorphism.

The diagram 8.29.1 extends to a diagram of $\widetilde{\underline{\Delta}}^+$–topoi

(8.29.3)
$$
\widetilde{U}_{.,\overline{K},\mathrm{et}}^{\widetilde{\underline{\Delta}}^+} \xrightarrow{\ j\ } U_{.,\overline{K},\mathrm{et}}^{\widetilde{\underline{\Delta}}^+} \xleftarrow{\ i\ } Y_{U,.,\overline{K},\mathrm{et}}^{\widetilde{\underline{\Delta}}^+}.
$$

Define $\overline{\Psi}_{\widetilde{\underline{\Delta}}^+}(F)$ to be the sheaf in $U_{.,\overline{K},\mathrm{et}}^{\widetilde{\underline{\Delta}}^+}$ whose restriction to $U_{r,\overline{K},\mathrm{et}}$ (for $r \geq -1$) is $\overline{\Psi}(F)^{r+1}$ and whose transition maps are given by the morphisms 7.21.2. Then $\overline{\Psi}(F)$ is equal to $\int \overline{\Psi}_{\widetilde{\underline{\Delta}}^+}(F)$. Similarly we have sheaves $\overline{\Psi}_{\widetilde{\underline{\Delta}}^+}(j^*F)$ and $\overline{\Psi}_{\widetilde{\underline{\Delta}}^+}(i^*F)$ on $\widetilde{U}_{.,\overline{K},\mathrm{et}}^{\widetilde{\underline{\Delta}}^+}$ and $Y_{U,.,\overline{K},\mathrm{et}}^{\widetilde{\underline{\Delta}}^+}$ respectively. The isomorphism 8.29.2 is then induced by an isomorphism $i^*\overline{\Psi}_{\widetilde{\underline{\Delta}}^+}(F) \to \overline{\Psi}_{\widetilde{\underline{\Delta}}^+}(i^*F)$.

LEMMA 8.30. *Assume F is an injective sheaf in $U^o_{\cdot,\overline{K},\mathrm{Fet}}$. Then the natural map of sheaves in $U^{\widetilde{\Delta}^+}_{\cdot,\overline{K},\mathrm{et}}$*

$$(8.30.1) \qquad \overline{\Psi}_{\underline{\widetilde{\Delta}}^+}(F) \otimes^{\mathbb{L}} Rj_*\Lambda \to Rj_*j^*\overline{\Psi}_{\underline{\widetilde{\Delta}}^+}(F)$$

is an isomorphism.

PROOF. It suffices to verify that 8.30.1 induces an isomorphism on stalks at a geometric point $\bar{y} \to U_{n,\overline{K}}$. Let W denote $\mathrm{Spec}(\mathscr{O}_{U_{n,\overline{K}},\bar{y}})$, let $Y_W \subset W$ denote the inverse image of Y, let $\widetilde{W}$ denote the inverse image of $\widetilde{U}$, and define W^o_A etc. as before. Let Δ denote the fundamental group of W^o, and let $\widetilde{\Delta}$ denote the fundamental group of $\widetilde{W}^o$. The fundamental group of W^o (resp. $\widetilde{W}^o$) is isomorphic to $\widehat{\mathbb{Z}}(1)^d$ (resp. $\widehat{\mathbb{Z}}(1)^{d+1}$) for some integer d, and the map from Δ (resp. $\widetilde{\Delta}$) to the fundamental group of $U^o_{n,\overline{K}}$ (resp. $\widetilde{U}^o_{n,\overline{K}}$) is injective since we have by assumption an étale morphism 7.1.1. It follows from 7.18 that the pullback of F to W^o is an injective Δ–representation, so it suffices to prove the lemma for W.

Let M be the representation corresponding to F. Then $\overline{\Psi}_{\underline{\widetilde{\Delta}}^+}(F)$ is a direct sum of sheaves of the form $j_{A*}j_A^*j_*F$ for subsets $A \subset \{1,\ldots,m\}$. Let $I_A \subset \Delta$ (resp. $\widetilde{I}_A \subset \widetilde{\Delta}$) denote the inertia group of D_A. Then the right side of 8.30.1 computes the direct sum of the groups

$$(8.30.2) \qquad H^*(\widetilde{\Delta}/\widetilde{I}_A, M^{I_A}).$$

On the other hand, the map $\widetilde{I}_A \to I_A$ is an isomorphism, so the restriction of M to $\widetilde{I}_A$ is an injective representation. Therefore 8.30.2 is isomorphic to

$$(8.30.3) \qquad H^*(\widetilde{\Delta}, M).$$

From the Leray spectral sequence associated to the map $\widetilde{\Delta} \to \Delta$ (whose kernel is isomorphic to $\widehat{\mathbb{Z}}(1)$) and the fact that M is an injective Δ–representation we find that

$$(8.30.4) \qquad H^0(\widetilde{\Delta}, M) \simeq M^{\Delta}, \quad H^1(\widetilde{\Delta}, M) \simeq H^0(\Delta, M \otimes H^1(\widehat{\mathbb{Z}}(1), \Lambda)),$$

and all other cohomology groups are zero.

To compute the left side of 8.30.1, note that the distinguished triangle

$$(8.30.5) \qquad \Lambda \to Rj_*\Lambda \to i_*\Lambda(-1)[-1] \to \Lambda[1],$$

where $i : W_Y \hookrightarrow W$ denotes the inclusion shows that we have

$$(8.30.6) \qquad H^0(W, \overline{\Psi}_{\underline{\widetilde{\Delta}}^+}(F) \otimes^{\mathbb{L}} Rj_*\Lambda) = \oplus_{A\subset\{1,\ldots,m\}}M^{\Delta},$$

and

$$\begin{aligned} H^i(W, \overline{\Psi}_{\underline{\widetilde{\Delta}}^+}(F) \otimes^{\mathbb{L}} Rj_*\Lambda) &= H^{i-1}(W, \overline{\Psi}_{\underline{\widetilde{\Delta}}^+}(F) \otimes i_*\Lambda(-1)) \\ &\simeq H^{i-1}(W_Y, i^*\overline{\Psi}_{\underline{\widetilde{\Delta}}^+}(F))(-1). \end{aligned}$$

for $i > 0$. Since W_Y is strictly henselian local, we get that

$$(8.30.7) \qquad H^i(W, \overline{\Psi}_{\underline{\widetilde{\Delta}}^+}(F) \otimes^{\mathbb{L}} Rj_*\Lambda) = 0$$

for $i > 1$, and that

$$(8.30.8)$$
$$H^1(W, \overline{\Psi}_{\underline{\widetilde{\Delta}}^+}(F) \otimes^{\mathbb{L}} Rj_*\Lambda) = H^0(W_Y, i^*\overline{\Psi}_{\underline{\widetilde{\Delta}}^+}(F))(-1) = \oplus_{A\subset\{1,\ldots,m\}}M^{\Delta}(-1).$$

□

8.31. Assume now that F is an injective sheaf in $U_{\cdot,\overline{K},\mathrm{Fet}}$. In $U_{\cdot,\overline{K},\mathrm{et}}^{\widetilde{\underline{\Delta}}^+}$ we then obtain a distinguished triangle

$$(8.31.1) \qquad \overline{\Psi}_{\widetilde{\underline{\Delta}}^+}(F) \to Rj_*\overline{\Psi}_{\widetilde{\underline{\Delta}}^+}(j^*F) \to i_*\overline{\Psi}_{\widetilde{\underline{\Delta}}^+}(i^*F)(-1)[-1] \to \overline{\Psi}_{\widetilde{\underline{\Delta}}^+}(F)[1]$$

by tensoring the usual distinguished triangle

$$(8.31.2) \qquad \Lambda \to Rj_*\Lambda \to i_*\Lambda(-1)[-1] \to \Lambda[1]$$

with $\overline{\Psi}_{\widetilde{\underline{\Delta}}^+}(F)$ and applying 8.30. Applying the total complex functor we obtain a distinguished triangle

$$(8.31.3) \qquad \overline{\Psi}(F) \to Rj_*\overline{\Psi}(j^*F) \to i_*\overline{\Psi}(i^*F)(-1)[-1] \to \overline{\Psi}(F)[1]$$

in $U_{\cdot,\overline{K},\mathrm{et}}$.

8.32. Let $s_{!\widetilde{\underline{\Delta}}^+}F$ denote the sheaf in $U_{\cdot,\overline{K},\mathrm{et}}^{\widetilde{\underline{\Delta}}^+}$ whose degree -1-sheaf is $s_!F$ and 0 in all degrees ≥ 0, and define $s_{!\widetilde{\underline{\Delta}}^+}(j^*F)$ and $s_{!\widetilde{\underline{\Delta}}^+}(i^*F)$ similarly. The maps

$$(8.32.1) \qquad s_!F \to \overline{\Psi}(F)^0, \quad s_!j^*F \to \overline{\Psi}(j^*F)^0, \quad s_!i^*F \to \overline{\Psi}(i^*F)^0$$

induce morphisms of sheaves

$$(8.32.2) \quad s_{!\widetilde{\underline{\Delta}}^+}(F) \to \overline{\Psi}_{\widetilde{\underline{\Delta}}^+}(F), \quad s_{!\widetilde{\underline{\Delta}}^+}j^*F \to \overline{\Psi}_{\widetilde{\underline{\Delta}}^+}(j^*F), \quad s_{!\widetilde{\underline{\Delta}}^+}(i^*F) \to \overline{\Psi}_{\widetilde{\underline{\Delta}}^+}(i^*F).$$

These maps induce a diagram in the derived category of Λ–modules on $U_{\cdot,\overline{K},\mathrm{et}}^{\widetilde{\underline{\Delta}}^+}$

$$(8.32.3)$$

$$
\begin{array}{ccccccc}
s_{!\widetilde{\underline{\Delta}}^+}(F) & \longrightarrow & Rj_*s_{!\widetilde{\underline{\Delta}}^+}j^*F & \longrightarrow & i_*s_{!\widetilde{\underline{\Delta}}^+}(i^*F)(-1)[-1] & \longrightarrow & s_{!\widetilde{\underline{\Delta}}^+}(F)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\overline{\Psi}_{\widetilde{\underline{\Delta}}^+}(F) & \longrightarrow & Rj_*\overline{\Psi}_{\widetilde{\underline{\Delta}}^+}(j^*F) & \longrightarrow & i_*\overline{\Psi}_{\widetilde{\underline{\Delta}}^+}(i^*F)(-1)[-1] & \longrightarrow & \overline{\Psi}_{\widetilde{\underline{\Delta}}^+}(F)[1].
\end{array}
$$

LEMMA 8.33. *The diagram 8.32.3 commutes.*

PROOF. This follows from the construction. □

8.34. Similarly, we can consider the diagram of $\widetilde{\underline{\Delta}}^+$-topoi

$$(8.34.1) \qquad\qquad \widetilde{\mathscr{X}_{\overline{K}}^{o\underline{\Delta}^+}} \xrightarrow{\ j\ } \mathscr{X}_{\overline{K}}^{o\widetilde{\underline{\Delta}}^+} \xleftarrow{\ i\ } \mathscr{Y}_{\overline{K}}^{o\widetilde{\underline{\Delta}}^+}.$$

Let G be a sheaf of Λ-modules in $\mathscr{X}_{\overline{K}}^o$. The complexes $\Phi^a(\mathcal{O}_{\mathscr{X}_{\overline{K}}^o})$ and $\Psi^a(G)$ defined in 8.15 and 8.3 are then also obtained from $\widetilde{\underline{\Delta}}^+$-versions $\Phi^a_{\widetilde{\underline{\Delta}}^+}(\mathcal{O}_{\mathscr{X}_{\overline{K}}^o})$ and $\Psi^a_{\widetilde{\underline{\Delta}}^+}(G)$. Namely, let $\Phi^a_{\widetilde{\underline{\Delta}}^+}(\mathcal{O}_{\mathscr{X}_{\overline{K}}^o})$ be the sheaf whose degree r–component ($r \geq -1$) is equal to $\Phi^a(\mathcal{O}_{\mathscr{X}_{\overline{K}}^o})^{r+1}$ and whose transition maps are defined as in 8.15.4. Note that the sheaf $\Phi^a_{\widetilde{\underline{\Delta}}^+}(\mathcal{O}_{\mathscr{X}_{\overline{K}}^o})$ is actually a sheaf of rings. The sheaf $\Psi^a_{\widetilde{\underline{\Delta}}^+}(G)$ is the sheaf whose degree r component is $\Psi^a(G)^{r+1}$ and again the transition maps are defined as in 7.26.

Assume now further that every $(U, N) \in \mathscr{S}$ (object of the site defining $\mathscr{X}_{\overline{K}}^o$) the corresponding sheaf $G_{(U,N)}$ on N_{Fet} is a locally constant constructible sheaf of

flat Λ-modules. Then using the same argument that proved 8.22, one sees that there is a distinguished triangle
(8.34.2)
$$G \otimes \Phi^a_{\underline{\widetilde{\Delta}}^+}(\mathscr{O}_{\mathscr{X}^o_{\overline{K}}}) \to Rj_*(j^*G \otimes \Phi^a_{\underline{\widetilde{\Delta}}^+}(\mathscr{O}_{\widetilde{\mathscr{X}}^o_{\overline{K}}})) \to i_*(i^*G \otimes \Phi^a_{\underline{\widetilde{\Delta}}^+}(\mathscr{O}_{\mathscr{Y}^o_{\overline{K}}}))(-1)[-1] \overset{+1}{\to}$$

in the almost derived category of sheaves in $\mathscr{X}^{o\widetilde{\underline{\Delta}}^+}_{\overline{K}}$.

Let $\mathscr{J}^{\widetilde{\underline{\Delta}}^+}$ denote the sheaf in $\mathscr{X}^{o\widetilde{\underline{\Delta}}^+}_{\overline{K}}$ whose -1–component is $\mathscr{J}$ (the ideal defining the boundary) and whose components for $r \geq 0$ are zero, and similarly define $\widetilde{\mathscr{J}}^{\widetilde{\underline{\Delta}}^+}$ and $\mathscr{J}^{\widetilde{\underline{\Delta}}^+}_Y$ in $\widetilde{\mathscr{X}}^{o\widetilde{\underline{\Delta}}^+}_{\overline{K}}$ and $\mathscr{Y}^{o\widetilde{\underline{\Delta}}^+}_{\overline{K}}$ respectively. We then have maps

(8.34.3) $\mathscr{J}^{\widetilde{\underline{\Delta}}^+} \to \Phi^a_{\underline{\widetilde{\Delta}}^+}(\mathscr{O}_{\mathscr{X}^o_{\overline{K}}}), \quad \widetilde{\mathscr{J}}^{\widetilde{\underline{\Delta}}^+} \to \Phi^a_{\underline{\widetilde{\Delta}}^+}(\mathscr{O}_{\widetilde{\mathscr{X}}^o_{\overline{K}}}), \quad \mathscr{J}^{\widetilde{\underline{\Delta}}^+}_Y \to \Phi^a_{\underline{\widetilde{\Delta}}^+}(\mathscr{O}_{\mathscr{Y}^o_{\overline{K}}})$

inducing the maps

(8.34.4) $\mathscr{J} \to \Phi^a(\mathscr{O}_{\mathscr{X}^o_{\overline{K}}}), \quad \widetilde{\mathscr{J}} \to \Phi^a(\mathscr{O}_{\widetilde{\mathscr{X}}^o_{\overline{K}}}), \quad \mathscr{J}_Y \to \Phi^a(\mathscr{O}_{\mathscr{Y}^o_{\overline{K}}})$

defined in 8.15.5 by normalization. It follows from the construction of the distinguished triangles 8.22.1 and 8.34.2 that the diagram

(8.34.5)

$$
\begin{array}{ccc}
G \otimes \Phi^a_{\underline{\widetilde{\Delta}}^+}(\mathscr{O}_{\mathscr{X}^o_{\overline{K}}}) & \longleftarrow & G \otimes \mathscr{J}^{\widetilde{\underline{\Delta}}^+} \\
\downarrow & & \downarrow \\
Rj_*(j^*G \otimes \Phi^a_{\underline{\widetilde{\Delta}}^+}(\mathscr{O}_{\widetilde{\mathscr{X}}^o_{\overline{K}}})) & \longleftarrow & Rj_*(j^*G \otimes \widetilde{\mathscr{J}}^{\widetilde{\underline{\Delta}}^+}) \\
\downarrow & & \downarrow \\
i_*(i^*G \otimes \Phi^a_{\underline{\widetilde{\Delta}}^+}(\mathscr{O}_{\mathscr{Y}^o_{\overline{K}}}))(-1)[-1] & \longleftarrow & i_*(i^*G \otimes \mathscr{J}^{\widetilde{\underline{\Delta}}^+}_Y)(-1)[-1] \\
\downarrow & & \downarrow \\
G \otimes \Phi^a_{\underline{\widetilde{\Delta}}^+}(\mathscr{O}_{\mathscr{X}^o_{\overline{K}}})[1] & \longleftarrow & G \otimes \mathscr{J}^{\widetilde{\underline{\Delta}}^+}[1]
\end{array}
$$

commutes and becomes an isomorphism between the two distinguished triangles, after applying $R\nu_*$.

Note also that if $\Lambda = \mathbb{Z}/p^r \to \Phi^a_{\underline{\widetilde{\Delta}}^+}(\mathscr{O}_{\mathscr{X}^o_{\overline{K}}})/p^r$ is the unique morphism of rings in $\mathscr{X}^{o\widetilde{\underline{\Delta}}^+}_{\overline{K}}$ (where Λ denotes the ring which in every degree is Λ), then there is a canonical map

(8.34.6) $\Psi^a_{\underline{\widetilde{\Delta}}^+}(\Lambda) \to \mathbb{Z}/p^r \otimes \Phi^a_{\underline{\widetilde{\Delta}}^+}(\mathscr{O}_{\mathscr{X}^o_{\overline{K}}}).$

This defines for every sheaf G of Λ–modules a morphism

(8.34.7) $\Psi^a(G) \to G \otimes \Psi^a_{\underline{\widetilde{\Delta}}^+}(\Lambda) \to G \otimes \Phi^a_{\underline{\widetilde{\Delta}}^+}(\mathscr{O}_{\mathscr{X}^o_{\overline{K}}}).$

It follows from the construction that this map induces 8.18.3 by normalization.

8.35. Next we need the analogue of the sequence 8.31.1 for the sheaves $\Psi^a(F)$ in $\mathscr{X}^o_{\overline{K}}$.

Let $U \to X$ be either an étale morphism such that the conditions of 7.1 hold (i.e. each of the D^o_A as well as $\widetilde{U}_{\overline{K}} := U \times_X \widetilde{X}^o_{\overline{K}}$ are $K(\pi,1)$'s) or the strict henselization

of X at a geometric point (in which case the conditions of 7.1 as well as condition (R) with respect to $D \cup Y$ hold). Let $Y_U \subset U$ be the inverse image of Y.

Let Δ denote the fundamental group of $U^o_{\overline{K}}$ (with respect to a geometric generic point), $\widetilde{\Delta}$ the fundamental group of $\widetilde{U}_{\overline{K}}$, let Δ_{Y_U} denote the fundamental group of $Y^o_{U,\overline{K}}$ and the choice of a specialization morphism defines a homomorphism $\Delta_{Y_U} \to \Delta$ (well–defined up to conjugation). If condition (R) with respect to $D \cup Y$ holds then this map $\Delta_{Y_U} \to \Delta$ is an inclusion.

Consider the diagram of topoi

(8.35.1)
$$
\begin{array}{ccccc}
\widetilde{U}^o_{\overline{K},\mathrm{et}} & \xrightarrow{\ j\ } & U^o_{\overline{K},\mathrm{et}} & \xleftarrow{\ i\ } & Y^o_{U,\overline{K},\mathrm{et}} \\
{\scriptstyle \tilde{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_Y} \\
\widetilde{U}^o_{\overline{K},\mathrm{Fet}} & \xrightarrow{\ \bar{j}\ } & U^o_{\overline{K},\mathrm{Fet}} & \xleftarrow{\ \bar{i}\ } & Y^o_{U,\overline{K},\mathrm{Fet}}.
\end{array}
$$

LEMMA 8.36. *Let F be a sheaf of Λ–modules in $U^o_{\overline{K},\mathrm{Fet}}$. Then there is a canonical isomorphism*

(8.36.1)
$$
R^1\bar{j}_*\bar{j}^*F \simeq \bar{i}_*\bar{i}^*F(-1).
$$

If condition (R) holds then $R^s\bar{j}_*\bar{j}^*F = 0$ for all $s > 1$.

PROOF. Since $\widetilde{U}^o_{\overline{K}}$ is a $K(\pi,1)$ we have

(8.36.2)
$$
R^s\bar{j}_*\bar{j}^*F \simeq R^s(\pi \circ j)_*j^*\pi^*F,
$$

and since $Y^o_{U,\overline{K}}$ is a $K(\pi,1)$ we have

(8.36.3)
$$
R^s\bar{i}_*\bar{i}^*F \simeq R^s\pi_*(i_*i^*\pi^*F).
$$

The purity triangle on $U^o_{\overline{K},\mathrm{et}}$

(8.36.4)
$$
\pi^*F \to Rj_*j^*\pi^*F \to i_*i^*\pi^*F(-1)[-1]
$$

induces an exact sequence

(8.36.5)
$$
R^1\pi_*\pi^*F \to R^1(\pi \circ j)_*j^*\pi^*F \to \pi_*i_*i^*\pi^*F(-1) \to R^2\pi_*\pi^*F.
$$

Since $U^o_{\overline{K}}$ is a $K(\pi,1)$ we have $R^i\pi_*\pi^*F = 0$ for $i > 0$ which implies the first part of the lemma, and in fact shows that for $s \geq 1$ we have

(8.36.6)
$$
R^s\bar{j}_*\bar{j}^*F \simeq R^{s-1}\bar{i}_*\bar{i}^*F(-1).
$$

If condition (R) holds then $\bar{i}_*$ is given in terms of group cohomology by induction via the inclusion $\Delta_{Y_U} \hookrightarrow \Delta$ which is an exact functor. Hence the higher cohomology groups vanish in this case. $\square$

COROLLARY 8.37. *Let Σ denote the kernel of the surjective homomorphism $\widetilde{\Delta} \to \Delta$. There is a canonical isomorphism of Δ–representations*

(8.37.1)
$$
H^1(\Sigma,\Lambda) \simeq \mathrm{Ind}^\Delta_{\Delta_{Y_U}}(\Lambda)(-1).
$$

PROOF. For a Δ–representation M let $M^{\tilde{}}$ denote the corresponding sheaf on U^o_{Fet}. We then have

$$
\begin{array}{rcl}
H^1(\Sigma,\Lambda)^{\tilde{}} & \simeq & R^1\bar{j}_*\Lambda \\
& \simeq & \bar{i}_*\Lambda(-1) \\
& \simeq & \mathrm{Ind}^\Delta_{\Delta_{Y_U}}(\Lambda)^{\tilde{}}(-1).
\end{array}
$$

$\square$

8.38. Let $I_A \subset H_A \subset \Delta$ and $\widetilde{I}_A \subset \widetilde{H}_A \subset \widetilde{\Delta}$ be the decomposition and inertia groups of $D_A \subset U$ and $\widetilde{D}_A := D_A \cap \widetilde{U} \subset \widetilde{U}$.

The structure of the inertia groups is very simple. Write $U_{\overline{K}} = \mathrm{Spec}(R)$, and let $\overline{R}$ be the integral closure of R in the compositum of all finite extensions L of $\mathrm{Frac}(R)$ (in the fixed geometric generic point) for which the normalization of R in L is étale over $U_{\overline{K}}^o$. The decomposition groups H_A is defined by the choice of a geometric point $\bar{y} \to \mathrm{Spec}(\overline{R})$ mapping to the generic point of D_A^o. The inertia group I_A is equal to the subgroup of Δ of elements which fix $\bar{y}$ and act trivially on the residue field of $\mathscr{O}_{\mathrm{Spec}(\overline{R}),\bar{y}}$. Let $\widehat{\mathscr{O}}_{R,\bar{y}}$ denote the completion of the strict henselization of R at $\bar{y}$ and let $\widetilde{R}_{\bar{y}}$ denote the local ring of $\bar{y}$ in $\mathrm{Spec}(\overline{R} \otimes_R \widehat{\mathscr{O}}_{R,\bar{y}})$. If $|A| = r$, then $\widehat{R}_{\bar{y}}$ is isomorphic to $S[[t_1, \ldots, t_r]]$ for some ring S with D_A defined by $t_1 = t_2 = \cdots = t_r = 0$. The extension $\widetilde{R}_{\bar{y}}$ is then by Abhyankar's lemma equal to the extensions obtained by taking roots of the variables t_i. This defines an isomorphism $I_A \simeq \widehat{\mathbb{Z}}(1)^r$. Applying the same analysis to $\widetilde{U}$ and Y_U we obtain:

COROLLARY 8.39. *The projection* $\widetilde{I}_A \to I_A$ *is an isomorphism. The projection* $\Sigma \cap \widetilde{H}_A \to \widetilde{H}_A/\widetilde{I}_A$ *is injective and identifies* $\Sigma \cap \widetilde{H}_A$ *with the kernel of the map* $\widetilde{H}_A/\widetilde{I}_A \to H_A/I_A$. *In particular, if* Σ_A *denotes the kernel of the homomorphism* $\widetilde{\Delta}_A \to \Delta_A$ *(where* Δ_A *denotes the fundamental group of* $D_{A,\overline{K}}^o$ *etc.), then we have a commutative diagram*

$$(8.39.1) \quad \begin{array}{ccccccccc} 1 & \longrightarrow & \Sigma_A & \longrightarrow & \widetilde{\Delta}_A & \longrightarrow & \Delta_A & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Sigma \cap \widetilde{H}_A & \longrightarrow & \widetilde{H}_A/\widetilde{I}_A & \longrightarrow & H_A/I_A & \longrightarrow & 1, \end{array}$$

where the rows are exact and the vertical maps are surjective.

If condition (R) holds, then the vertical maps in 8.39.1 are isomorphisms, and there is a commutative diagram with exact rows

$$(8.39.2) \quad \begin{array}{ccccccccc} 1 & \longrightarrow & I_A & \longrightarrow & H_A \cap \Delta_{Y_U} & \longrightarrow & \Delta_{Y_U,A} \cap \Delta_A & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & I_A & \longrightarrow & H_A & \longrightarrow & \Delta_A & \longrightarrow & 1. \end{array}$$

COROLLARY 8.40. *Let* $H_{A,Y}$ *denote the decomposition group of* $D_A \cap Y_U \subset Y_U$, *and let* $\Delta_{Y_U,A}$ *denote the fundamental group of* $D_A^o \cap Y_U^o$. *The natural map of* Δ_A*–representations*

$$(8.40.1) \quad \mathrm{Ind}_{H_{A,Y}}^{H_A}(\Lambda) \simeq \mathrm{Ind}_{H_{A,Y}/I_A}^{H_A/I_A}(\Lambda) \to \mathrm{Ind}_{\Delta_{Y_U,A}}^{\Delta_A}(\Lambda)$$

is an isomorphism if condition (R) holds.

COROLLARY 8.41. *There is a canonical morphism*

$$(8.41.1) \quad H^1(\Sigma \cap \widetilde{H}_A, \Lambda) \to \mathrm{Ind}_{\Delta_{Y_U,A}}^{\Delta_A}(\Lambda)(-1)$$

which is an isomorphism if condition (R) holds. Furthermore, if (R) holds then $H^s(\Sigma \cap \widetilde{H}_A, \Lambda) = 0$ *for all* $s > 1$.

PROOF. Applying 8.37 to

(8.41.2) $$\Sigma_A := \mathrm{Ker}(\widetilde{\Delta}_A \to \Delta_A)$$

we obtain an isomorphism

(8.41.3) $$H^1(\Sigma_A, \Lambda) \simeq \mathrm{Ind}_{\Delta_{Y_U, A}}^{\Delta_A}(\Lambda)(-1).$$

From the map $\Sigma_A \to \Sigma \cap \widetilde{H}_A$ we then obtain a morphism

(8.41.4) $$H^1(\Sigma \cap \widetilde{H}_A, \Lambda) \to H^1(\Sigma_A, \Lambda) \simeq \mathrm{Ind}_{\Delta_{Y_U, A}}^{\Delta_A}(\Lambda)(-1).$$

That it is an isomorphism when (R) holds follows from 8.40 and the fact that when (R) holds the map $\Sigma_A \to \Sigma \cap \widetilde{H}_A$ is an isomorphism. The last statement follows from 8.36. $\qquad\square$

8.42. Now let F be an injective sheaf in $U^o_{\overline{K}, \mathrm{Fet}}$ and consider the sheaf $\psi_A(F)$ on $U^o_{\overline{K}, \mathrm{Fet}}$. Recall (see 7.11) that if F corresponds to an injective Δ–representation M then $\psi_A(F)$ corresponds to the induced representation $\mathrm{Ind}_{H_A}^{\Delta}(M|_{H_A})$. Consider again the diagram

(8.42.1) $$\widetilde{U}^o_{\overline{K}, \mathrm{Fet}} \xrightarrow{\ \overline{j}\ } U^o_{\overline{K}, \mathrm{et}} \xleftarrow{\ \overline{i}\ } Y^o_{U, \overline{K}, \mathrm{Fet}}.$$

The group $H^s(\widetilde{U}_{\overline{K}, \mathrm{Fet}}, \psi_A(j^*F))$ is equal to

(8.42.2) $$H^s(\widetilde{H}_A, M).$$

The Leray spectral sequence corresponding to $\widetilde{H}_A \to H_A$ gives a spectral sequence

(8.42.3) $$H^p(H_A, H^q(\Sigma \cap \widetilde{H}_A, M)) \implies H^{p+q}(\widetilde{H}_A, M),$$

which we can also write as

(8.42.4) $$H^p(H_A, M \otimes H^q(\Sigma \cap \widetilde{H}_A, \Lambda)) \implies H^{p+q}(\widetilde{H}_A, M),$$

since M is a flat Λ-module by 7.15.

In particular, for $s = 1$ we have $H^1(H_A, M) = 0$ since M is injective so we get an isomorphism

(8.42.5) $$H^1(\widetilde{H}_A, M) \simeq H^0(H_A, M \otimes H^1(\Sigma \cap \widetilde{H}_A, \Lambda)).$$

We then obtain a diagram

(8.42.6)
$$
\begin{array}{ccc}
H^1(\widetilde{H}_A, M) & \longrightarrow & H^0(\Delta_A, M^{I_A} \otimes \mathrm{Ind}_{\Delta_{Y_U, A}}^{\Delta_A}(\Lambda)(-1)) \\
 & & \uparrow \\
 & & H^0(H_A, M \otimes \mathrm{Ind}_{H_{A, Y}}^{H_A}(\Lambda)(-1)) \\
 & & \simeq \uparrow \\
 & & H^0(H_{A, Y}, M(-1)).
\end{array}
$$

If condition (R) holds then all these maps are isomorphisms.

8.43. Let F be an injective sheaf in $\mathscr{X}^o_{\overline{K}}$ and consider the diagram

(8.43.1) $$\widetilde{\mathscr{X}}^{o\widetilde{\Delta}^+}_{\overline{K}} \xrightarrow{\ j\ } \mathscr{X}^{o\widetilde{\Delta}^+}_{\overline{K}} \xleftarrow{\ i\ } \mathscr{Y}^{o\widetilde{\Delta}^+}_{\overline{K}}.$$

Let

$$(8.43.2) \quad \nu_{\widetilde{\overline{X}}}^{\widetilde{\underline{\Delta}}^+} : \widetilde{\mathcal{X}_{\overline{K}}^{o\widetilde{\underline{\Delta}}^+}} \to X_{\text{et}}^{\widetilde{\underline{\Delta}}^+}, \quad \nu_{\overline{X}}^{\widetilde{\underline{\Delta}}^+} : \mathcal{X}_{\overline{K}}^{o\widetilde{\underline{\Delta}}^+} \to X_{\text{et}}^{\widetilde{\underline{\Delta}}^+}, \quad \nu_{\overline{Y}}^{\widetilde{\underline{\Delta}}^+} : \mathcal{Y}_{\overline{K}}^{o\widetilde{\underline{\Delta}}^+} \to X_{\text{et}}^{\widetilde{\underline{\Delta}}^+}$$

be the projections.

LEMMA 8.44. *We have canonical isomorphisms*

$$R^s\nu_{\widetilde{\overline{X}}*}^{\widetilde{\underline{\Delta}}^+} \Psi_{\underline{\widetilde{\Delta}}^+}^a(j^*F) = \begin{cases} \nu_{\overline{X}*}^{\widetilde{\underline{\Delta}}^+} \Psi_{\underline{\widetilde{\Delta}}^+}^a(F) & \text{if } s = 0, \\ \nu_{\overline{Y}*}^{\widetilde{\underline{\Delta}}^+} \Psi_{\underline{\widetilde{\Delta}}^+}^a(i^*F)(-1) & \text{if } s = 1, \\ 0 & \text{if } s > 1, \end{cases}$$

and

$$R^t\nu_{\overline{Y}*}^{\widetilde{\underline{\Delta}}^+} \Psi_{\underline{\widetilde{\Delta}}^+}^a(i^*F) = 0$$

for $t > 0$.

PROOF. Fix $[r] \in \underline{\widetilde{\Delta}}^+$. The restriction of $R^s\nu_{\widetilde{\overline{X}}*}^{\widetilde{\underline{\Delta}}^+} \Psi_{\underline{\widetilde{\Delta}}^+}^a(j^*F)$ to the $[r]$-component of $X_{\text{et}}^{\widetilde{\underline{\Delta}}^+}$ is the sheaf $R^s\nu_{\widetilde{\overline{X}}*}\Psi_{\underline{\widetilde{\Delta}}^+}^a(j^*F)_r$ where $\Psi_{\underline{\widetilde{\Delta}}^+}^a(j^*F)_r$ is the sheaf on $\widetilde{\mathcal{X}_{\overline{K}}^o}$ associated to the presheaf whose restriction to $\widetilde{U}_{\overline{K},\text{Fet}}^o$ is equal to

$$\oplus_{A \subset \{1,\ldots,m\}, |A|=r}\psi_A(j^*F_U),$$

and

$$\nu_{\widetilde{\overline{X}}} : \widetilde{\mathcal{X}_{\overline{K}}^o} \to X_{\text{et}}$$

is the natural morphism of topoi. By 6.9 the sheaf $R^s\nu_{\widetilde{\overline{X}}*}\Psi_{\underline{\widetilde{\Delta}}^+}^a(j^*F)_r$ is equal to the sheaf associated to the presheaf which to any étale $U \to X$, satisfying the conditions in 7.1, assigns

$$\oplus_{A \subset \{1,\ldots,m\}, |A|=r}H^s(\widetilde{H}_A, M_U),$$

where $\widetilde{H}_A$ is as in 8.38 and M_U denotes the Δ-representation corresponding to F_U. From this the case $s = 0$ follows immediately, and the vanishing for $s > 1$ follows from 8.41 by a standard limit argument.

Similarly the sheaf $R^t\nu_{\overline{Y}*}^{\widetilde{\underline{\Delta}}^+} \Psi_{\underline{\widetilde{\Delta}}^+}^a(i^*F)_r$ is equal to the sheaf associated to the presheaf which to $U \to X$ as above associates

$$\oplus_{A \subset \{1,\ldots,m\}, |A|=r}H^t(H_{A,Y}, M_U).$$

From this description, a limit argument again, and 7.18 it follows that

$$R^t\nu_{\overline{Y}*}^{\widetilde{\underline{\Delta}}^+} \Psi_{\underline{\widetilde{\Delta}}^+}^a(i^*F)_r = 0$$

for $t > 0$.

Furthermore the maps in 8.42.6 and a limit argument using the fact that for any geometric point $\bar{x} \to X$ condition (R) holds for $\text{Spec}(\mathcal{O}_{X,\bar{x}} \otimes_V \overline{K})$ yield the isomorphism

$$R^1\nu_{\widetilde{\overline{X}}*}^{\widetilde{\underline{\Delta}}^+} \Psi_{\underline{\widetilde{\Delta}}^+}^a(j^*F)_r \simeq \nu_{\overline{Y}*}^{\widetilde{\underline{\Delta}}^+} \Psi_{\underline{\widetilde{\Delta}}^+}^a(i^*F)(-1)_r.$$

We leave to the reader the verification that these isomorphisms are compatible with the simplicial structure, as r varies. □

COROLLARY 8.45. *Let F be a sheaf of Λ-modules in $\mathscr{X}_{\overline{K}}^o$. Then there is a canonical distinguished triangle in $X_{\mathrm{et}}^{\widetilde{\underline{\Delta}}^+}$*

(8.45.1)
$$R\nu_{X*}^{\widetilde{\underline{\Delta}}^+}\Psi_{\underline{\Delta}^+}^a(F) \to R\nu_{\widetilde{X}*}^{\widetilde{\underline{\Delta}}^+}\Psi_{\underline{\Delta}^+}^a(j^*F) \to R\nu_{Y*}^{\widetilde{\underline{\Delta}}^+}\Psi_{\underline{\Delta}^+}^a(i^*F)(-1)[-1] \to R\nu_{X*}^{\widetilde{\underline{\Delta}}^+}\Psi_{\underline{\Delta}^+}^a(F)[1].$$

In particular, we obtain by normalization a distinguished triangle in X_{et}

(8.45.2)
$$R\nu_{X*}\Psi^a(F) \to R\nu_{\widetilde{X}*}\Psi^a(j^*F) \to R\nu_{Y*}\Psi^a(i^*F)(-1)[-1] \to R\nu_{X*}\Psi^a(F)[1],$$

where

(8.45.3)
$$\nu_{\widetilde{X}} : \widetilde{\mathscr{X}_{\overline{K}}^o} \to X_{\mathrm{et}}, \quad \nu_X : \mathscr{X}_{\overline{K}}^o \to X_{\mathrm{et}}, \quad \nu_Y : \mathscr{Y}_{\overline{K}}^o \to X_{\mathrm{et}}$$

are the projections.

PROOF. In the case when F is an injective sheaf of Λ-modules this follows from 8.44.

To show the general case, it suffices to construct the triangle for a bounded below complex $F^\bullet$ of injective sheaves in $\mathscr{X}_{\overline{K}}^o$, instead of just a single injective sheaf. To handle this case, choose a bicomplex $J^{\bullet\bullet}$ of injectives in $\widetilde{\mathscr{X}_{\overline{K}}^o}$ and a map of bicomplexes $j^*F^\bullet \to J^{\bullet\bullet}$ such that for every j the map $j^*F^j \to J^{j\bullet}$ is a quasi–isomorphism. Let $\tau_{\geq 1}\nu_{\widetilde{X}*}\Psi_{\underline{\Delta}^+}(J^{\bullet\bullet})$ be the truncation with respect to the second index. We then obtain from 8.44 a quasi–isomorphism

(8.45.4)
$$\tau_{\geq 1}\nu_{\widetilde{X}*}\Psi_{\underline{\Delta}^+}^a(J^{\bullet\bullet}) \simeq \nu_{Y*}\Psi_{\underline{\Delta}^+}^a(i^*F^\bullet)$$

defining the desired triangle 8.45.1. $\square$

8.46. For any sheaf F of Λ–modules in $\mathscr{X}_{\overline{K}}^o$ we obtain a diagram

(8.46.1)
$$
\begin{array}{ccc}
R\Gamma(\mathscr{X}_{\overline{K}}^o, \Psi^a(F)) & \longrightarrow & R\Gamma(\mathscr{X}_{\overline{K}}^o, F \otimes \Phi_{\underline{\Delta}^+}^a(\mathscr{O}_{\mathscr{X}_{\overline{K}}^o})) \\
\downarrow & & \downarrow \\
R\Gamma(\widetilde{\mathscr{X}_{\overline{K}}^o}, \Psi^a(j^*F)) & \longrightarrow & R\Gamma(\widetilde{\mathscr{X}_{\overline{K}}^o}, j^*F \otimes \Phi_{\underline{\Delta}^+}^a(\mathscr{O}_{\widetilde{\mathscr{X}_{\overline{K}}^o}})) \\
\downarrow & & \downarrow \\
R\Gamma(\mathscr{Y}_{\overline{K}}^o, \Psi^a(i^*F))(-1)[-1] & \longrightarrow & R\Gamma(\mathscr{Y}_{\overline{K}}^o, i^*F \otimes \Phi_{\underline{\Delta}^+}^a(\mathscr{O}_{\mathscr{Y}_{\overline{K}}^o}))(-1)[-1], \\
{\scriptstyle +1}\downarrow & & {\scriptstyle +1}\downarrow \\
\end{array}
$$

where the left triangle is obtained from 8.45.2, the right triangle is obtained from 8.34.2, and the horizontal arrows are the maps 8.34.7. It follows from the construction that this diagram commutes.

8.47. Let L be a locally constant constructible sheaf on $X_{\overline{K}}^o$, and let $\mathscr{L} := u_{X*}L$ denote the induced sheaf on $\mathscr{X}_{\overline{K}}^o$. Choose an injective resolution $\mathscr{L} \to I^\bullet$. For a hypercover $U. \in HR(X)'$ such that each $U_{n,\overline{K}}$ also satisfies the conditions of 7.1 with respect to the divisor $D \cup Y$, we then have by 8.31.1 a distinguished triangle

(8.47.1) $\overline{\Psi}(I^\bullet|_{U_{\cdot,\overline{K}}^o}) \to Rj_*\overline{\Psi}(I^\bullet|_{\widetilde{U}_{\cdot,\overline{K}}^o}) \to i_*\overline{\Psi}(I^\bullet|_{U_{\cdot,Y,\overline{K}}^o})(-1)[-1] \to \overline{\Psi}(I^\bullet|_{U_{\cdot,\overline{K}}^o})[1]$

on $U._{,\overline{K},\text{et}}$, where $U._{,Y}$ denotes the base change $U. \times_X Y$ and $I^\bullet|_{U^o._{,\overline{K}}}$ denotes the pushforward of $I^\bullet$ to $U^o._{,\overline{K},\text{Fet}}$.

We obtain a diagram
(8.47.2)

$$
\begin{array}{ccc}
\text{hocolim} R\Gamma(U._{,\overline{K}}, \overline{\Psi}(I^\bullet|_{U^o._{,\overline{K}}})) & \longleftarrow & \text{hocolim} R\Gamma(U^o._{,\overline{K}}, \Psi(I^\bullet|_{U^o._{,\overline{K}}})) \\
\downarrow & & \downarrow \\
\text{hocolim} R\Gamma(\widetilde{U}._{,\overline{K}}, \overline{\Psi}(I^\bullet|_{\widetilde{U}^o._{,\overline{K}}})) & \longleftarrow & \text{hocolim} R\Gamma(\widetilde{U}^o._{,\overline{K}}, \Psi(I^\bullet|_{\widetilde{U}^o._{,\overline{K}}})) \\
\downarrow & & \downarrow \\
\text{hocolim} R\Gamma(U._{,Y,\overline{K}}, \overline{\Psi}(I^\bullet|_{U^o._{,Y,\overline{K}}})) & \longleftarrow & \text{hocolim} R\Gamma(U^o._{,Y,\overline{K}}, \Psi(I^\bullet|_{U^o._{,Y,\overline{K}}})) \\
\downarrow & & \downarrow \\
\text{hocolim} R\Gamma(U._{,\overline{K}}, \overline{\Psi}(I^\bullet|_{U^o._{,\overline{K}}}))[1] & \longleftarrow & \text{hocolim} R\Gamma(U^o._{,\overline{K}}, \Psi(I^\bullet|_{U^o._{,\overline{K}}}))[1],
\end{array}
$$

where the columns are the distinguished triangles obtained from 8.47.1 and 8.45.2, and the horizontal arrows are the morphisms obtained from 8.8.1, and the homotopy colimits are taken over $U. \in HR(X)'$.

PROPOSITION 8.48. *The diagram 8.47.2 commutes.*

PROOF. Let pt denote the punctual topos, and let T denote the topos $(\text{pt}^{\underline{\Delta}})^{\widetilde{\underline{\Delta}}^+}$. So a Λ–module in T consists of a Λ–module $F_{n,m}$ for every pair of objects $[n] \in \underline{\Delta}$ and $[m] \in \widetilde{\underline{\Delta}}^+$ together with transition morphisms. We refer to $F_{n,m}$ as the (n,m)-*component of F.*

For a simplicial scheme $Z.$ there is a canonical morphism of topoi

$$(8.48.1) \qquad\qquad q : Z.^{\widetilde{\underline{\Delta}}^+} \to T.$$

For a sheaf $F \in Z.^{\widetilde{\underline{\Delta}}^+}$ we write $\widetilde{\Gamma}(F)$ for q_*F and $R\widetilde{\Gamma}(F)$ for Rq_*F. Taking the total complex (in both the Δ–direction and $\widetilde{\underline{\Delta}}^+$–direction) defines a triangulated functor

$$(8.48.2) \qquad\qquad \text{Tot} : D(T, \Lambda) \to D(\Lambda).$$

The diagram 8.47.2 is obtained by applying Tot to a diagram in $D(T, \Lambda)$. Namely, we have complexes of sheaves $\Psi_{\widetilde{\underline{\Delta}}^+}(I^\bullet)$ and $\overline{\Psi}_{\widetilde{\underline{\Delta}}^+}(I^\bullet)$ in $U^{o\widetilde{\underline{\Delta}}^+}._{,\overline{K},\text{Fet}}$ and $U.^{\widetilde{\underline{\Delta}}^+}_{,\overline{K},\text{et}}$ respectively. The diagram 8.47.2 is then obtained by applying Tot to the diagram
(8.48.3)

$$
\begin{array}{ccc}
\text{hocolim} R\widetilde{\Gamma}(U._{,\overline{K}}, \overline{\Psi}_{\widetilde{\underline{\Delta}}^+}(I^\bullet|_{U^o._{,\overline{K}}})) & \longleftarrow & \text{hocolim} R\widetilde{\Gamma}(U^o._{,\overline{K}}, \Psi_{\widetilde{\underline{\Delta}}^+}(I^\bullet|_{U^o._{,\overline{K}}})) \\
\downarrow & & \downarrow \\
\text{hocolim} R\widetilde{\Gamma}(\widetilde{U}._{,\overline{K}}, \overline{\Psi}_{\widetilde{\underline{\Delta}}^+}(I^\bullet|_{\widetilde{U}^o._{,\overline{K}}})) & \longleftarrow & \text{hocolim} R\widetilde{\Gamma}(\widetilde{U}^o._{,\overline{K}}, \Psi_{\widetilde{\underline{\Delta}}^+}(I^\bullet|_{\widetilde{U}^o._{,\overline{K}}})) \\
\downarrow & & \downarrow \\
\text{hocolim} R\widetilde{\Gamma}(U._{,Y,\overline{K}}, \overline{\Psi}_{\widetilde{\underline{\Delta}}^+}(I^\bullet|_{U^o._{,Y,\overline{K}}})) & \longleftarrow & \text{hocolim} R\widetilde{\Gamma}(U^o._{,Y,\overline{K}}, \Psi_{\widetilde{\underline{\Delta}}^+}(I^\bullet|_{U^o._{,Y,\overline{K}}})) \\
\downarrow & & \downarrow \\
\text{hocolim} R\widetilde{\Gamma}(U._{,\overline{K}}, \overline{\Psi}_{\widetilde{\underline{\Delta}}^+}(I^\bullet|_{U^o._{,\overline{K}}}))[1] & \longleftarrow & \text{hocolim} R\widetilde{\Gamma}(U^o._{,\overline{K}}, \Psi_{\widetilde{\underline{\Delta}}^+}(I^\bullet|_{U^o._{,\overline{K}}}))[1],
\end{array}
$$

where the homotopy colimits are taken over $U_\cdot \in HR(X)'$.

Choose a bicomplex $J^{\bullet\bullet}$ of injective objects in $\widetilde{\mathcal{X}^o_{\overline{K}}}$ and a morphism $I^\bullet|_{\widetilde{\mathcal{X}^o_{\overline{K}}}} \to J^{\bullet\bullet}$ such that for every j the map

$$(8.48.4) \qquad\qquad\qquad\qquad I^j \to J^{j\bullet}$$

is a quasi–isomorphism. For a hypercover $U_\cdot \in HR(X)'$ and for any $j \geq 0$, the complex in T

$$(8.48.5) \qquad\qquad\qquad \widetilde{\Gamma}(\widetilde{U}^o_{\cdot,\overline{K},\mathrm{Fet}}, \Psi_{\widetilde{\underline{\Delta}}^+}(J^{j\bullet}|_{U_\cdot}))$$

has (n,m)-component a complex computing the cohomology $R\Gamma(\widetilde{U}^o_{n,\overline{K},\mathrm{Fet}}, \Psi(I^j)^m)$.

Also choose a cofinal projective system of étale hypercovers $\{U^\alpha_\cdot\}$ of X.

From 8.36 we have:

LEMMA 8.49. *There are canonical isomorphisms*
(i) $\varinjlim_\alpha R^0\Gamma(\widetilde{U}^{\alpha o}_{n,\overline{K},\mathrm{Fet}}, \Psi(I^j|_{\widetilde{U}^o_{\cdot,\overline{K}}})^m) \simeq \varinjlim_\alpha \Gamma(U^{\alpha o}_{n,\overline{K},\mathrm{Fet}}, \Psi(I^j)^m),$
(ii) $\varinjlim_\alpha R^1\Gamma(\widetilde{U}^{\alpha o}_{n,\overline{K},\mathrm{Fet}}, \Psi(I^j|_{\widetilde{U}^o_{\cdot,\overline{K}}})^m) \simeq \varinjlim_\alpha \Gamma(Y^{\alpha o}_{n,\overline{K},\mathrm{Fet}}, \Psi(I^j|_{U^o_{\cdot,Y,\overline{K}}})^m)(-1),$
and
(iii) for any $i > 1$ we have

$$\varinjlim_\alpha R^i\Gamma(\widetilde{U}^{\alpha o}_{n,\overline{K},\mathrm{Fet}}, \Psi(I^j|_{\widetilde{U}^o_{\cdot,\overline{K}}})^m) = 0.$$

Let $\Theta(J^{j\bullet})^m$ denote the kernel of the map

$$(8.49.1) \qquad\qquad\qquad \Psi(J^{j1})^m \to \Psi(J^{j2})^m$$

induced by the morphism $J^{j1} \to J^{j2}$. By the above we then have an exact sequence (in the vertical direction) of complexes in T

$$(8.49.2)$$

A very similar argument can be carried out with the complexes $\overline{\Psi}(I^\bullet)$ and $\overline{\Psi}(J^{\bullet\bullet})$ (we leave the details to the reader). Let

$$(8.49.3) \qquad\qquad \overline{\Theta}(J^{j\bullet})^m := \mathrm{Ker}(\overline{\Psi}(J^{j1})^m \to \overline{\Psi}(J^{j2})^m).$$

We then also obtain an exact sequence of complexes in T

(8.49.4)

$$
\begin{array}{ccc}
& 0 & 0 \\
& \downarrow & \downarrow \\
\varinjlim_\alpha \widetilde{\Gamma}(U^\alpha_{\cdot,\overline{K},\mathrm{et}}, \overline{\Psi}_{\underline{\widetilde{\Delta}}^+}(I^\bullet)) & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
\varinjlim_\alpha \widetilde{\Gamma}(\widetilde{U}^\alpha_{\cdot,\overline{K},\mathrm{et}}, \overline{\Psi}_{\underline{\widetilde{\Delta}}^+}(J^{\bullet 0})) & \longrightarrow & \varinjlim_\alpha \widetilde{\Gamma}(U^\alpha_{\cdot,\overline{K},\mathrm{et}}, \overline{\Theta}(J^{\bullet\bullet})) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \varinjlim_\alpha \widetilde{\Gamma}(Y^\alpha_{\cdot,\overline{K},\mathrm{et}}, \overline{\Psi}_{\underline{\widetilde{\Delta}}^+}(i^* I^\bullet))(-1) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

The construction of 8.8.1 gives isomorphisms between the individual terms of 8.49.1 and 8.49.4, and to prove 8.48 we have to check that in fact these maps give a morphism of diagrams (for then the diagram 8.47.2 is obtained by taking total complexes).

To prove that we in fact get a morphism of diagrams from 8.49.4 to 8.49.1, it suffices to show that for every $j \geq 0$, $m \geq -1$, and $n \geq 0$ the diagram

(8.49.5)

$$
\begin{array}{ccc}
\varinjlim_\alpha \Gamma(U^\alpha_{n,\overline{K},\mathrm{et}}, \overline{\Theta}(J^{j\bullet})^m) & \longrightarrow & \varinjlim_\alpha \Gamma(Y^\alpha_{n,\overline{K},\mathrm{et}}, \overline{\Psi}_{\underline{\widetilde{\Delta}}^+}(i^* I^j)^m)(-1) \\
\downarrow & & \downarrow \\
\varinjlim_\alpha \Gamma(U^{\alpha o}_{n,\overline{K},\mathrm{Fet}}, \Theta(J^{j\bullet})^m) & \longrightarrow & \varinjlim_\alpha \Gamma(Y^{\alpha o}_{n,\overline{K},\mathrm{Fet}}, \Psi_{\underline{\widetilde{\Delta}}^+}(i^* I^\bullet)^m)(-1)
\end{array}
$$

commutes (the other parts of the verification that we get a map of diagrams from 8.49.4 to 8.49.1 are immediate). Equivalently we need to show that the diagram

(8.49.6)

$$
\begin{array}{ccc}
\varinjlim_\alpha H^1(U^\alpha_{n,\overline{K},\mathrm{et}}, \overline{\Psi}(I^j)^m) & \longrightarrow & \varinjlim_\alpha \Gamma(Y^\alpha_{n,\overline{K},\mathrm{et}}, \overline{\Psi}_{\underline{\widetilde{\Delta}}^+}(i^* I^j)^m)(-1) \\
\downarrow & & \downarrow \\
\varinjlim_\alpha H^1(U^{\alpha o}_{n,\overline{K},\mathrm{Fet}}, \Psi(I^j)^m) & \longrightarrow & \varinjlim_\alpha \Gamma(Y^{\alpha o}_{n,\overline{K},\mathrm{Fet}}, \Psi_{\underline{\widetilde{\Delta}}^+}(i^* I^\bullet)^m)(-1)
\end{array}
$$

commutes, where the horizontal arrows are obtained from 8.49 and the vertical arrows are given by 8.8.1. The sheaves $\overline{\Psi}(I^j)^m$ and $\Psi(I^j)^m$ are by construction direct sums of sheaves

(8.49.7) $$\Psi(I^j)^m = \oplus \Psi_A(I^j)^m, \quad \overline{\Psi}(I^j)^m = \oplus \overline{\Psi}_A(I^j)^m,$$

and 8.49.6 is obtained by taking the direct sum of diagrams

(8.49.8)

$$
\begin{array}{ccc}
\varinjlim_\alpha H^1(U^\alpha_{n,\overline{K},\mathrm{et}}, \overline{\Psi}_A(I^j)^m) & \longrightarrow & \varinjlim_\alpha \Gamma(Y^\alpha_{n,\overline{K},\mathrm{et}}, \overline{\Psi}_{A\underline{\widetilde{\Delta}}^+}(i^* I^j)^m)(-1) \\
\downarrow & & \downarrow \\
\varinjlim_\alpha H^1(U^{\alpha o}_{n,\overline{K},\mathrm{Fet}}, \Psi_A(I^j)^m) & \longrightarrow & \varinjlim \Gamma(Y^{\alpha o}_{n,\overline{K},\mathrm{Fet}}, \Psi_{A\underline{\widetilde{\Delta}}^+}(i^* I^\bullet)^m)(-1).
\end{array}
$$

The proof of 8.48 is then completed by showing that 8.49.8 commutes, which is a straightforward verification (which we leave to the reader). This completes the proof of 8.48, and hence also of 8.21 in the case of a divisor. $\square$

8.50. To treat the case of a higher codimension inclusion $i : Y \hookrightarrow X$ we first need some computations.

If $\mathscr{L}$ is a line bundle on X we define its first Chern class $c_1(\mathscr{L}) \in H^2(\mathscr{X}_{\overline{K}}^o, \mu_{p^s})$ as follows. Raising to the p^s–th power induces an exact sequence

(8.50.1) $0 \to \mu_{p^s} \to \mathscr{O}_{\mathscr{X}_{\overline{K}}^o}^* \xrightarrow{p^s} \mathscr{O}_{\mathscr{X}_{\overline{K}}^o}^* \to 0,$

and we define $c_1(\mathscr{L})$ to be the image of the class of $\mathscr{L}$ under the induced map

(8.50.2) $H^1(X, \mathscr{O}_X^*) \to H^1(\mathscr{X}_{\overline{K}}^o, \mathscr{O}_{\mathscr{X}_{\overline{K}}^o}^*) \to H^2(\mathscr{X}_{\overline{K}}^o, \mu_{p^s}).$

By 6.5 there is a canonical isomorphism $H^2(\mathscr{X}_{\overline{K}}^o, \mu_{p^s}) \simeq H^2(X_{\overline{K}, \mathrm{et}}^o, \mu_{p^s})$ and it follows from the definition that under this isomorphism $c_1(\mathscr{L})$ is equal to the class obtained from the Kummer sequence $0 \to \mu_{p^s} \to \mathbb{G}_m \to \mathbb{G}_m \to 0$.

COMPUTATION 8.51. Let $X = \mathbb{P}^1$, $D = \emptyset$, $j : \mathbb{G}_m \hookrightarrow \mathbb{A}^1$, and $i : \mathrm{Spec}(V) \cup \mathrm{Spec}(V) \hookrightarrow \mathbb{P}^1$ be the inclusions of 0 and ∞. By 8.22 there is then a distinguished triangle in $\widetilde{D}(\mathscr{X}_{\overline{K}})$

(8.51.1) $\mathscr{O}_{\mathscr{X}_{\overline{K}}} \to Rj_* \mathscr{O}_{\mathscr{X}_{\overline{K}}^o} \to i_* \mathscr{O}_{\mathscr{D}_{\overline{K}}^o}(-1)[-1] \to \mathscr{O}_{\mathscr{X}_{\overline{K}}}[1],$

where $D = D_0 \coprod D_\infty \subset X$ denotes the inclusion $\mathrm{Spec}(V) \coprod \mathrm{Spec}(V) \hookrightarrow \mathbb{P}^1$ of 0 and ∞. By 4.7 we know that $H^*(\mathscr{X}_{\overline{K}}, Rj_* \mathscr{O}_{\mathscr{X}_{\overline{K}}^o}) \simeq \Lambda^\bullet \overline{V}/p^s(-1)$ (almost isomorphism). Looking at the long exact sequence corresponding to 8.51.1 one sees that $H^0(\mathscr{X}_{\overline{K}}, \mathscr{O}_{\mathscr{X}_{\overline{K}}^o}/p^s) \simeq \overline{V}/p^s$, $H^1(\mathscr{X}_{\overline{K}}, \mathscr{O}_{\mathscr{X}_{\overline{K}}^o}/p^s) = 0$, and $H^2(\mathscr{X}_{\overline{K}}, \mathscr{O}_{\mathscr{X}_{\overline{K}}^o}/p^s) \simeq \overline{V}/p^s(-1)$ with generator the image under the boundary map of a generator of $H^0(\mathscr{D}_{0,\overline{K}}, \mathscr{O}_{\mathscr{D}_{\overline{K}}})$. A straightforward verification shows that this generator is equal to $-c_1(\mathscr{O}_{\mathbb{P}^1}(1))$.

COMPUTATION 8.52. Let $d \geq 1$ be an integer, and let $X = \mathbb{P}^d$. Denote by $H_0, \ldots, H_d$ the standard hyperplanes (so H_i is the locus of points where the i–th coordinate is zero). For $0 \leq i \leq d$ let $\mathscr{X}_{\overline{K},i}^o$ denote the topos obtained by taking $D = H_0 \cup \cdots \cup H_i$. We claim that then $H^*(\mathscr{X}_{\overline{K},i}^o, \overline{\mathscr{O}}_X/p^s)$ is isomorphic to the exterior algebra $\Lambda^\bullet(\overline{V}/p^s)^i(-1)$. This we prove by descending induction on i.

For $i = d$, the scheme X^o is equal to $\mathbb{G}_m^d$. Let Δ denote the fundamental group of $\mathbb{G}_{m,\overline{K}}^d$ so that $\Delta \simeq \widehat{\mathbb{Z}}(1)^d$. Let $U_i \subset \mathbb{P}^d$ denote the standard affine opens (U_i is the open set $\{X_i \neq 0\}$) so that $U_i^o = \mathbb{G}_m^d$. Then as in 4.7 one finds that

(8.52.1) $H^*(\mathscr{U}_{\overline{K},i}^o, \mathscr{O}_{\mathscr{X}_{\overline{K}}^o}/p^s) \simeq \Lambda^\bullet \Omega_X^1(\log H_0 \cup \cdots H_d)(U_i)(-1) \otimes \overline{V}/p^s,$

where $\Omega_X^1(\log H_0 \cup \cdots H_d)$ denotes the sheaf of differentials with logarithmic poles along $H_0 \cup \cdots H_d$. Since $\Omega_X^1(\log H_0 \cup \cdots H_d)$ is trivial (for example trivialized by the forms dX_i/X_i) it follows that $H^*(\mathscr{X}_{\overline{K}}^o, \mathscr{O}_{\mathscr{X}_{\overline{K}}^o}/p^s)$ is computed by the complex

(8.52.2) $\mathscr{C}(\{U_i\}, \mathscr{O}_X/p^s) \otimes (\Lambda^\bullet \overline{V}/p^s)^d(-1),$

where $\mathscr{C}(\{U_i\}, \mathscr{O}_X/p^s)$ denotes the Cech complex computing the coherent cohomology of $\mathscr{O}_X/p^s$. Since $H^*(\mathbb{P}^d, \mathscr{O}_{\mathbb{P}^d}/p^s) \simeq V/p^s$ we obtain the case $i = d$.

For the general case note that the distinguished triangle 8.22 induces a long exact sequence
(8.52.3)
$$\to H^j(\mathscr{X}^o_{\overline{K},i}, \mathscr{O}_{\mathscr{X}^o_{\overline{K}}}) \to H^j(\mathscr{X}^o_{\overline{K},i+1}, \mathscr{O}_{\mathscr{X}^o_{\overline{K}}}) \to H^{j-1}(\mathbb{P}^{d-1}-\{i \text{ hyperplanes}\})(-1) \to .$$

Using this and also induction on d one obtains the general case.

Note in particular that for $D = H_0$ we find that $H^i(\mathscr{X}^o_{\overline{K}}, \overline{\mathscr{O}}_X/p^s) = 0$ for $i > 0$ and for $i = 0$ we get $\overline{V}/p^s$. Using this we can also compute the cohomology of projective space $\mathbb{P}^d$ with $D = \emptyset$. We claim that in this case

$$(8.52.4) \qquad H^*(\mathscr{X}_{\overline{K}}, \mathscr{O}_{\mathscr{X}^o_{\overline{K}}}/p^s) \simeq \oplus_{i=0}^d \overline{V}/p^s(-i) \cdot \xi^i$$

where $\xi \in H^2(\mathscr{X}_{\overline{K}}, \mathscr{O}_{\mathscr{X}^o_{\overline{K}}}/p^s)(1)$ denotes the Chern class of $\mathscr{O}_X(1)$. This is proven by induction on d, the case $d = 0$ being trivial. For general d, let X^o denote $X - H$ where H is a hyperplane so that by 8.22 we have a long exact sequence
(8.52.5)
$$\cdots \to H^i(\mathscr{X}_{\overline{K}}, \mathscr{O}_{\mathscr{X}_{\overline{K}}}/p^s) \to H^i(\mathscr{X}^o_{\overline{K}}, \mathscr{O}_{\mathscr{X}^o_{\overline{K}}}/p^s) \to H^{i-1}(\mathscr{H}_{\overline{K}}, \mathscr{O}_{\mathscr{H}_{\overline{K}}}/p^s)(-1) \to \cdots .$$

By the above we have $H^i(\mathscr{X}^o_{\overline{K}}, \mathscr{O}_{\mathscr{X}^o_{\overline{K}}}/p^s) = 0$ for $i > 0$, and by the above the boundary map

$$(8.52.6) \qquad H^{i-1}(\mathscr{H}, \mathscr{O}_{\mathscr{H}_{\overline{K}}}/p^s)(-1) \to H^{i+1}(\mathscr{X}_{\overline{K}}, \mathscr{O}_{\mathscr{X}_{\overline{K}}}/p^s)$$

is equal to i_*. Now we have $i_*(\xi^j) = \xi^{j+1}$. Indeed we have already shown that i_* agrees with the étale pushforward i_*^{et} for the inclusion of smooth divisors, so it suffices to verify this formula in the étale theory and by the projection formula it in turn suffices to show that $i_*(1) = \xi$ which is true for example by [**9**, Cycle, 2.9].

PROPOSITION 8.53. *Let $\mathscr{E}$ be a vector bundle of rank $r + 1$ on X and let $\pi : Y = \mathbb{P}(\mathscr{E}) \to X$ be the corresponding projective bundle. Let $\widetilde{D} \subset Y$ be the inverse image of D, and for a decomposition $D = E \cup F$ let $\widetilde{D} = \widetilde{E} \cup \widetilde{F}$ be the decomposition obtained by taking inverse images of E and F. Let $\mathscr{O}(1)$ denote the tautological line bundle on Y, and let $\xi \in H^2(\mathscr{Y}^0, \mu_{p^s})$ denote the Chern class $c_1(\mathscr{O}(1))$. Then for any integer s and sheaf L the cup product map*

$$(8.53.1) \qquad \oplus_{i=0}^r \cup \xi^i : \bigoplus_{i=0}^r H^{s-2i}(\mathscr{X}^o_{\overline{K}}, L \otimes \mathscr{J}_E)(-i) \to H^s(\mathscr{Y}^o_{\overline{K}}, \pi^*L \otimes \mathscr{J}_{\widetilde{E}})$$

is an almost isomorphism.

PROOF. It suffices to prove the proposition after making an étale localization on X. We may therefore consider the analogous statement for X the spectrum of a strictly henselian local ring. In this case Y is isomorphic to $X \times \mathbb{P}^r$. In this case the result follows from the computation 8.52 and the observation that if P denotes $\mathbb{P}^d$ then $H^*(\mathscr{Y}^o, \mathscr{J}_{\widetilde{E}}) \simeq J_E \otimes_V H^*(\mathscr{P}_{\overline{K}}, \mathscr{O}_{\mathscr{P}_{\overline{K}}})$, which follows from a similar argument to the one used in 8.52 using 3.12. $\square$

COROLLARY 8.54. *With notation as in 8.53, the image of ξ^r under the map*

$$(8.54.1) \qquad H^{2d}(\mathscr{Y}^o_{\overline{K}}, \overline{\mathscr{O}}_Y/p^s)(d) \xrightarrow{\pi_*} H^0(\mathscr{X}^o_{\overline{K}}, \mathscr{O}_{\mathscr{X}^o_{\overline{K}}}/p^s) \simeq \overline{V}/p^s\overline{V}$$

is equal to 1.

PROOF. Everything we have done is functorial with respect to morphisms $f : (X', D') \to (X, D)$ such that $f^{-1}(D) \subset D'$. After making a finite extension of V, we may assume that there exists a section $\mathrm{Spec}(V) \to X^o$. Base changing to $\mathrm{Spec}(V)$ it therefore suffices to consider the case when $X = \mathrm{Spec}(V)$ and D is trivial so $Y = \mathbb{P}^d$. $\qquad\square$

We can now finally prove 8.21 in general. Let $i : Y \hookrightarrow X$ be as in the theorem and let $\widetilde{X}$ denote the blow-up of X along Y. Let $\widetilde{D} \subset \widetilde{X}$ be the inverse image of D, and let $E \subset \widetilde{X}$ be the inverse image of Y so we have a cartesian square

$$
(8.54.2) \qquad
\begin{array}{ccc}
E & \xrightarrow{\;\tilde{i}\;} & \widetilde{X} \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\
Y & \xrightarrow{\;i\;} & X.
\end{array}
$$

Let $r + 1$ denote the codimension of Y in X so that E is isomorphic to the projective bundle associated to the rank $r + 1$–normal bundle $\mathscr{N}_{Y/X}$ on Y. Let $\xi \in H^{2r}(E^o_{\overline{K},\mathrm{et}}, \mathbb{Z}/p^s(r))$ denote the r-th power of the first Chern class of the tautological line bundle on $E = \mathbb{P}_Y(\mathscr{N}_{Y/X})$. Since $\pi_*\xi = 1$ (this follows for example from [**24**, VII.2.2.6], for a class $\alpha \in H^i_{E_Y, F_Y}(Y^0_{\overline{K},\mathrm{et}}, f^*u_X^*L)$ we then have

$$
(8.54.3) \qquad\qquad \alpha = \pi_*(\pi^*\alpha \cup \xi)
$$

and similarly for a class $\beta \in H^i(\mathscr{Y}^o_{\overline{K}}, f^*L \otimes \mathscr{J}_{E_Y})$ we have

$$
(8.54.4) \qquad\qquad \beta = \pi_*(\pi^*\beta \cup \xi).
$$

By [**24**, VII.8.4.3], in étale cohomology $H^*_{E_Y, F_Y}(Y^o_{\overline{K}}, i^*u_X^*L)$ we have $\pi^*i_*(x) = \tilde{i}_*(\pi^*(x) \cup \xi)$. Since the transformations σ_Y and σ_X commute with $\tilde{i}_*$ by the case of a smooth divisor already treated we get that $\pi^*\sigma_X(i_*x) = \tilde{i}_*(\pi^*(\sigma_Y(x)) \cup \xi)$. Since $\pi_*\pi^* = \mathrm{id}$ on $H^*(\mathscr{X}^o_{\overline{K}}, L \otimes \mathscr{J}_E)$ we can apply π_* to obtain

$$
(8.54.5) \qquad\qquad \sigma_X(i_*x) = \pi_*\tilde{i}_*(\pi^*(\sigma_Y(x)) \cup \xi).
$$

We then find that

$$
(8.54.6) \qquad
\begin{array}{rcl}
\sigma_X(i_*x) & = & \pi_*\tilde{i}_*(\pi^*(\sigma_Y(x)) \cup \xi) \\
& = & i_*\pi_*(\pi^*(\sigma_Y(x)) \cup \xi) \\
& = & i_*\sigma_Y(x).
\end{array}
$$

This completes the proof of 8.21. $\qquad\square$

8.55. The next step in the proof of 6.17 is to prove some facts about the behavior of cohomology under blowups of the boundary. So let $D_1, D_2 \subset D$ be two smooth divisors meeting tranversally, and let $\pi : \widetilde{X} \to X$ be the blow-up of their intersection. Let $\widetilde{D} \subset \widetilde{X}$ be the inverse image of D. The subscheme $\widetilde{D} \subset \widetilde{X}$ is also a divisor with simple normal crossings. In fact, $\widetilde{D}$ is equal to the union of the strict transforms of the components of D together with the exceptional divisor $N \subset \widetilde{X}$.

Now let $D = E \cup F$ be a decomposition of the divisor D such that $D_1 \in E$ and $D_2 \in F$, and let $\widetilde{D} = E_1 \cup F_1$ be the decomposition obtained by taking E_1 equal to the union of the strict transforms of the divisors in E together with N, and let $\widetilde{D} = E_2 \cup F_2$ be the decomposition obtained by taking E_2 to be the union

of the strict transforms of the components in E (so $N \subset F_2$). In $\mathscr{X}_{\overline{K}}^{\cdot o}$ we then have morphisms

$$(8.55.1) \qquad \pi^* \mathscr{J}_E \to \mathscr{J}_{E_2} \leftarrow \mathscr{J}_{E_1}.$$

PROPOSITION 8.56. *Let L be a sheaf of flat Λ-modules in $\mathscr{X}_{\overline{K}}^{\cdot o}$. If $\pi : \widetilde{\mathscr{X}_{\overline{K}}^{\cdot o}} \to \mathscr{X}_{\overline{K}}^{\cdot o}$ is the induced morphism of topoi and $\nu_X : \mathscr{X}_{\overline{K}}^{\cdot o} \to X_{\text{et}}$ the projection, then the maps*

$$(8.56.1) \qquad R\nu_{X*}L \otimes \mathscr{J}_E \to R\nu_{X*}R\pi_*\pi^*L \otimes \mathscr{J}_{E_2} \leftarrow R\nu_{X*}R\pi_*\pi^*L \otimes \mathscr{J}_{E_1}.$$

are all almost isomorphisms. In particular the natural maps

$$(8.56.2) \qquad H^*(\mathscr{X}_{\overline{K}}^{\cdot o}, L \otimes \mathscr{J}_E) \to H^*(\widetilde{\mathscr{X}_{\overline{K}}^{\cdot o}}, \pi^*L \otimes \mathscr{J}_{E_2}) \leftarrow H^*(\widetilde{\mathscr{X}_{\overline{K}}^{\cdot o}}, \pi^*L \otimes \mathscr{J}_{E_1})$$

are all almost isomorphisms.

PROOF. We may work étale locally on X, and hence can assume that $X = \text{Spec}(V[X_1, \ldots, X_r, X, Y])$ where D is defined by $(X_1 \cdots X_r XY)$ and $\widetilde{X}$ is the blow-up of the ideal (X, Y) with $D_1 = (X)$ and $D_2 = (Y)$. We can further assume that $E = D_1$ and by the projection formula that L is the trivial sheaf.

Write $R = V[X_1, \ldots, X_r, X, Y]$ and

$$(8.56.3) \qquad \widetilde{X} = \text{Proj}(R[U, V]/XV = YU)$$

and let $U_1 \subset \widetilde{X}$ be the open set

$$\text{Spec}(R[u]/X = Yu) \simeq \text{Spec}(V[X_1, \ldots, X_r, Y, u])$$

and let U_2 be the other open set

$$U_2 = \text{Spec}(R[v]/Xv = Y) \simeq \text{Spec}(V[X_1, \ldots, X_r, X, v]).$$

Let R_i denote the coordinate ring of U_i and let R_{12} denote the coordinate ring of the intersection $U_1 \cap U_2$. Let R_∞, $R_{i\infty}$ and $R_{12\infty}$ be the rings obtained by taking all the p^n–th power roots of the variables ($n \geq 1$), and let $J_{i,\infty} \subset R_{i,\infty}$ be the ideal defined by the roots of the variable X on U_2 and the variables Y and u on U_1 (the inverse image of D_1). Also let $J_\infty \subset R_\infty$ be the ideal defined by the roots of X. The ideals $J_{i,\infty}$ define the same ideal in $R_{12,\infty}$ which we denote simply by $J_{12,\infty}$. Also define $I_{i,\infty} \subset R_{i,\infty}$ to be the unit ideal on U_2 and the ideal defined by the p^n–th roots of u on U_1 (the strict transform of D_1). The ideals I_i define the same ideal in $R_{12,\infty}$ which we denote by $I_{12,\infty}$ (the unit ideal).

Note that the automorphism groups of $U_{i,\infty}^o$ (resp. $U_{12,\infty}^o$, $\text{Spec}(R_\infty)^o$) over U_i^o (resp. U_{12}^o, $\text{Spec}(R)^o$) are all isomorphic via the natural maps. Call this group Δ. Then the cohomology of $\mathscr{J}_E$ is given by $H^*(\Delta, J_\infty)$, the cohomology of $\mathscr{J}_{E_1}$ is given by the cohomology $H^*(\Delta, J_{1,\infty} \oplus J_{2,\infty} \to J_{12,\infty})$, and the cohomology of $\mathscr{J}_{E_2}$ is given by $H^*(\Delta, I_{1,\infty} \oplus I_{2,\infty} \to I_{12,\infty})$. Then the lemma follows from observing that the natural maps

$$(8.56.4) \qquad J_\infty \to (J_{1,\infty} \oplus J_{2,\infty} \to J_{12,\infty})$$

and

$$(8.56.5) \qquad J_\infty \to (I_{1,\infty} \oplus I_{2,\infty} \to I_{12,\infty})$$

are quasi–isomorphism which is an immediate verification. $\qquad\square$

8.57. The analogous result also holds for étale cohomology. Consider again the commutative diagram

$$
\begin{array}{ccc}
X - E & \xrightarrow{\ j_E\ } & X \\
{\scriptstyle j_F}\big\uparrow & & \big\uparrow{\scriptstyle j_F} \\
X - D & \xrightarrow{\ j_E\ } & X - F.
\end{array}
$$
(8.57.1)

Let L be a locally constant constructible sheaf on $X - D$. Set $M = Rj_{E*}j_{F!}L$. On $\widetilde{X}$ we can then also consider $M_1 = Rj_{E_1*}j_{F_1!}\pi^*L$ and $M_2 = Rj_{E_2*}j_{F_2!}\pi^*L$. We then have maps

$$\pi^*M \to M_2 \leftarrow M_1.$$
(8.57.2)

PROPOSITION 8.58. *The induced maps*

$$M \to R\pi_*M_2 \leftarrow R\pi_*M_1$$
(8.58.1)

are isomorphisms.

PROOF. By the proper base change theorem [**1**, XII.5.1] it suffices to show that for any geometric point $\bar{x} \to X$ with $e : N_{\bar{x}} = \widetilde{X} \times_X \bar{x} \to \widetilde{X}$ the fiber, the maps

$$M_{\bar{x}} \to H^*(N_{\bar{x}}, e^*M_2) \leftarrow H^*(N_{\bar{x}}, e^*M_1)$$
(8.58.2)

are isomorphisms. This is clear if $\bar{x}$ has image in the complement of $D_1 \cap D_2$ so it suffices to consider the case when $\bar{x}$ has image in $D_1 \cap D_2$. In this case $N_{\bar{x}}$ is isomorphic to $\mathbb{P}^1$, $M_{\bar{x}} = 0$, and e^*M_2 is also trivial. Thus the only issue is e^*M_1.

The pair (X, D) is étale locally isomorphic to $X_1 \times \mathbb{A}^2$ together with a divisor $D' \subset X_1$ such that $D_1 = X_1 \times \mathbb{A}^1 \times \{0\}$, $D_2 = X_1 \times \{0\} \times \mathbb{A}^1$, and $D = D_1 \cup D_2 \cup D' \times \mathbb{A}^2$. Furthermore, étale locally on X the sheaf L is isomorphic to the sheaf associated to a representation of $\widehat{\mathbb{Z}}(1)^r$ (the Galois group of the extension of X obtained by taking roots of the components of the divisor D). By considering the distinguished triangles obtained from filtering L it suffices to consider the case when L is a simple sheaf, and hence of the form L_μ for some character μ of $\widehat{\mathbb{Z}}(1)^d$ (notation as in 3.5). Furthermore, by the projection formula one reduces to the case when L is obtained by pullback via the second projection $X_1 \times \mathbb{G}_m^2 \to \mathbb{G}_m^2$ from a rank 1 sheaf on $\mathbb{G}_m^2$. By the proper base change theorem this reduces the proof to the case when $X = \mathbb{P}^1 \times \mathbb{P}^1$ with divisor $D = \{0 \times \mathbb{P}^1, \mathbb{P}^1 \times 0, \infty \times \mathbb{P}^1, \mathbb{P}^1 \times \infty\}$ and $D_1 = \infty \times \mathbb{P}^1$ and $D_2 = \mathbb{P}^1 \times \infty$.

With these assumptions, consider first the case of the constant sheaf Λ. The exceptional fiber $e : N \hookrightarrow \widetilde{X}$ is equal to $\mathbb{P}^1$ and we can compute the pullback $e^*j_{E_1!}Rj_{F_1*}\Lambda$ explicitly. Indeed let X' denote $X - E_1$ and consider the diagram

$$
\begin{array}{ccccc}
X' - N \cup F_1 & \xrightarrow{\ a\ } & X' - F_1 & \xrightarrow{\ b\ } & X' \\
& & {\scriptstyle e}\big\uparrow & & \big\uparrow{\scriptstyle e} \\
& & N - \{0, \infty\} & \xrightarrow{\ j_0\ } & N - \{\infty\}.
\end{array}
$$
(8.58.3)

Then $e^*j_{E_1!}Rj_{F_1*}\Lambda$ is isomorphic to the extension by zero along $N - \infty \hookrightarrow N$ of $e^*Rb_*Ra_*\Lambda$. By the proper base change theorem we have $e^*Rb_*Ra_*\Lambda \simeq Rj_{0*}e^*Ra_*\Lambda$. By cohomological purity [**1**, XVI.3.9] there is a distinguished triangle on $X' - F_1$

$$\cdots \to \Lambda \to Ra_*\Lambda \to e_*\Lambda(-1)[-1] \to \cdots$$
(8.58.4)

which upon applying e^* gives a distinguished triangle on $N - \{0, \infty\}$

$$(8.58.5) \qquad \cdots \to \Lambda \to e^* Ra_* \Lambda \to \Lambda(-1)[-1] \to \cdots .$$

Applying $j_{\infty!}$ and Rj_{0*} we obtain a distinguished triangle on N

$$(8.58.6) \qquad \cdots \to j_{\infty!} Rj_{0*} \Lambda \to e^* j_{E_1!} Rj_{F_1*} \Lambda \to j_{\infty!} Rj_{0*} \Lambda(-1)[-1] \to \cdots ,$$

where j_0 and j_∞ are the inclusions of the complements of the points $0, \infty \in \mathbb{P}^1$ respectively. Thus it suffices to show that $H_c^*(\mathbb{A}^1, Rj_{0*}\Lambda)$ is zero. For this consider again the distinguished triangle on $\mathbb{A}^1$ (where $i : \{0\} \hookrightarrow \mathbb{A}^1$ is the inclusion)

$$(8.58.7) \qquad \Lambda \to Rj_{0*}\Lambda \to i_*\Lambda(-1)[-1] \to \Lambda[1].$$

Consideration of the associated long exact sequence then gives the result.

For the case of a general sheaf L, we show that $H^i(N_{\bar{x}}, e^* M_1)$ is zero by induction on i. For the base case we take $i = -1$ where the result is vacuous.

For the inductive step, fix i and assume that $H^j(N_{\bar{x}}, e^* M_1) = 0$ for $j < i$ and any locally constant constructible sheaf L on $X - D$. We then show that $H^i(N_{\bar{x}}, e^* M_1) = 0$. By the same argument used above it suffices to consider the case when L has rank 1. Let N be an integer such that the pullback of L under the map $\rho : \mathbb{G}_m^2 \to \mathbb{G}_m^2$ given by multiplication by N on each factor is trivial. Then we get an inclusion

$$L \hookrightarrow \rho_* \Lambda$$

giving a short exact sequence

$$0 \to L \to \rho_* \Lambda \to Q \to 0.$$

This short exact sequence induces a distinguished triangle

$$Rj_{E_1*} j_{F_1!} \pi^* L \to Rj_{E_1*} j_{F_1!} \pi^* \rho_* \Lambda \to Rj_{E_1*} j_{F_1!} \pi^* Q \to Rj_{E_1*} j_{F_1!} \pi^* L[1]$$

and hence an exact sequence

$$H^{i-1}(N_{\bar{x}}, e^* Rj_{E_1*} j_{F_1!} \pi^* Q) \to H^i(N_{\bar{x}}, e^* Rj_{E_1*} j_{F_1!} \pi^* L) \to H^i(N_{\bar{x}}, e^* Rj_{E_1*} j_{F_1!} \pi^* \rho_* \Lambda).$$

Since $H^{i-1}(N_{\bar{x}}, e^* Rj_{E_1*} j_{F_1!} \pi^* Q) = 0$ by the induction hypothesis it therefore suffices to consider $L = \rho_* \Lambda$.

Let $\rho : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ also denote the map raising the coordinates to the N–th power. Then there is a commutative diagram

$$(8.58.8) \qquad
\begin{array}{ccccc}
\mathbb{G}_m^2 & \longrightarrow & \widetilde{\mathbb{P}^1 \times \mathbb{P}^1} & \overset{\pi}{\longrightarrow} & \mathbb{P}^1 \times \mathbb{P}^1 \\
\rho \downarrow & & \downarrow \rho & & \downarrow \rho \\
\mathbb{G}_m^2 & \longrightarrow & \widetilde{\mathbb{P}^1 \times \mathbb{P}^1} & \overset{\pi}{\longrightarrow} & \mathbb{P}^1 \times \mathbb{P}^1.
\end{array}$$

Using this one we are then reduced to the case of the constant sheaf Λ which was already shown. $\qquad \square$

8.59. We apply all this to complete the proof of 6.17 as follows. Let (X, D) be as in 8.2. Let Y be the blow-up of $X \times X$ along the intersections $D_i \times D_i$ of the divisors $D_i \times X$ and $X \times D_i$. Then the diagonal map $X \to X \times X$ induces a canonical morphism $i : X \hookrightarrow Y$ such that D meets the boundary divisor D_Y of Y (the inverse image of $D \times X \cup X \times D$) transversally. Let $\pi : Y \to X \times X$ be the projection. Let $D_Y = E_1 \cup F_1$ be the decomposition obtained by taking E_1 equal to the union of the strict transforms of the divisors $X \times D_i$, and let $D_Y = E_2 \cup F_2$ be the decomposition obtained by taking E_2 equal to the union of the exceptional

divisors with the strict transforms of the $X \times D_i$. Then cup product for étale cohomology is given by the composite

(8.59.1)
$$
\begin{array}{rcl}
H^*(X_{\overline{K}}^o, L) \otimes H_c^*(X_{\overline{K}}^o, L^*) & \simeq & H_{E_1, F_1}^*(Y_{\overline{K}}^o, L \boxtimes L^*) \\
& \underset{\simeq}{\simeq} & H_{E_2, F_2}^*(Y_{\overline{K}}^o, L \boxtimes L^*) \\
& \overset{i^*}{\rightarrow} & H_c^*(X_{\overline{K}}^o, \mathrm{End}(L)),
\end{array}
$$

where $L \boxtimes L^*$ denotes the pullback to Y of the sheaf $\mathrm{pr}_1^* L \otimes \mathrm{pr}_2^* L^*$ on $X \times X$. In other words, let $\delta_{\mathrm{et}} \in H_{F_2, E_2}^{2d}(Y_{\overline{K}}^o, L \boxtimes L^*)(d)$ denote the class $i_*(1)$, where $1 \in H^0(X_{\overline{K}}^o, \mathrm{End}(L))$ is the class corresponding to the identity map $L \to L$. Then the cup product of $\alpha \in H^i(X_{\overline{K}}^o, L)$ and $\beta \in H_c^{2d-i}(X_{\overline{K}}^o, L^*)$ is given by

(8.59.2) $\qquad (p_1^* \alpha \otimes p_2^* \beta) \cup \delta_{\mathrm{et}} \in H_c^{4d}(Y, \mathbb{Z}/p^r)(d) \overset{\mathrm{tr}}{\to} \mathbb{Z}/p^r(-d).$

Similarly, for $\alpha \in H^i(\mathscr{X}_{\overline{K}}^o, L \otimes \mathscr{O}_{\mathscr{X}_{\overline{K}}^o})$ and $\beta \in H^{2d-i}(\mathscr{X}_{\overline{K}}^o, L^* \otimes \mathscr{J}_X)$ we have

(8.59.3) $\qquad\qquad\qquad \alpha \cup \beta = (p_1^* \alpha \otimes p_2^* \beta) \cup \delta,$

where $\delta \in H^{2d}(\mathscr{Y}_{\overline{K}}^o, L \boxtimes L^* \otimes \mathscr{J}_{E_2})(d)$ is the class $i_*(1)$. If $\sigma : H_{F_2, E_2}^*(Y_{\overline{K}}, \mathbb{Z}/p^r) \to H^*(\mathscr{Y}_{\overline{K}}^o, \mathscr{J}_{F_2})$ is the transformation 8.19.2 we have by 8.21 that $\sigma(\delta_{\mathrm{et}}) = \delta$.

Now since the trace map in either theory has the property that it sends the cohomology class of a point in the interior to 1, the diagram

$$
\begin{array}{ccc}
H_{E,F}^i(X_{\overline{K}}^o, L) \otimes \overline{V}/(p^r) \times H_{F,E}^{2d-i}(X_{\overline{K}}^o, L^*) \otimes \overline{V}/(p^r) & \overset{\cup}{\longrightarrow} & H_c^{2d}(X_{\overline{K}}^o \otimes \overline{V}/(p^r) \\
\Big\downarrow{\scriptstyle \sigma} & & \Big\downarrow{\scriptstyle \mathrm{tr}} \\
& & \overline{V}/(p^r) \\
& & \Big\uparrow{\scriptstyle \mathrm{tr}} \\
H^i(\mathscr{X}_{\overline{K}}^o, L \otimes \mathscr{J}_E) \times H^{2d-i}(\mathscr{X}_{\overline{K}}^o, L^* \otimes \mathscr{J}_F) & \overset{\cup}{\longrightarrow} & H^i(\mathscr{X}_{\overline{K}}^o, \mathscr{J})
\end{array}
$$

commutes (somewhat abusively we say that σ is an "isometry").

Let $\gamma : H^i(\mathscr{X}_{\overline{K}}^o, L \otimes \mathscr{J}_E) \to H_{E,F}^i(X_{\overline{K}}^o, L) \otimes \overline{V}/p^r$ denote the adjoint of σ.

LEMMA 8.60. *For any $\alpha \in H^i(\mathscr{X}_{\overline{K}}^o, L \otimes \mathscr{J}_E)$ and $\beta \in H^j(\mathscr{X}_{\overline{K}}^o, L^* \otimes \mathscr{J}_F)$ we have*

(8.60.1) $\qquad\qquad \gamma(p_1^* \alpha \otimes p_2^* \beta) = p_1^*(\gamma(\alpha)) \otimes p_2^*(\gamma(\beta)).$

PROOF. Let $s \in H_{F_2, E_2}^{4d-i-j}(Y_{\overline{K}, \mathrm{et}}^o, L \boxtimes L^*) \otimes \overline{V}/p^r$ be a class in complementary degree. We show that the two elements in question cup with s to give the same answer. By Poincaré duality this shows that they are equal. In fact it suffices to consider s of the form $p_1^* s_1 \otimes p_2^* s_2$ with $s_1 \in H_{F,E}^{2d-i}(X_{\overline{K}, \mathrm{et}}^o, L) \otimes \overline{V}/p^r$ and $s_2 \in H_{E,F}^{2d-i}(X_{\overline{K}, \mathrm{et}}^o, L^*) \otimes \overline{V}/p^r$. Computing we find that
(8.60.2)
$\gamma(p_1^* \alpha \otimes p_2^* \beta) \cup s = (p_1^* \alpha \otimes p_2^* \beta) \cup (\sigma(p_1^*(s_1)) \otimes (\sigma(p_2^*(s_2)))) = p_1^*(\alpha \cup \sigma(s_1)) \otimes p_2^*(\beta \cup \sigma(s_2)).$

We also have

(8.60.3) $\quad (p_1^*(\gamma(\alpha)) \otimes p_2^*(\gamma(\beta))) \cup p_1^*(s_1) \otimes p_2^*(s_2) = p_1^*(\alpha \cup \sigma(s_1)) \otimes p_2^*(\beta \cup \sigma(s_2)).$

<div align="right">□</div>

8.61. Let $\alpha \in H^i(\mathscr{X}^{\text{o}}_{\overline{K}}, L \otimes \mathscr{O}_{\mathscr{X}^{\text{o}}_{\overline{K}}})$ be a class and $\beta \in H^{2d-i}(\mathscr{X}^{\text{o}}_{\overline{K}}, L^* \otimes \mathscr{J}_X)$ a class in complementary degree. Then

$$(8.61.1) \qquad \begin{aligned} \gamma(\alpha \cup \beta) &= \gamma(p_1^*\alpha \otimes p_2^*\beta \cup \sigma(\delta^{\text{et}})) \\ &= \gamma(p_1^*\alpha \otimes p_2^*\beta) \cup \delta^{\text{et}} \\ &= p_1^*\gamma(\alpha) \otimes p_2^*\gamma(\beta) \cup \delta^{\text{et}} \\ &= \gamma(\alpha) \cup \gamma(\beta). \end{aligned}$$

It follows that γ is also an isometry, and hence a monomorphism in every degree. Since $\gamma \circ \sigma$ is the identity it follows that both σ and γ are isomorphisms. This completes the proof of 6.17.

9. The topos $\widehat{\mathscr{X}^{\text{o}}_{\overline{K}}}$

9.1. Let (X, D) be as in 6.1 and assume in addition that X is proper over V. Let $\widehat{X}$ denote the formal scheme over $\text{Spf}(V)$ obtained by taking the p–adic completion of X and define another site $\widehat{\mathscr{S}}$ as follows. The objects of $\widehat{\mathscr{S}}$ are pairs (U, W) where $U \to \widehat{X}$ is an étale morphism of formal schemes with U affine, and $W \to \text{Spec}(\Gamma(U, \mathscr{O}_U) \otimes_V \overline{K})^{\text{o}}$ is a finite étale morphism of schemes. Morphisms $(U', W') \to (U, W)$ in $\widehat{\mathscr{S}}$ are commutative diagrams

$$(9.1.1) \qquad \begin{array}{ccc} W' & \xrightarrow{\;f\;} & W \\ \downarrow & & \downarrow \\ U' & \xrightarrow{\;g\;} & U, \end{array}$$

where g is an $\widehat{X}$–morphism. A collection of morphisms $\{(U_i, W_i) \to (U, W)\}$ is a covering if the collections $\{U_i \to U\}$ and $\{W_i \to W\}$ are étale coverings in the usual sense. We denote by $\widehat{\mathscr{X}^{\text{o}}_{\overline{K}}}$ the associated topos. There is a sheaf of rings $\mathscr{O}_{\widehat{\mathscr{X}^{\text{o}}_{\overline{K}}}}$ in $\widehat{\mathscr{X}^{\text{o}}_{\overline{K}}}$ given by

$$(9.1.2) \qquad (U, W) \mapsto \Gamma(\overline{W}, \mathscr{O}_{\overline{W}}),$$

where $\overline{W}$ denotes the normalization of $\text{Spec}(\Gamma(U, \mathscr{O}_U))$ in W. More generally, for any decomposition $D = E \cup F$ of the boundary we have a sheaf of ideals $\mathscr{J}_{\widehat{E}} \subset \mathscr{O}_{\widehat{\mathscr{X}^{\text{o}}_{\overline{K}}}}$ which to any (U, W) associates the ideal of the fiber product $(\overline{W} \times_{\text{Spec}(\Gamma(U, \mathscr{O}_U))} \widehat{E})_{\text{red}} \subset \overline{W}$, where $\widehat{E} \subset \widehat{X}$ denotes the p-adic completion of E.

9.2. There is a natural morphism of topoi

$$(9.2.1) \qquad p : \widehat{\mathscr{X}^{\text{o}}_{\overline{K}}} \to \mathscr{X}^{\text{o}}_{\overline{K}}$$

induced from the morphism of sites $\mathscr{S} \to \widehat{\mathscr{S}}$ sending a pair (U, W), with U affine, to the p–adic completion $\widehat{U}$ with the morphism $W \times_U \text{Spec}(\Gamma(\widehat{U}, \mathscr{O}_{\widehat{U}}) \otimes_V \overline{K})^{\text{o}} \to \text{Spec}(\Gamma(\widehat{U}, \mathscr{O}_{\widehat{U}}) \otimes_V \overline{K})^{\text{o}}$. For any decomposition $D = E \cup F$ there is a natural morphism $p^* \mathscr{J}_E \to \mathscr{J}_{\widehat{E}}$ inducing a morphism

$$(9.2.2) \qquad \mathscr{J}_E \to Rp_* \mathscr{J}_{\widehat{E}}.$$

Note also that the map $U \mapsto (U, \text{Spec}(\Gamma(U, \mathscr{O}_U))^{\text{o}}_{\overline{K}})$ induces a morphism of sites $\text{Et}(\widehat{X}) \to \widehat{\mathscr{S}}$ inducing a morphism of topoi $\nu_{\widehat{X}} : \widehat{\mathscr{X}^{\text{o}}_{\overline{K}}} \to \widehat{X}_{\text{et}}$ sitting in a commutative

diagram

$$(9.2.3) \qquad \begin{array}{ccc} \widehat{\mathscr{X}^o_{\overline{K}}} & \xrightarrow{\ p\ } & \mathscr{X}^o_{\overline{K}} \\ \nu_{\widehat{X}} \downarrow & & \downarrow \nu_X \\ \widehat{X}_{\mathrm{et}} & \xrightarrow{\ \ \ \ } & X_{\mathrm{et}}. \end{array}$$

For any affine $U = \mathrm{Spf}(R) \in \mathrm{Et}(\widehat{X})$ there is also a canonical morphism of topoi

$$(9.2.4) \qquad\qquad \epsilon_U : \widehat{\mathscr{X}^o_{\overline{K}}} \to \mathrm{Spec}(R \otimes \overline{K})^o_{\mathrm{Fet}}$$

induced by the morphism of sites sending $W \to \mathrm{Spec}(R \otimes \overline{K})^o$ to $(U, W) \in \widehat{\mathscr{S}}$.

PROPOSITION 9.3. *Let L be a locally constant sheaf of $\mathbb{Z}/(p^r)$-modules on $X^o_{\overline{K}, et}$, and let $\mathscr{L}$ (resp. $\widehat{\mathscr{L}}$) denote the induced sheaf on $\mathscr{X}^o_{\overline{K}}$ (resp. $\widehat{\mathscr{X}^o_{\overline{K}}}$). Then the natural map*

$$(9.3.1) \qquad\qquad p^* R\nu_{X*}\mathscr{L} \otimes \mathscr{J}_E \to R\nu_{\widehat{X}*}\widehat{\mathscr{L}} \otimes \mathscr{J}_{\widehat{E}}$$

is an almost isomorphism and for every i the sheaves

$$R^i \nu_{X*}(\mathscr{L} \otimes \mathscr{J}_E) \quad and \quad R^i \nu_{\widehat{X}*}\widehat{\mathscr{L}} \otimes \mathscr{J}_{\widehat{E}}$$

are almost isomorphic to coherent sheaves.

PROOF. We may work étale locally on X so can assume that $X = \mathrm{Spec}(R)$ and that there is a formally étale morphism $\mathrm{Spec}(R) \to \mathrm{Spec}(V[X_1, \ldots, X_d])$ as in 2.16.1. Let $\widehat{R}$ denote the p–adic completion of X, R_∞ the extension of R obtained by taking all the p^n–th roots of the variables X_i, and let $\widehat{R}_\infty$ denote the p–adic completion of $R_\infty \otimes_R \widehat{R}$. Let $J_\infty \subset R_\infty$ denote the ideal generated by the roots of the element $\prod_{i \in E} X_i$. Let $\Delta_\infty \simeq \mathbb{Z}_p(1)^d$ denote the Galois group of R_∞ over R, which is also the Galois group of $\widehat{R}_\infty$ over $\widehat{R}$. Then by 3.8 we may assume that L is obtained from a representation of Δ_∞ and then the morphism 9.3.1 is identified in $\widetilde{D}(\widehat{R})$ with the morphism of complexes of quasi–coherent sheaves associated to the morphism of complexes of $\widehat{R}$–modules

$$(9.3.2) \qquad R\Gamma_{\Delta_\infty}(L \otimes J_\infty) \otimes_R \widehat{R} \to R\Gamma_{\Delta_\infty}(L \otimes J_\infty \otimes_{R_\infty} \widehat{R}_\infty).$$

The proposition therefore follows from 3.10. $\qquad\qquad\qquad\qquad\qquad\square$

COROLLARY 9.4. *For any locally constant sheaf of $\mathbb{Z}/(p^r)$–modules L in $\mathscr{X}^o_{\overline{K}}$ the natural map*

$$(9.4.1) \qquad\qquad H^*(\mathscr{X}^o_{\overline{K}}, \mathscr{L} \otimes \mathscr{J}_E) \to H^*(\widehat{\mathscr{X}^o_{\overline{K}}}, p^*\mathscr{L} \otimes \mathscr{J}_{\widehat{E}})$$

is an almost isomorphism.

PROOF. Combine 9.3 with Grothendieck's comparison theorem [**11**, III.5.1.2].
$$\square$$

10. An aside on Galois cohomology

In order to compare étale cohomology to crystalline cohomology it is necessary to compute global cohomology using group cohomology. This is a standard application of simplicial techniques so we just sketch the necessary details here.

Let $\underline{\Delta}$ denote the standard simplicial category of finite order sets with order preserving morphisms.

10.1. Let us first recall the standard complex representing group cohomology. Let X be a connected normal scheme and let $\eta : \mathrm{Spec}(\Omega) \to X$ be a geometric point mapping to the generic point. Denote by Δ the fundamental group of X with respect to the base point η. Then the topos X_{Fet} is equivalent to the category of (discrete) sets with continuous action of the group Δ. It follows that for an abelian sheaf in X_{Fet} the cohomology groups $H^*(X_{\mathrm{Fet}}, L)$, or better the complex $R\Gamma(L)$, can be computed as follows.

Write also L for the continuous Δ–representation corresponding to the sheaf L. For an integer n let $\mathcal{C}^n(\Delta, L)$ denote the group

(10.1.1) $\qquad \mathcal{C}^n(\Delta, L) := \{\text{continuous } \Delta\text{–equivariant maps } \Delta^{n+1} \to L\},$

where Δ acts on Δ^{n+1} via the diagonal action on the left. For any morphism $\delta : [n] \to [m]$ there is an induced map

(10.1.2) $\qquad \Delta^{m+1} = \mathrm{Hom}([m], \Delta) \xrightarrow{\delta^*} \mathrm{Hom}([n], \Delta) = \Delta^{n+1}$

which induces a map $\delta^* : \mathcal{C}^n(\Delta, L) \to \mathcal{C}^m(\Delta, L)$. In this way $[n] \mapsto \mathcal{C}^n(\Delta, L)$ becomes a cosimplicial abelian group and we define $\mathcal{C}^\bullet(\Delta, L)$ to be the total complex of this cosimplicial group. It is well-known that this complex $\mathcal{C}^\bullet(\Delta, L)$ computes the group cohomology of L (see for example [**44**, 2.2]).

If $L^\bullet$ is a complex of Δ–modules, then we can apply the $\mathcal{C}^\bullet(\Delta, -)$ to each L^i to obtain a bicomplex. We denote by $\mathcal{C}^\bullet(\Delta, L^\bullet)$ the resulting total complex. Note that there is a canonical map of complexes $L^\Delta \to \mathcal{C}^\bullet(\Delta, L^\bullet)$.

10.2. The advantage of this construction is that it is functorial. Let $f : Y \to X$ be a morphism of connected normal schemes and $\epsilon : \mathrm{Spec}(\Omega) \to Y$ a lifting of η to Y. If Δ_Y (resp. Δ_X) denotes the fundamental group of Y (resp. X) with respect to the given geometric points, then there is an induced morphism $\Delta_Y \to \Delta_X$ of topological groups. This morphism induces a map

(10.2.1) $\qquad f^* : \mathcal{C}^n(\Delta_X, L) \to \mathcal{C}^n(\Delta_Y, L)$

compatible with the cosimplicial structure. The resulting map of complexes $f^* : \mathcal{C}^\bullet(\Delta_X, L) \to \mathcal{C}^\bullet(\Delta_Y, L)$ represents the pullback morphism on cohomology.

10.3. More generally, let $X = X_1 \coprod \cdots \coprod X_r$ be a disjoint union of connected normal schemes and assume that for each i we are given a geometric generic point $\eta_i : \mathrm{Spec}(\Omega) \to X_i$. We then define $\mathcal{C}^\bullet(\Delta, L)$ to be the product of the complexes $\mathcal{C}^\bullet(\Delta_{X_i}, L)$. This is again functorial with respect to morphisms of pointed schemes.

A convenient way to deal with non–connected schemes is as follows. If X is a disjoint union of normal schemes as above define a *base point* for X to be a disjoint union of geometric points $E = \coprod_{i=1}^r \mathrm{Spec}(\Omega)$ with a morphism of schemes $E \to X$ such that for each connected component X_i of X there exists a unique point of E mapping to the generic point of X_i. Then we can define $\mathcal{C}^\bullet(\Delta, L)$ using E and this complex is functorial with respect to commutative diagrams

(10.3.1)
$$
\begin{array}{ccc}
E_Y & \longrightarrow & Y \\
\downarrow & & \downarrow \\
E_X & \longrightarrow & X.
\end{array}
$$

In particular, if $U.$ is a simplicial scheme such that each U_n is a disjoint union of normal schemes and $E. \to U.$ is a morphism of simplicial schemes such that for

every n the morphism $E_n \to U_n$ is a base point, then we obtain a cosimplicial complex $\mathcal{C}^\bullet(U_\cdot, L)$.

If $X = \coprod X_i$ is a disjoint union of normal schemes as above and $E \to X$ is a base point, we define a *Galois module on X* to be the data of a continuous representation of each $\pi_1(X_i)$ (with respect to the base point given by E). For such a module L we can then define $\mathcal{C}^\bullet(\Delta, L)$

10.4. The main example we will consider is the following. Let X/V be a smooth proper geometrically connected V–scheme and $D \subset X$ a divisor with relative normal crossings. As usual we write X^o for $X - D$ and for any X–scheme U we write U^o for $U \times_X X^o$. Fix a geometric point $\mathrm{Spec}(\Omega) \to X$ mapping to the generic point of X. Let $U_\cdot \to X$ be a smooth hypercover by affine schemes such that for every n the scheme U_n is a disjoint union of open subschemes of X and such that each $U_{n,\overline{K}}^o$ is a $K(\pi, 1)$. Let $\pi_0(U_n)$ denote the set of connected components of $U_{n,\overline{K}}$. The set of connected components is functorial so we obtain a simplicial set $\pi_0(U_\cdot)$, and since each connected component of U_n is an open subset there is a canonical base point $\mathrm{Spec}(\Omega)^{\pi_0(U_\cdot)} \to U_\cdot$. Thus for a sheaf L on $U_{\cdot,\mathrm{Fet}}^o$ we obtain a complex $\mathcal{C}^\bullet(U_\cdot, L)$. We write $GC(U_\cdot, L)$ for the associated simple complex. If L is obtained from a locally constant constructible sheaf on $X_{\overline{K}}^o$ then since each $U_{n,\overline{K}}^o$ is a $K(\pi, 1)$ this complex computes $H^*(X_{\overline{K},\mathrm{et}}^o, L)$.

10.5. With $U_\cdot$ as in the previous paragraph, we will also need to consider the p–adic completion $U_\cdot^\wedge$ of $U_\cdot$. The scheme $U_n^\wedge$ is the spectrum of the p–adic completion of the coordinate ring of U_n. Let $\widehat{\Omega}$ be the p–adic completion of the field Ω. Then for any connected component P of $U_n^\wedge$ there exists a unique morphism $\mathrm{Spec}(\widehat{\Omega}) \to P$ such that the diagram

$$
(10.5.1) \qquad
\begin{array}{ccc}
\mathrm{Spec}(\widehat{\Omega}) & \longrightarrow & P \\
\downarrow & & \downarrow \\
\mathrm{Spec}(\Omega) & \longrightarrow & X
\end{array}
$$

commutes. We therefore also obtain a base point $E_\cdot \to U_\cdot^\wedge$. For a sheaf L on $U_{\cdot,\mathrm{Fet}}^\wedge$ we can then define the complex $GC(U_\cdot^\wedge, L)$ using this base point.

In what follows, we usually omit the initial choice of base point from the discussion. The careful reader should make the straightforward verification that all the results are independent of the choice of the geometric generic point of X.

REMARK 10.6. When dealing with algebraic spaces or Deligne–Mumford stacks, one must take extra care in dealing with the choice of base point as one cannot work Zariski topology and instead must work with the étale topology. The necessary technicalities for dealing with this problem are discussed in [**41**, §4].

11. Fontaine's big rings

11.1. Let R be a $\overline{V}$–algebra which is an integral domain. Choose an algebraic closure $\mathrm{Frac}(R) \hookrightarrow \Omega$ and assume that Frobenius on R/pR is surjective and that $\mathrm{Spec}(R/pR)$ is connected and nonempty. Then Fontaine's theory gives rings $\mathrm{A_{cris}}(R)$ and $\mathrm{B_{cris}}(R)$ defined as follows. First set

$$
(11.1.1) \qquad\qquad S_R := \varprojlim R/pR
$$

where the projective limit is taken with respect to the Frobenius morphism on R/pR (S_R is the *perfection* of R/pR). Since S_R is perfect, the ring of Witt vectors $W(S_R)$ has a canonical lift of Frobenius. An element $x \in W(S_R)$ can be represented by a vector $(x_0, x_1, x_2, \dots)$ where each $x_i = (x_{i0}, x_{i1}, \dots)$ is an infinite vector with $x_{ij} \in R/pR$ and such that $x_{i(j+1)}^p = x_{ij}$. There is a natural map

$$(11.1.2) \qquad\qquad \theta : W(S_R) \longrightarrow \widehat{R}$$

defined by sending x as above to

$$(11.1.3) \qquad \theta(x) = \varinjlim_m (\tilde{x}_{0m}^{p^m} + p\tilde{x}_{1m}^{p^{m-1}} + \cdots + p^m \tilde{x}_{mm}),$$

where $\tilde{x}_{ij} \in \widehat{R}$ is any lift of x_{ij}. Here $\widehat{R}$ denotes the p–adic completion of R. The assumption that Frobenius on R/pR is surjective ensures that the map θ is surjective [**49**, A1.1].

For $n \geq 0$ we write $W_n(S_R)$ for the ring of Witt vector of length $n+1$.

We set $J = \mathrm{Ker}(\theta)$ and define $\mathrm{A}_{\mathrm{cris}}(R)$ to be the p–adic completion of the divided power envelope $D_J(W(S_R))$. We thus obtain a diagram

$$(11.1.4) \qquad\qquad \begin{array}{ccc} \mathrm{Spec}(\widehat{R}) & \xrightarrow{\ \theta\ } & \mathrm{Spec}(\mathrm{A}_{\mathrm{cris}}(R)) \\ \downarrow & & \\ \mathrm{Spec}(R). & & \end{array}$$

Let $\mathrm{Fil}^{\cdot}\mathrm{A}_{\mathrm{cris}}(R)$ denote the filtration on $\mathrm{A}_{\mathrm{cris}}(R)$ obtained as the p–adic completion of the filtration on $D_J(W(S_R))$ defined by the PD–ideals $J^{[r]} \subset D_J(W(S_R))$. For an element $\alpha \in \mathrm{Fil}^1\mathrm{A}_{\mathrm{cris}}(R)$ and an integer $r \geq 1$ we write $\alpha^{[r]} \in \mathrm{Fil}^r\mathrm{A}_{\mathrm{cris}}(R)$ for the element obtained from the r-th divided power operation on $J^{[1]}$.

Choose elements $\epsilon_m \in \widehat{R}$ with $\epsilon_0 = 1$, $\epsilon_{m+1}^p = \epsilon_m$, and $\epsilon_1 \neq 1$. Let $\epsilon \in S_R$ denote the element obtained from the reductions of the ϵ_i, and let $[\epsilon] \in W(S_R)$ be the Teichmuller lift of ϵ. Set $\pi_\epsilon := [\epsilon] - 1 \in W(S_R)$. Then $\pi_\epsilon \in \mathrm{Fil}^1\mathrm{A}_{\mathrm{cris}}(R)$ so we can define

$$(11.1.5) \qquad t = \log([\epsilon]) = \sum_{m \geq 1} (-1)^{m-1} (m-1)! \pi_\epsilon^{[m]} \in \mathrm{A}_{\mathrm{cris}}(R).$$

Define

$$(11.1.6) \qquad\qquad \mathrm{B}_{\mathrm{cris}}(R)^+ := \mathrm{A}_{\mathrm{cris}}(R) \otimes \mathbb{Q},$$

and

$$(11.1.7) \qquad\qquad \mathrm{B}_{\mathrm{cris}}(R) := \mathrm{B}_{\mathrm{cris}}(R)^+[\tfrac{1}{t}].$$

In fact, by [**49**, A3.2] we have $t^{p-1} \in p\mathrm{A}_{\mathrm{cris}}(R)$ and therefore $\mathrm{B}_{\mathrm{cris}}(R) = \mathrm{A}_{\mathrm{cris}}(R)[1/t]$. The element $t \in \mathrm{A}_{\mathrm{cris}}(R)$ lies in $\mathrm{Fil}^1\mathrm{A}_{\mathrm{cris}}(R)$ and we obtain a filtration $\mathrm{Fil}_{\mathrm{B}_{\mathrm{cris}}(R)}$ on $\mathrm{B}_{\mathrm{cris}}(R)$ by declaring that $1/t$ has degree -1.

More canonically, we can describe this localization as follows. The group $\mathbb{Z}_p(1)$ is by definition the group of sequences $(\zeta_n)_{n \geq 0}$, where ζ_n is a p^n–th root of 1 and $\zeta_{n+1}^p = \zeta_n$. The above construction then induces a homomorphism

$$(11.1.8) \qquad\qquad \alpha : \mathbb{Z}_p(1) \to \mathrm{A}_{\mathrm{cris}}(R)^*$$

whose image is contained in the elements mapping to 1 in $\widehat{R}$. Taking the logarithm of this map we obtain a map

$$(11.1.9) \qquad\qquad \beta : \mathbb{Q}_p(1) \to \mathrm{B}_{\mathrm{cris}}(R)$$

whose image is the line spanned by t. This map induces for every i an isomorphism

$$(11.1.10) \qquad\qquad \beta^{\otimes i} : \mathrm{B}_{\mathrm{cris}}(R)(i) \to \mathrm{B}_{\mathrm{cris}}(R).$$

11.2. The ring $\mathrm{A}_{\mathrm{cris}}(R)$ has a lift of Frobenius $\varphi_{\mathrm{A}_{\mathrm{cris}}(R)}$ induced by the canonical lift of Frobenius to $W(S_R)$. This lifting of Frobenius induces a semi–linear automorphism $\varphi_{\mathrm{B}_{\mathrm{cris}}(R)}$ of $\mathrm{B}_{\mathrm{cris}}(R)$ and preserves the filtration $\mathrm{Fil}_{\mathrm{B}_{\mathrm{cris}}(R)}$.

Also define a second filtration $I^{[\cdot]}$ on $\mathrm{A}_{\mathrm{cris}}(R)$ by

$$(11.2.1) \qquad I^{[r]} := \{ x \in \mathrm{A}_{\mathrm{cris}}(R) | \varphi^n(x) \in \mathrm{Fil}^r \mathrm{A}_{\mathrm{cris}}(R) \text{ for all } n \geq 0 \}.$$

LEMMA 11.3. *Let S be a p-adically complete and separated ring flat over $\mathbb{Z}_p$. Let $\xi \in S$ be an element whose image in S/pS is a regular element. Then*
(i) *ξ is a regular element of S.*
(ii) *The sequence*

$$(11.3.1) \qquad 0 \longrightarrow S\langle u \rangle \xrightarrow{\ \xi - u\ } S\langle u \rangle \xrightarrow{\ u \mapsto \xi\ } S \otimes \mathbb{Q}$$

is exact where $S\langle u \rangle$ denotes the free PD–polynomial algebra on one variable.

PROOF. For (i) suppose $\beta \in S$ is an element with $\xi\beta = 0$. We show by induction on n that β is in $p^n S$ for all n. This implies that $\beta = 0$ since $\cap_{n \geq 1} p^n S = 0$. So assume that $\beta \in p^n S$ and write $\beta = p^n \beta'$. Then $\xi\beta' = 0$ since S is flat over $\mathbb{Z}_p$ and $p^n \xi\beta' = \xi\beta = 0$. Since ξ maps to a regular element in S/pS it follows that $\beta' \in pS$ and therefore $\beta \in p^{n+1} S$.

For (ii) note that the injectivity of multiplication by $\xi - u$ on $S\langle u \rangle$ follows from (i): If $f = \sum_{i \geq i_0} a_i u^{[i]}$ is an element with $a_{i_0} \neq 0$ then

$$(11.3.2) \qquad\qquad (\xi - u)f = a_{i_0} \xi u^{[i_0]} + \sum_{i > i_0} b_i u^{[i]}$$

for some $b_i \in S$. Since $\xi a_{i_0} \neq 0$ by (i) it follows that $(\xi - u)f \neq 0$.

For exactness in the middle of 11.3.1, suppose $f = \sum_i a_i u^{[i]}$ maps to zero in $S \otimes \mathbb{Q}$. We show by induction on the smallest integer n for which f can be written as

$$(11.3.3) \qquad\qquad f = a_{i_0} u^{[i_0]} + \cdots + a_{i_0+n} u^{[i_0+n]}$$

that f is in the image of $(\xi - u)$. So consider f mapping to zero in $S \otimes \mathbb{Q}$ and write f in the form 11.3.3. Since f maps to zero in $S \otimes \mathbb{Q}$ we have

$$(11.3.4) \qquad\qquad a_{i_0} \frac{\xi^{i_0}}{i_0!} + \cdots + a_{i_0+n} \frac{\xi^{i_0+n}}{(i_0+n)!} = 0$$

in $S \otimes \mathbb{Q}$, and since S is flat over $\mathbb{Z}_p$ this implies that

$$(11.3.5) \qquad\qquad \frac{(i_0+n)!}{i_0!} a_{i_0} \xi^{i_0} + \cdots + a_{i_0+n} \xi^{i_0+n} = 0$$

in S (note that the coefficients in this expression are integers). Since ξ is a regular element in S this implies that

$$(11.3.6) \qquad\qquad \frac{(i_0+n)!}{i_0!} a_{i_0} = \xi\beta$$

for some $\beta \in S$. Furthermore, since ξ maps to a regular element in S/pS it follows that $\beta = \frac{(i_0+n)!}{i_0!}\beta'$ for some $\beta' \in S$ and hence by the flatness of S over $\mathbb{Z}_p$ we have $a_{i_0} = \xi\beta'$ for some $\beta' \in S$. Subtracting $(\xi - u)\beta'u^{[i_0]}$ from f we are then reduced to the case when f can be written in the form

$$(11.3.7) \qquad f = a_{i_0}u^{[i_0+1]} + \cdots + a_{i_0+n-1}u^{[i_0+1+n-1]}$$

which by induction on n completes the proof of (ii). $\qquad\square$

11.4. Choose $\nu_m \in R$ such that $\nu_0 = -p$ and $\nu_{m+1}^p = \nu_m$ (note that this is possible since R is a $\overline{V}$–algebra). Let $\underline{-p} \in S_R$ denote the element defined by the reductions $\nu_m \pmod{p}$. Let $[\underline{-p}] \in \overline{W}(S_R)$ denote the Teichmuller lifting of $\underline{-p}$, and let $\xi := [\underline{-p}] + p$. By [**49**, A2.2] the element ξ generates the kernel of $\theta : W(S_R) \to \widehat{R}$ and ξ satisfies the assumptions of 11.3. In particular the sequence

$$(11.4.1) \qquad 0 \longrightarrow W(S_R)\langle u \rangle \xrightarrow{\xi - u} W(S_R)\langle u \rangle \xrightarrow{\tau} W(S_R) \otimes \mathbb{Q}$$

is exact, where τ is the map sending $u^{[i]}$ to $\xi^i/i!$.

As before let $D_J(W(S_R))$ denote the divided power envelope of the surjection $W(S_R) \to \widehat{R}$. By [**49**, A2.8] there exists a unique inclusion $D_J(W(S_R)) \hookrightarrow W(S_R)\otimes\mathbb{Q}$ whose composite with $W(S_R) \to D_J(W(S_R))$ is the inclusion $W(S_R) \hookrightarrow W(S_R) \otimes \mathbb{Q}$, and via this inclusion $D_J(W(S_R))$ is identified with the image of τ. It follows from this that $D_J(W(S_R))$ is p-torsion free, and that for every n there is an exact sequence

$(11.4.2)$
$$0 \longrightarrow W(S_R)/p^n\langle u \rangle \xrightarrow{\xi - u} W(S_R)/p^n\langle u \rangle \xrightarrow{\tau} D_J(W(S_R))/p^n \longrightarrow 0.$$

Also for all integers $r < n$ we obtain a commutative diagram with exact rows
$(11.4.3)$

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \\
& & \downarrow & & \downarrow & & & & \\
0 & \to & W(S_R)/p^n\langle u \rangle & \xrightarrow{\xi-u} & W(S_R)/p^n\langle u \rangle & \xrightarrow{\tau} & D_J(W(S_R))/p^n & \to & 0 \\
& & \downarrow{\scriptstyle \times p^r} & & \downarrow{\scriptstyle \times p^r} & & \downarrow{\scriptstyle \times p^r} & & \\
0 & \to & W(S_R)/p^{n+r}\langle u \rangle & \xrightarrow{\xi-u} & W(S_R)/p^{n+r}\langle u \rangle & \xrightarrow{\tau} & D_J(W(S_R))/p^{n+r} & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & W(S_R)/p^r\langle u \rangle & \xrightarrow{\xi-u} & W(S_R)/p^r\langle u \rangle & \xrightarrow{\tau} & D_J(W(S_R))/p^r & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

By the snake lemma and the fact that the first two columns are exact we conclude that the sequence
$(11.4.4)$

$$0 \longrightarrow D_J(W(S_R))/p^n \xrightarrow{\times p^r} D_J(W(S_R))/p^{n+r} \longrightarrow D_J(W(S_R))/p^r \longrightarrow 0$$

is also exact. As n varies the short exact sequences 11.4.4 form an exact sequence of projective systems and passing to the limit we obtain an exact sequence

(11.4.5) $\quad 0 \longrightarrow \mathrm{A}_{\mathrm{cris}}(R) \xrightarrow{\times p^r} \mathrm{A}_{\mathrm{cris}}(R) \longrightarrow D_J(W(S_R))/p^r \longrightarrow 0.$

In particular, we can rewrite 11.4.2 as

(11.4.6)

$\quad 0 \longrightarrow W(S_R)/p^n\langle u\rangle \xrightarrow{\xi-u} W(S_R)/p^n\langle u\rangle \xrightarrow{\tau} \mathrm{A}_{\mathrm{cris}}(R)/p^n \longrightarrow 0.$

LEMMA 11.5. *For any integer n the natural map*

(11.5.1) $\qquad W_n(S_R) \otimes_{W_n(S_{\overline{V}})} \mathrm{A}_{\mathrm{cris}}(\overline{V}) \to \mathrm{A}_{\mathrm{cris}}(R)/p^{n+1}$

is an isomorphism.

PROOF. This follows from the exactness of 11.4.6 and consideration of the commutative diagram

$$0$$
$$\downarrow$$

(11.5.2)

$$W(S_R)/p^{n+1}\langle u\rangle \xleftarrow{\quad a \quad} W_n(S_R)_n \otimes_{W_n(S_{\overline{V}})} W_n(S_{\overline{V}})\langle u\rangle$$
$$\xi-u \downarrow \qquad\qquad\qquad \downarrow \xi-u$$
$$W(S_R)/p^{n+1}\langle u\rangle \xleftarrow{\quad b \quad} W_n(S_R)_n \otimes_{W_n(S_{\overline{V}})} W_n(S_{\overline{V}})\langle u\rangle$$
$$\tau \downarrow \qquad\qquad\qquad \downarrow$$
$$\mathrm{A}_{\mathrm{cris}}(R)/p^{n+1} \xleftarrow{\qquad} W_n(S_R) \otimes_{W_n(S_{\overline{V}})} \mathrm{A}_{\mathrm{cris}}(\overline{V})/p^{n+1}$$
$$\downarrow \qquad\qquad\qquad \downarrow$$
$$0 \qquad\qquad\qquad 0,$$

where the maps labelled a and b are clearly isomorphisms. $\qquad\square$

LEMMA 11.6. *For any integer $n \geq 0$ the map*

(11.6.1) $\qquad R^\wedge \to \mathrm{gr}^n_{\mathrm{Fil}_{\mathrm{A}_{\mathrm{cris}}}} \mathrm{A}_{\mathrm{cris}}(R^\wedge),\ f \mapsto f \cdot \xi^{[n]}$

is an isomorphism. In particular, the natural map

(11.6.2) $\qquad \mathrm{gr}_{\mathrm{Fil}_{\mathrm{A}_{\mathrm{cris}}}} \mathrm{A}_{\mathrm{cris}}(\overline{V}) \otimes_{\overline{V}^\wedge} R^\wedge \to \mathrm{gr}_{\mathrm{Fil}_{\mathrm{A}_{\mathrm{cris}}}} \mathrm{A}_{\mathrm{cris}}(R^\wedge)$

is an isomorphism.

PROOF. This is shown in [**49**, A2.11]. $\qquad\square$

11.7. Since the image of ξ in $\mathrm{A}_{\mathrm{cris}}(\overline{V})$ is in the divided power ideal of $\mathrm{A}_{\mathrm{cris}}(\overline{V})$, the map

(11.7.1) $\qquad W(S_{\overline{V}})/(p) \to \mathrm{A}_{\mathrm{cris}}(\overline{V})/p$

factors through $W(S_{\overline{V}})/(p, \xi^p)$ since $\xi^p = p!\xi^{[p]} = 0$ in $\mathrm{A}_{\mathrm{cris}}(\overline{V})/p$.

LEMMA 11.8. *Let $i \in \mathbb{N}$ be an integer. For any integer $1 \leq j \leq p$ the map*

(11.8.1) $\qquad \times \xi^{[ip]} : W(S_{\overline{V}})/(p, \xi^j) \to J^{[ip]}/(p, J^{[ip+j]})$

is an isomorphism, where $J = \mathrm{Ker}(\theta : \mathrm{A}_{\mathrm{cris}}(\overline{V}) \to \overline{V}^\wedge)$.

PROOF. The case $j = 1$ follows from [**49**, A2.9]. The general case follows by induction and consideration of the diagrams

(11.8.2)

$$0 \longrightarrow W(S_{\overline{V}})/(p, \xi) \xrightarrow{\xi^j} W(S_{\overline{V}})/(p, \xi^{j+1}) \longrightarrow W(S_{\overline{V}})/(p, \xi^j) \longrightarrow 0$$

$$\downarrow \simeq \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$0 \longrightarrow J^{[ip]}/(p, J^{[ip+1]}) \xrightarrow{\xi^j} J^{[ip]}/(p, J^{[ip+j+1]}) \longrightarrow J^{[ip]}/(p, J^{[ip+j]}) \longrightarrow 0.$$

Here the top row is exact since ξ maps to a regular element in $W(S_{\overline{V}})/(p)$, and the bottom row is exact since for $j < p$ we have

$$\xi^{[ip+j]} = \frac{(ip)!}{(ip+j)!} \xi^j \cdot \xi^{[ip]},$$

and $(ip)!/(ip+j)!$ has p-adic valuation 0. □

LEMMA 11.9. *The map of $W(S_{\overline{V}})/(p, \xi^p)$–modules*

(11.9.1) $$\oplus_{i \geq 0} W(S_{\overline{V}})/(p, \xi^p) \xrightarrow{\oplus \xi^{[ip]}} \mathrm{A}_{\mathrm{cris}}(\overline{V})/p$$

is an isomorphism. In particular, $\mathrm{A}_{\mathrm{cris}}(\overline{V})/p$ is flat over $W(S_{\overline{V}})/(p, \xi^p)$.

PROOF. The map is clearly surjective. That it is injective follows from the preceding lemma. □

11.10. In what follows we will need to consider "almost mathematics" over the ring $\mathrm{B}_{\mathrm{cris}}(\overline{V})$. Let $\Lambda \subset \mathbb{Q}$ be the subring $\mathbb{Z}[1/p]$, and let $\Lambda_+ := \Lambda \cap \mathbb{Q}_{>0}$. Fix a sequence $(\tau_m)_{m \geq 0}$ of elements of $\overline{V}$ with $\tau_0 = p$ and $\tau_{m+1}^p = \tau_m$ for all $m \geq 0$. We define $\tau_m := 0$ for $m < 0$. Then for any $n \in \mathbb{Z}$ define λ_{1/p^n} to be the element $(a_m)_{m \geq 0} \in S_{\overline{V}}$ with

(11.10.1) $$a_m = \tau_{m+n}.$$

Note that $\lambda_{1/p^n}^p = \lambda_{1/p^{n-1}}$.

LEMMA 11.11. *For every $m \in \mathbb{Z}$ the map*

(11.11.1) $$\times \lambda_{1/p^n} : S_{\overline{V}} \to S_{\overline{V}}$$

is injective.

PROOF. Let $F : S_{\overline{V}} \to S_{\overline{V}}$ be the Frobenius morphism (which is a bijection). Then $F(\lambda_{1/p^n}) = \lambda_{1/p^{n-1}}$ so the diagram

(11.11.2)
$$\begin{array}{ccc} S_{\overline{V}} & \xrightarrow{\cdot \lambda_{1/p^n}} & S_{\overline{V}} \\ F \uparrow & & \uparrow F \\ S_{\overline{V}} & \xrightarrow{\cdot \lambda_{1/p^{n+1}}} & S_{\overline{V}} \end{array}$$

commutes. It follows that multiplication by λ_{1/p^n} is injective if and only if multiplication by either $\lambda_{1/p^{n-1}}$ or $\lambda_{1/p^{n+1}}$ is injective. It follows that it suffices to consider the case $n = 0$. Now with the notation of [**49**, A2] we have $-\lambda_1 = -p$, and therefore the result in the case $n = 0$ follows from [**49**, A2.1]. □

11.12. For $n \in \mathbb{Z}$, let $\delta_{1/p^n} \in W(S_{\overline{V}})$ denote the Teichmuller lifting $[\lambda_{1/p^n}]$ of λ_{1/p^n}. Since the Teichmuller lifting is multiplicative we have

$$(11.12.1) \qquad\qquad \delta_{1/p^n}^p = \sigma(\delta_{1/p^n}) = \delta_{1/p^{n-1}},$$

where $\sigma : W(S_{\overline{V}}) \to W(S_{\overline{V}})$ is the canonical lifting of Frobenius.

Since $W(S_{\overline{V}})$ is p-torsion free and p-adically separated, it follows from 11.11 that

$$(11.12.2) \qquad\qquad \times \, \delta_{1/p^n} : W(S_{\overline{V}}) \to W(S_{\overline{V}})$$

is injective.

For $\alpha \in \Lambda_+$, we define $\delta_\alpha \in W(S_{\overline{V}})$ as follows. Write $\alpha = s/p^n$ for some $s \in \mathbb{Z}_{>0}$ and $n \in \mathbb{Z}$, and define

$$(11.12.3) \qquad\qquad \delta_\alpha := \delta_{1/p^n}^s.$$

Note that if $s = p^r s'$ with $(s', p) = 1$ then

$$(11.12.4) \qquad\qquad \delta_{1/p^n}^s = (\delta_{1/p^n}^{p^r})^{s'} = \delta_{1/p^{n-r}}^{s'},$$

which implies that this definition of δ_α is independent of how we write α as a fraction s/p^n.

Similarly we can define $\lambda_\alpha \in S_{\overline{V}}$ for any $\alpha \in \Lambda_+$. Note that since δ_{1/p^n} (resp. λ_{1/p^n}) is not a zero divisor in $W(S_{\overline{V}})$ (resp. $S_{\overline{V}}$), the element δ_α (resp. λ_α) is not a zero divisor in $W(S_{\overline{V}})$ (resp. $S_{\overline{V}}$) either.

LEMMA 11.13. *For every $\alpha, \beta \in \Lambda_+$ we have $\lambda_\alpha \cdot \lambda_\beta = \lambda_{\alpha+\beta}$ and $\delta_\alpha \cdot \delta_\beta = \delta_{\alpha+\beta}$.*

PROOF. Since the Teichmuller lifting is multiplicative, it suffices to prove the statement about the λ's. Write $\alpha = s/p^m$ and $\beta = t/p^m$ (note that without loss of generality we may assume that the denominators are the same). Then by definition

$$(11.13.1) \qquad \lambda_\alpha = \lambda_{1/p^m}^s, \quad \lambda_\beta = \lambda_{1/p^m}^t, \quad \text{and} \quad \lambda_{\alpha+\beta} = \lambda_{1/p^m}^{s+t}$$

which implies the result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

11.14. For $\alpha \in \Lambda_+$ let $\mathfrak{m}_\alpha \subset W(S_{\overline{V}})$ denote the ideal generated by δ_α, and let $\mathfrak{m} := \cup_{\alpha \in \Lambda_+} \mathfrak{m}_\alpha$. We can then apply the almost theory of section 2 for modules over $W(S_{\overline{V}})$.

Since $A_{\mathrm{cris}}(\overline{V})$ is an integral domain, for any $\alpha \in \Lambda_+$ the element δ_α is also not a zero divisor in $A_{\mathrm{cris}}(\overline{V})$ and $B_{\mathrm{cris}}(\overline{V})$.

Thus we can also apply the almost theory of section 2 with $B_{\mathrm{cris}}(\overline{V})$ and the ideals $\mathfrak{m}_{\mathrm{cris},\alpha} \subset B_{\mathrm{cris}}(\overline{V})$ generated by δ_α. In what follows, when we consider almost mathematics of modules over the ring $B_{\mathrm{cris}}(\overline{V})$ (resp. $W(S_{\overline{V}})$) it will always be with respect to the ideals $\mathfrak{m}_{\mathrm{cris},\alpha}$ (resp. $\mathfrak{m}_\alpha$).

Let $\widetilde{B}_{\mathrm{cris}}(\overline{V})$ denote $B_{\mathrm{cris}}(\overline{V})[\delta_\alpha^{-1}]_{\alpha \in \Lambda_+}$.

11.15. The action of G_K on $\overline{V}$ induces an action of G_K on $S_{\overline{V}}$, $W(S_{\overline{V}})$, and $A_{\mathrm{cris}}(\overline{V})$ by functoriality. Let $\chi : G_K \to \mathbb{Z}_p^*$ denote the cyclotomic character. Then it follows from the construction that G_K acts on t by

$$(11.15.1) \qquad\qquad g * t = \chi(g)t.$$

In particular, the action of G_K on $A_{\mathrm{cris}}(\overline{V})$ induces an action on $B_{\mathrm{cris}}(\overline{V})$. Also the choice of the elements τ_m defines a homomorphism

$$(11.15.2) \qquad\qquad \rho : G_K \to \mathbb{Z}_p(1) = \varprojlim_n \mu_{p^n}.$$

If $g \in G_K$ then the image $\rho(g) = (\zeta_n)$ in $\mathbb{Z}_p(1)$ is characterized by the equality

(11.15.3) $$g(\tau_n) = \zeta_n \tau_n.$$

One verifies immediately from the construction that for $g \in G_K$ we have

(11.15.4) $$g * \delta_1 = \alpha(\rho(g)) \cdot \delta_1,$$

where α is the map 11.1.8. In particular, the G_K-action on $\mathrm{B}_{\mathrm{cris}}(\overline{V})$ induces an action of G_K on $\widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V})$.

11.16. We will apply the preceding discussion as follows. Let X/V be a smooth proper scheme and $D \subset X$ a divisor with normal crossings relative to V. For any étale morphism $\mathrm{Spec}(R) \to X$ for which there exists a morphism as in 2.16.1, let $R^\wedge$ denote the p–adic completion of R, and let $\overline{R}^\wedge$ denote the integral closure of $R^\wedge$ in the maximal subextension of the field of fractions of $R^\wedge$ which is unramified over $\mathrm{Spec}(R^\wedge) \times_X X^o_{\overline{K}}$. We then apply the preceding discussion to $\overline{R}^\wedge$, and denote the resulting ring by $\mathrm{A}_{\mathrm{cris}}(R^\wedge)$. Note that there is a natural action of $\mathrm{Gal}(\overline{R}^\wedge/R^\wedge)$ on $S := S_{\overline{R}^\wedge}$ which induces an action of $\mathrm{Gal}(\overline{R}^\wedge/R^\wedge)$ on $\mathrm{A}_{\mathrm{cris}}(R^\wedge)$ which in turn induces an action $\rho_{\mathrm{B}_{\mathrm{cris}}(R^\wedge)}$ of Galois on $\mathrm{B}_{\mathrm{cris}}(R^\wedge)$. This action is continuous and compatible with the filtration. Furthermore the induced action on the diagram 11.1.4 commutes with the lift of Frobenius.

If $s : \mathrm{Spec}(\Omega') \to \mathrm{Spec}(\Omega)$ is a morphism of geometric generic points of $\mathrm{Spec}(R)$, then there is a natural isomorphism

(11.16.1) $$\iota_s : s^* \mathrm{B}_{\mathrm{cris}}(R^\wedge) \to \mathrm{B}_{\mathrm{cris}}(R^\wedge)',$$

where $\mathrm{B}_{\mathrm{cris}}(R^\wedge)'$ denotes the $\mathrm{Gal}(\overline{R}'^\wedge/R^\wedge)$–module obtained by replacing Ω with Ω' in the above construction.

11.17. There is a natural log structure $M_{\mathrm{A}_{\mathrm{cris}}(R^\wedge)}$ on $\mathrm{Spec}(\mathrm{A}_{\mathrm{cris}}(R^\wedge))$ defined as follows. Choose an étale map as in 2.16, and write $t_1, \ldots, t_s \in R$ for the images of the T_i ($i = 1, \ldots, s$). For each i and l, the extension $R[X]/(X^{p^l} - t_i)$ is étale over $R[1/(pt_1 \cdots t_s)]$. It follows that for each i, we can choose a sequence $\tau_{i,n}$ of elements in $\overline{R}^\wedge$ such that $\tau_{i,n}^p = \tau_{i,n-1}$ and $\tau_{i,0} = t_i$. Let $\tau_i \in S$ denote the corresponding element. We then get a map

(11.17.1) $$\lambda : \mathbb{N}^s \to W(S), \quad e_i \mapsto [\tau_i],$$

where $[\tau_i]$ denotes the Teichmuller lift of τ_i. This defines a log structure on $W(S)$ and hence in turn also a log structure on $\mathrm{A}_{\mathrm{cris}}(R^\wedge)$. Note that the log structure on $\overline{R}^\wedge$ induced by this map λ composed with θ is simply the log structure induced by pulling back M_R via the map $\mathrm{Spec}(\overline{R}^\wedge) \to \mathrm{Spec}(R)$.

We show that the above log structure on $\mathrm{A}_{\mathrm{cris}}(R^\wedge)$ is independent of the choices as follows. Consider a second map as in 2.16 giving elements $t_1', \ldots, t_s' \in R$ defining the log structure, and let $\tau_{i,n}'$ be a choice of roots of the t_i'. Then there exists a unique sequence $u_{i,n} \in \overline{R}^{\wedge*}$ such that $u_{i,n}^p = u_{i,n-1}$ and such that $\tau_{i,n} = u_{i,n} \tau_{i,n}'$. Letting u_i denote the corresponding element of S, we see that $[\tau_i] = [u_i] \cdot [\tau_i']$ in $\mathrm{A}_{\mathrm{cris}}(R^\wedge)$, and hence we get a canonical isomorphism between the associated log structures.

It follows from the above discussion that the enlargement 11.1.4 has a natural structure of a logarithmic enlargement

$$(11.17.2) \quad \begin{array}{ccc} (\mathrm{Spec}(\overline{R}^\wedge/p\overline{R}^\wedge), M_R|_{\overline{R}^\wedge/p\overline{R}^\wedge}) & \longrightarrow & (\mathrm{Spec}(\mathrm{A}_{\mathrm{cris}}(R^\wedge)), M_{\mathrm{A}_{\mathrm{cris}}(R^\wedge)}) \\ \downarrow & & \\ (\mathrm{Spec}(R/p), M_{R/p}). & & \end{array}$$

Note also that the action of $\mathrm{Gal}(\overline{R}^\wedge/R^\wedge)$ extends naturally to an action on the log scheme $(\mathrm{Spec}(\mathrm{A}_{\mathrm{cris}}(R^\wedge)), M_{\mathrm{A}_{\mathrm{cris}}(R^\wedge)})$.

11.18. The rings $\mathrm{A}_{\mathrm{cris}}(R^\wedge)$ define a sheaf of rings $\mathscr{A}_{\mathrm{cris}}$ on $\widehat{\mathscr{X}_{\overline{K}}^o}$ as follows. For any object $(U, N) \in \widehat{\mathscr{S}}$ with $U = \mathrm{Spf}(R^\wedge)$ as in 2.16, choose a geometric generic point $\eta : \mathrm{Spec}(\Omega) \to N$ and set $\mathscr{A}_{\mathrm{cris}}(U, N)$ equal to $\mathrm{A}_{\mathrm{cris}}(R^\wedge)^{\pi_1(N,\eta)}$ with the induced filtration. Then this is up to canonical isomorphism independent of the choice of geometric point and $\mathscr{A}_{\mathrm{cris}}$ is defined to be the associated sheaf.

The filtrations $I^{[\cdot]}$ and $\mathrm{Fil}_{\mathrm{A}_{\mathrm{cris}}}$ define two filtrations $\mathscr{I}^{[\cdot]}$ and $\mathrm{Fil}_{\mathscr{A}_{\mathrm{cris}}}$ on $\mathscr{A}_{\mathrm{cris}}$.

11.19. We will also need to consider certain ideals to study cohomology with (partial) compact support. Let $D = E \cup F$ be a decomposition of the boundary so that we have ideals $J_E \subset R$ and $\overline{J}_E \subset \overline{R}$. Since $\overline{J}_E \cap R_\infty = J_{E,\infty}$ (notation as in the proof of 8.16) the natural map

$$(11.19.1) \qquad\qquad J_{E,\infty} \otimes_{R_\infty} \overline{R} \to \overline{J}_E$$

is by 3.4 an almost isomorphism. Since the Frobenius morphism of R_∞/pR_∞ clearly induces a surjection on $J_{E,\infty}/pJ_{E,\infty}$ this implies that the Frobenius morphism on $\overline{R}/p\overline{R}$ induces an almost surjection on $\overline{J}_E/p\overline{J}_E$. Also $R_\infty/J_{E,\infty}$ is p-torsion free, so the sequence

$$(11.19.2) \qquad\qquad 0 \to \overline{J}_E/p\overline{J}_E \to \overline{R}/p\overline{R} \to \overline{R}/(p, \overline{J}_E) \to 0$$

is almost exact. Passing to the inverse limit with respect to the Frobenius morphisms we get by 2.22 an almost exact sequence

$$(11.19.3) \qquad\qquad 0 \to \varprojlim_{\mathrm{Frob}} \overline{J}_E/p\overline{J}_E \to S_{\overline{R}/p\overline{R}} \to S_{\overline{R}/(p,\overline{J}_E)} \to 0.$$

Define

$$(11.19.4) \qquad\qquad J_{n,E}^{\mathrm{inf}} := \mathrm{Ker}(W_n(S_{\overline{R}}) \to W_n(S_{\overline{R}/(p,\overline{J}_E)}))$$

and

$$(11.19.5) \qquad\qquad J_{n,E}^{\mathrm{cris}} := \mathrm{Ker}(\mathrm{A}_{\mathrm{cris}}(\overline{R})/p^{n+1} \to \mathrm{A}_{\mathrm{cris}}(\overline{R}/\overline{J}_E)/p^{n+1}).$$

LEMMA 11.20. *For every n the sequence*

$$(11.20.1) \qquad\qquad 0 \to J_{n,E}^{\mathrm{inf}} \to W_n(S_{\overline{R}}) \to W_n(S_{\overline{R}/(p,\overline{J}_E)}) \to 0$$

is almost exact.

PROOF. The case $n = 0$ is 11.19.3. The general case follows by induction and consideration of the commutative diagrams
(11.20.2)

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \\
& & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & J_{0,E}^{\mathrm{inf}} & \longrightarrow & W_0(S_{\overline{R}}) & \longrightarrow & W_0(S_{\overline{R}/(p,\overline{J}_E)}) & \longrightarrow & 0 \\
& & {\scriptstyle \times p^{n+1}}\downarrow & & \downarrow{\scriptstyle \times p^{n+1}} & & \downarrow{\scriptstyle \times p^{n+1}} & & \\
0 & \longrightarrow & J_{n+1,E}^{\mathrm{inf}} & \longrightarrow & W_{n+1}(S_{\overline{R}}) & \longrightarrow & W_{n+1}(S_{\overline{R}/(p,\overline{J}_E)}) & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J_{n,E}^{\mathrm{inf}} & \longrightarrow & W_n(S_{\overline{R}}) & \longrightarrow & W_n(S_{\overline{R}/(p,\overline{J}_E)}) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0. & & & &
\end{array}
$$

$\square$

COROLLARY 11.21. *For every integer n the sequence*

$$(11.21.1) \qquad 0 \to J_{n,E}^{\mathrm{cris}} \to \mathrm{A}_{\mathrm{cris}}(\overline{R})/p^{n+1} \to \mathrm{A}_{\mathrm{cris}}(\overline{R}/\overline{J}_E)/p^{n+1} \to 0$$

is almost exact.

PROOF. The only issue is the almost surjectivity of the map $\mathrm{A}_{\mathrm{cris}}(\overline{R})/p^{n+1} \to \mathrm{A}_{\mathrm{cris}}(\overline{R}/\overline{J}_E)/p^{n+1}$. By 11.5 this map can be identified with the map

$$(11.21.2) \qquad \mathrm{A}_{\mathrm{cris}}(\overline{V}) \otimes_{W_n(S_{\overline{V}})} W_n(S_{\overline{R}}) \to \mathrm{A}_{\mathrm{cris}}(\overline{V}) \otimes_{W_n(S_{\overline{V}})} W_n(S_{\overline{R}/\overline{J}_E})$$

so the almost surjectivity follows from the almost exactness of 11.20.1. $\square$

LEMMA 11.22. *For every integer n the sequence*

$$(11.22.1) \qquad 0 \longrightarrow J_{0,E}^{\mathrm{inf}} \xrightarrow{\ \times p^n\ } J_{n,E}^{\mathrm{inf}} \longrightarrow J_{n-1,E}^{\mathrm{inf}} \longrightarrow 0$$

is almost exact.

PROOF. This follows from consideration of the commutative diagram 11.20.2. $\square$

LEMMA 11.23. *For every integer $r \geq 1$ the sequence*

$$(11.23.1) \qquad 0 \to J_{0,E}^{\mathrm{inf}}/\xi^r J_{0,E}^{\mathrm{inf}} \to S_{\overline{R}}/(\xi^r) \to S_{\overline{R}/\overline{J}_E}/(\xi^r) \to 0$$

is almost exact.

PROOF. This follows from the almost exactness of 11.20.1 (with $n = 0$) and the fact that ξ is a regular element in $S_{\overline{R}/\overline{J}_E}$. $\square$

LEMMA 11.24. *For every integer n the sequence*

$$(11.24.1) \quad 0 \to \mathrm{A}_{\mathrm{cris}}(\overline{V}) \otimes J_{n,E}^{\mathrm{inf}} \to \mathrm{A}_{\mathrm{cris}}(\overline{V}) \otimes W_n(S_{\overline{R}}) \to \mathrm{A}_{\mathrm{cris}}(\overline{V}) \otimes W_n(S_{\overline{R}/\overline{J}_E}) \to 0$$

is almost exact, where the tensor products are taken over $W(S_{\overline{V}})$

PROOF. Note that for $n = 0$ the sequence 11.24.1 can be identified with the sequence obtained by applying $\mathrm{A_{cris}}(\overline{V})/(p) \otimes_{S_{\overline{V}}/(\xi^p)} (-)$ to the almost exact sequence

$$(11.24.2) \qquad 0 \to J_{0,E}^{\mathrm{inf}}/\xi^p J_{0,E}^{\mathrm{inf}} \to S_{\overline{R}}/(\xi^p) \to S_{\overline{R}/\overline{J}_E}/(\xi^p) \to 0.$$

Since $S_{\overline{V}}/(\xi^p) \to \mathrm{A_{cris}}(\overline{V})/p$ is a flat morphism by 11.9 this proves the case $n = 0$.

The result for general n follows by induction on n and consideration of the diagram (where all tensor products are taken over $W(S_{\overline{V}})$)
(11.24.3)

$$
\begin{array}{ccccccccc}
& & 0 & & & & 0 & & \\
& & \downarrow & & & & \downarrow & & \\
0 \longrightarrow & \mathrm{A_{cris}} \otimes J_{0,E}^{\mathrm{inf}} & \longrightarrow & \mathrm{A_{cris}} \otimes W_0(S_{\overline{R}}) & \longrightarrow & \mathrm{A_{cris}} \otimes W_0(S_{\overline{R}/\overline{J}_E}) & \longrightarrow 0 \\
& \downarrow{\scriptstyle \times p^n} & & \downarrow{\scriptstyle \times p^n} & & \downarrow{\scriptstyle \times p^n} & \\
& \mathrm{A_{cris}} \otimes J_{n,E}^{\mathrm{inf}} & \longrightarrow & \mathrm{A_{cris}} \otimes W_n(S_{\overline{R}}) & \longrightarrow & \mathrm{A_{cris}} \otimes W_n(S_{\overline{R}/\overline{J}_E}) & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \mathrm{A_{cris}} \otimes J_{n-1,E}^{\mathrm{inf}} & \longrightarrow & \mathrm{A_{cris}} \otimes W_{n-1}(S_{\overline{R}}) & \longrightarrow & \mathrm{A_{cris}} \otimes W_{n-1}(S_{\overline{R}/\overline{J}_E}) & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& 0 & & 0 & & 0, &
\end{array}
$$

where we write $\mathrm{A_{cris}}$ for $\mathrm{A_{cris}}(\overline{V})$, the top and bottom rows are exact by induction and the middle and right columns are exact since they can by 11.5 be identified with the almost exact sequences (note that $\mathrm{A_{cris}}(\overline{R})$ and $\mathrm{A_{cris}}(\overline{R}/\overline{J}_E)$ are p-torsion free)

$$(11.24.4) \qquad 0 \longrightarrow \mathrm{A_{cris}}(\overline{R})/p \xrightarrow{\times p^n} \mathrm{A_{cris}}(\overline{R})/p^{n+1} \longrightarrow \mathrm{A_{cris}}(\overline{R})/p^n \longrightarrow 0$$

and
(11.24.5)

$$0 \longrightarrow \mathrm{A_{cris}}(\overline{R}/\overline{J}_E)/p \xrightarrow{\times p^n} \mathrm{A_{cris}}(\overline{R}/\overline{J}_E)/p^{n+1} \longrightarrow \mathrm{A_{cris}}(\overline{R}/\overline{J}_E)/p^n \longrightarrow 0.$$

$\square$

COROLLARY 11.25. *For every integer n, the natural map*

$$(11.25.1) \qquad J_{n,E}^{\mathrm{inf}} \otimes_{W_n(S_{\overline{V}})} \mathrm{A_{cris}}(\overline{V})/p^{n+1} \to J_{n,E}^{\mathrm{cris}}$$

is an almost isomorphism.

PROOF. Consider the commutative diagram with almost exact columns (where the tensor products are taken over $W_n(S_{\overline{V}})$)

(11.25.2)

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
J_{n,E}^{\inf} \otimes \mathrm{A}_{\mathrm{cris}}(\overline{V})/p^{n+1} & \longrightarrow & J_{n,E}^{\mathrm{cris}} \\
\downarrow & & \downarrow \\
W_n(S_{\overline{R}}) \otimes \mathrm{A}_{\mathrm{cris}}(\overline{V})/p^{n+1} & \longrightarrow & \mathrm{A}_{\mathrm{cris}}(\overline{R})/p^{n+1} \\
\downarrow & & \downarrow \\
W_n(S_{\overline{R/J}_E}) \otimes \mathrm{A}_{\mathrm{cris}}(\overline{V})/p^{n+1} & \longrightarrow & \mathrm{A}_{\mathrm{cris}}(\overline{R/J}_E)/p^{n+1} \\
\downarrow & & \downarrow \\
0 & & 0,
\end{array}
$$

where the right column is almost exact by 11.21, and the bottom two horizontal arrows are isomorphisms by 11.5. $\qquad\square$

12. More computations of cohomology

We continue with the notation of 11.16.

THEOREM 12.1. *Let L be a smooth sheaf of $\mathbb{Z}/p^n$–modules on $X_{\overline{K}}^o$, let $U. \to X$ be an étale hypercover with each U_n admitting a morphism as in 2.16.1 and affine, and let $\widehat{U}. \to \widehat{X}$ be the p-adic completion of $U..$ Then the transformation*

(12.1.1) $$\mathrm{A}_{\mathrm{cris}}(\overline{V}) \otimes_{\mathbb{Z}_p} GC(U_{\cdot,\overline{K}}^o, L) \to GC(\widehat{U}_{\cdot,\overline{K}}^o, L \otimes_{\mathbb{Z}_p} \mathrm{A}_{\mathrm{cris}}(\widehat{U}.))$$

is an almost isomorphism.

PROOF. Consideration of the long exact sequences associated to the short exact sequences

(12.1.2) $$0 \longrightarrow L/pL \xrightarrow{p^n} L/p^{n+1}L \longrightarrow L/p^nL \longrightarrow 0$$

reduces the proof to the case when $n = 1$.

Let $S_{\overline{\mathscr{O}}_{\widehat{U}.}}$ be the Galois representation which in degree n is given by applying the construction in 11.1 to the coordinate ring of $\widehat{U}_n$.

LEMMA 12.2. *For any integer $j \geq 1$ the transformation*

(12.2.1) $$(S_{\overline{V}}/(\xi^j)) \otimes_{\mathbb{F}_p} GC(U_{\cdot,\overline{K}}^o, L) \to GC(\widehat{U}_{\overline{K}}^o, S_{\overline{\mathscr{O}}_{\widehat{U}.}}/(\xi^j) \otimes L)$$

is an almost isomorphism.

PROOF. For $j = 1$ this is 9.4 and 6.17. The general case follows by induction on j and consideration of the short exact sequences

(12.2.2) $$0 \longrightarrow (S_{\overline{V}}/(\xi)) \xrightarrow{\times \xi^j} (S_{\overline{V}}/(\xi^{j+1})) \longrightarrow (S_{\overline{V}}/(\xi^j)) \longrightarrow 0,$$

and
(12.2.3)

$$0 \longrightarrow (S_{\overline{\mathscr{O}}_{\widehat{U}.}}/(\xi)) \xrightarrow{\times \xi^j} (S_{\overline{\mathscr{O}}_{\widehat{U}.}}/(\xi^{j+1})) \longrightarrow (S_{\overline{\mathscr{O}}_{\widehat{U}.}}/(\xi^j)) \longrightarrow 0.$$

$\square$

To deduce that 12.1.1 is an isomorphism, note that by the flatness of $S_{\overline{V}}/(\xi^p) \to A_{\mathrm{cris}}(\overline{V})/p$ (11.9), the map

$$(12.2.4) \qquad A_{\mathrm{cris}}(\overline{V})/(p) \otimes_{S_{\overline{V}}/(\xi^p)} GC(\widehat{U}_{\overline{K}}^o, S_{\overline{\mathscr{O}}_{\widehat{U}.}}/(\xi^p) \otimes L)$$

$$\downarrow$$

$$GC(\widehat{U}_{\overline{K}}^o, A_{\mathrm{cris}}(\overline{V})/(p) \otimes_{S_{\overline{V}}/\xi^p} S_{\overline{\mathscr{O}}_{\widehat{U}.}}/(\xi^p) \otimes L)$$

is an isomorphism. Since

$$(12.2.5) \qquad A_{\mathrm{cris}}(\widehat{U}.)/p \simeq A_{\mathrm{cris}}(\overline{V})/(p) \otimes_{S_{\overline{V}}/\xi^p} S_{\overline{\mathscr{O}}_{\widehat{U}.}}/(\xi^p),$$

theorem 12.1 follows from the fact that 12.2.1 and 12.2.4 are quasi-isomorphisms.

$\square$

THEOREM 12.3. *Let $L = \varprojlim L_n$ be a smooth sheaf of $\mathbb{Z}_p$–modules on $X_{\overline{K}}^o$ (where L_n is a smooth sheaf of $\mathbb{Z}/p^n$–modules), let $U. \to X$ be an étale hypercover with each U_n admitting a morphism as in 2.16.1 and affine, and let $\widehat{U}. \to \widehat{X}$ be the p-adic completion of $U.$. Then the transformation*

$$(12.3.1) \qquad A_{\mathrm{cris}}(\overline{V}) \otimes_{\mathbb{Z}_p} GC(U_{.,\overline{K}}^o, L) \to GC(\widehat{U}_{\overline{K}}^o, L \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}(\widehat{U}.))$$

is an almost quasi-isomorphism.

PROOF. The map of projective systems of complexes

$$(12.3.2) \qquad \{A_{\mathrm{cris}}(\overline{V}) \otimes_{\mathbb{Z}_p} GC(U_{.,\overline{K}}^o, L_n)\}_n \to \{GC(\widehat{U}_{\overline{K}}^o, L_n \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}(\widehat{U}.))\}_n$$

is an almost quasi-isomorphism in each degree, and therefore the assumptions of 2.23 are satisfied for both projective systems in 12.3.2 (since the left projective system clearly satisfies the assumptions of 2.23). Therefore

$$(12.3.3) \qquad H^i(GC(\widehat{U}_{\overline{K}}^o, L \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}(\widehat{U}.))) \to \varprojlim H^i(GC(\widehat{U}_{\overline{K}}^o, L_n \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}(\widehat{U}.)))$$

is an almost isomorphism, and by 2.21 the composite map

$$\begin{aligned}
A_{\mathrm{cris}}(\overline{V}) \otimes_{\mathbb{Z}_p} H^i(GC(U_{.,\overline{K}}^o, L)) &= \varprojlim(A_{\mathrm{cris}}(\overline{V}) \otimes H^i(GC(U_{.,\overline{K}}^o, L_n))) \\
&\to \varprojlim H^i(GC(\widehat{U}_{\overline{K}}^o, L_n \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}(\widehat{U}.)))
\end{aligned}$$

is also an almost isomorphism. $\square$

The preceding result also extends to the filtered context:

COROLLARY 12.4. *With notation as in 12.3, for any integer $n \geq 0$ the natural map*

$$(12.4.1) \qquad \mathrm{Fil}_{A_{\mathrm{cris}}(\overline{V})}^n \otimes_{\mathbb{Z}_p} GC(U_{.,\overline{K}}^o, L) \to GC(\widehat{U}_{\overline{K}}^o, L \otimes_{\mathbb{Z}_p} \mathrm{Fil}_{A_{\mathrm{cris}}(\widehat{U}.)}^n)$$

is an almost quasi-isomorphism.

PROOF. By induction on n. The case $n = 0$ is 12.3. For the inductive step consider the short exact sequences

$$(12.4.2) \qquad 0 \to \mathrm{Fil}^{n+1}_{\mathrm{A_{cris}}(\overline{V})} \to \mathrm{Fil}^{n}_{\mathrm{A_{cris}}(\overline{V})} \to \overline{V}^{\wedge} \cdot \xi^{[n]} \to 0,$$

and

$$(12.4.3) \qquad 0 \to \mathrm{Fil}^{n+1}_{\mathrm{A_{cris}}(\widehat{U}_{\cdot})} \to \mathrm{Fil}^{n}_{\mathrm{A_{cris}}(\widehat{U}_{\cdot})} \to \overline{\mathscr{O}}_{\widehat{U}_{\cdot}} \cdot \xi^{[n]} \to 0.$$

$\square$

COROLLARY 12.5. *With notation as in 12.3, the natural map*

$$(12.5.1) \qquad \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V}) \otimes_{\mathbb{Q}_p} GC(U^o_{\cdot,\overline{K}}, L \otimes \mathbb{Q}_p) \to GC(\widehat{U}^o_{\cdot,\overline{K}}, L \otimes_{\mathbb{Z}_p} \widetilde{\mathrm{B}}_{\mathrm{cris}}(\widehat{U}_{\cdot}))$$

is a filtered quasi–isomorphism.

12.6. We will also need variants of the above results for cohomology with partial compact support along the boundary.

For simplicity we explain just the situation for cohomology with compact support along the entire boundary, leaving to the reader to make the necessary modifications to treat partial compact support. Also fix an ordering of the boundary components $D = D_1 \cup \cdots \cup D_m$. Write $\widehat{U}_n := \mathrm{Spec}(R_n)$ and let $\overline{R}_n$ be the normalization of R_n in $\widehat{U}_n \times_X X^o_{\overline{K}}$. For $A \subset \{1, \ldots, m\}$ let $\overline{R}_{n,A}$ be as in 8.18. We can then apply the functor $\mathrm{A_{cris}}(-)$ to get a Galois module over $\widehat{U}^o_{n,\overline{K}}$ which we denote by $\mathrm{A_{cris}}(\widehat{U}_n/\overline{J}_A)$. This construction is functorial in $\widehat{U}_n$ so we obtain a sheaf $\mathrm{A_{cris}}(\widehat{U}_{\cdot}/\overline{J}_A)$ on $\widehat{U}^o_{\overline{K},\mathrm{Fet}}$. Define a complex $\Phi(\mathrm{A_{cris}}(\widehat{U}_{\cdot}))^{\bullet}$ by setting

$$(12.6.1) \qquad \Phi(\mathrm{A_{cris}}(\widehat{U}_{\cdot}))^{r} := \oplus_{A \subset \{1,\ldots,m\},|A|=r} \mathrm{A_{cris}}(\widehat{U}_{\cdot}/\overline{J}_A)$$

with transition maps defined as in 8.15. Let $J^{\mathrm{cris}}_n \subset \mathrm{A_{cris}}(\widehat{U}_{\cdot})/p^{n+1}$ be the ideal corresponding to D as in 11.19.5.

There is a similar construction with $W_r(S_-)$ instead of $\mathrm{A_{cris}}(-)$. Namely, for $[n] \in \underline{\Delta}$ and $A \subset \{1, \ldots, m\}$, let $W_r(S_{\widehat{U}_n/\overline{J}_A})$ denote the truncated Witt ring of the perfection of $\overline{R}_{n,A}$. Then we again obtain a complex $\Phi(W_r(S_{\widehat{U}_{\cdot}}))^{\bullet}$ with

$$(12.6.2) \qquad \Phi(W_r(S_{\widehat{U}_{\cdot}}))^{k} := \oplus_{A \subset \{1,\ldots,m\},|A|=k} W_r(S_{\widehat{U}_{\cdot}/\overline{J}_A}).$$

Note also that there is a natural augmentation

$$(12.6.3) \qquad J^{\mathrm{inf}}_r \to \Phi(W_r(S_{\widehat{U}_{\cdot}}))^{\bullet}.$$

If $r = 0$ we write simply $\Phi(S_{\widehat{U}_{\cdot}})^{\bullet}$ for $\Phi(W_0(S_{\widehat{U}_{\cdot}}))^{\bullet}$.

LEMMA 12.7. *For every integer $j \geq 1$ the map*

$$(12.7.1) \qquad J^{\mathrm{inf}}_0/\xi^j J^{\mathrm{inf}}_0 \to \Phi(S_{\widehat{U}_{\cdot}})/(\xi^j)^{\bullet}.$$

is an almost quasi–isomorphism.

PROOF. We proceed by induction on j.

For the case $j = 1$ note that $S_{\widehat{U}_{\cdot}}/(\xi) = \overline{R}_{\cdot}/(p)$, and if $\overline{J} \subset \overline{R}_{\cdot}$ denotes the ideal of the boundary as in 3.11 then $J^{\mathrm{inf}}_0/\xi J^{\mathrm{inf}}_0 \simeq \overline{J}/p\overline{J}$ (by 11.23). The result in the case $j = 1$ then follows from an argument similar to the one used in the proof of 8.16.

The inductive step is obtained by devissage and noting that for every integer j we have exact sequences (the first being termwise exact)

(12.7.2)
$$0 \longrightarrow \Phi(S_{\widehat{U.}})/(\xi)^\bullet \xrightarrow{\times \xi^j} \Phi(S_{\widehat{U.}})/(\xi^{j+1})^\bullet \longrightarrow \Phi(S_{\widehat{U.}})/(\xi^j)^\bullet \longrightarrow 0,$$

and

(12.7.3)
$$0 \longrightarrow J_0^{\mathrm{inf}}/\xi J_0^{\mathrm{inf}} \xrightarrow{\times \xi^j} J_0^{\mathrm{inf}}/\xi^{j+1} J_0^{\mathrm{inf}} \longrightarrow J_0^{\mathrm{inf}}/\xi^j J_0^{\mathrm{inf}} \longrightarrow 0.$$

$\square$

PROPOSITION 12.8. *The inclusion $J_n^{\mathrm{cris}} \subset \mathrm{A}_{\mathrm{cris}}(\widehat{U.})/p^{n+1} = \Phi(\mathrm{A}_{\mathrm{cris}}(\widehat{U.}))^0/p^{n+1}$ induces an almost quasi-isomorphism*

(12.8.1)
$$J_n^{\mathrm{cris}} \to \Phi(\mathrm{A}_{\mathrm{cris}}(\widehat{U.}))^\bullet/p^{n+1}.$$

PROOF. By devissage one is reduced to the case $n = 0$. In this case 12.8.1 is obtained from the morphism of complexes

(12.8.2)
$$J_0^{\mathrm{inf}}/\xi^p J_0^{\mathrm{inf}} \to \Phi(S_{\widehat{U.}})/(\xi^p)^\bullet.$$

by making the flat base change $S_{\overline{V}}/(\xi^p) \to \mathrm{A}_{\mathrm{cris}}(\overline{V})/(p)$ (and using the identifications provided by 11.25). It therefore suffices to show that 12.8.2 is an almost quasi–isomorphism which is 12.7. $\square$

12.9. Let L be a smooth sheaf of $\mathbb{Z}/p^{n+1}$–modules on $X_{\overline{K}}^o$, and let $\Psi(L)$ be as in 8.3. As in 8.18.3, there is a natural map

(12.9.1)
$$\Psi(L)|_{\widehat{U.}, \overline{K}, \mathrm{Fet}} \to L \otimes \Phi(\mathrm{A}_{\mathrm{cris}}(\widehat{U.})/p^{n+1}).$$

We then obtain a diagram

(12.9.2) $GC(\widehat{U.}, L \otimes J_n^{\mathrm{cris}}) \to GC(\widehat{U.}, L \otimes \Phi(\mathrm{A}_{\mathrm{cris}}(\widehat{U.})/p^{n+1})) \leftarrow GC(U_{\overline{K}}^o, \Psi(L)),$

where the first map is an almost quasi-isomorphism. Passing to the limit over all hypercovers $U.$ we get maps

(12.9.3) $H^*(GC(\widehat{U.}, L \otimes J_n^{\mathrm{cris}})) \longrightarrow H^*(GC(\widehat{U.}, L \otimes \Phi(\mathrm{A}_{\mathrm{cris}}(\widehat{U.})/p^{n+1})))$

$$\Big\uparrow$$

$$H_c^*(X_{\overline{K}}^o, L) \otimes \mathrm{A}_{\mathrm{cris}}(\overline{V})/p^{n+1}.$$

THEOREM 12.10. *All the maps in 12.9.3 are almost isomorphisms.*

PROOF. By the usual devissage one is reduced to the case when $n = 0$. As in the proof of 12.1, by the flatness of $S_{\overline{V}}/(\xi^p) \to \mathrm{A}_{\mathrm{cris}}(\overline{V})/p$ it then suffices to show that the map

(12.10.1) $H_c^*(X_{\overline{K}}^o, L) \otimes S_{\overline{V}}/\xi^p \to H^*(GC(\widehat{U.}, L \otimes \Phi(S_{\widehat{U.}})/\xi^p))$

is an almost isomorphism. Filtering again by powers of ξ we are reduced to 6.17. $\square$

As in 12.3, we then also obtain results for adic sheaves:

THEOREM 12.11. *Let $L = \varprojlim L_n$ be a smooth $\mathbb{Z}_p$–sheaf on $X_{\overline{K}}^o$. Then there is a natural almost isomorphism*

$$(12.11.1) \qquad H^*(GC(\widehat{U}., L \otimes J^{\mathrm{cris}})) \simeq H_c^*(X_{\overline{K}}^o, L) \otimes \mathrm{A}_{\mathrm{cris}}(\overline{V}).$$

In particular there is an actual isomorphism

$$(12.11.2) \qquad H^*(GC(\widehat{U}., L \otimes J^{\mathrm{cris}} \otimes_{\mathrm{A}_{\mathrm{cris}}(\widehat{U}.)} \widetilde{\mathrm{B}}_{\mathrm{cris}}(\widehat{U}.))) \simeq H_c^*(X_{\overline{K}}^o, L) \otimes \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V}).$$

13. Crystalline cohomology and crystalline sheaves

13.1. Since we are interested in $\mathbb{Q}_p$–coefficients and isocrystals we will work throughout with the convergent topos as opposed to the crystalline topos. The reference for the convergent topos in the non–logarithmic setting is [**39**] and in the logarithmic setting the paper [**46**].

Let X/V be a smooth proper scheme, $D \subset X$ a divisor with relative normal crossings defining a log structure M_X on X, and let (Y, M_Y) denote the closed fiber of (X, M_X). Recall that the convergent topos $((Y, M_Y)/V)_{\mathrm{conv}}$ is the topos associated to the site $\mathrm{Conv}((Y, M_Y)/V)$ whose objects are strict closed immersions $(U, M_U) \hookrightarrow (T, M_T)$, where $U \to X$ is an étale morphism, M_U is the pullback of M_X, and (T, M_T) is a formal log scheme flat over $\mathrm{Spf}(V)$ with the π-adic topology. The diagram

$$
\begin{array}{ccc}
(U, M_U) & \lhook\joinrel\longrightarrow & (T, M_T) \\
\downarrow & & \\
(X, M_X) & &
\end{array}
$$

is called an *enlargement.* We often write simply (T, M_T) for such an enlargement. A morphism $((U', M_{U'}) \hookrightarrow (T', M_{T'})) \to ((U, M_U) \hookrightarrow (T, M_T))$ is a commutative diagram of log schemes

$$(13.1.1) \qquad
\begin{array}{ccc}
(U', M_{U'}) & \longrightarrow & (T', M_{T'}) \\
a \downarrow & & \downarrow b \\
(U, M_U) & \longrightarrow & (T, M_T),
\end{array}
$$

where a is a X–morphism. Coverings are defined to be collections of morphisms $\{(T_i, M_{T_i}) \to (T, M_T)\}_i$ such that the map $\coprod T_i \to T$ is an étale covering.

We write $\mathscr{O}_{(Y, M_Y)/V}$ for the structure sheaf $(T, M_T) \mapsto \Gamma(T, \mathscr{O}_T)$ and $\mathscr{K}_{(Y, M_Y)/V}$ for the sheaf $(T, M_T) \mapsto \Gamma(T, \mathscr{O}_T) \otimes \mathbb{Q}$. For a sheaf $\mathscr{F}$ of $\mathscr{K}_{(Y, M_Y)/V}$–modules and an object (T, M_T) of $\mathrm{Conv}((Y, M_Y)/V)$ we write $\mathscr{F}_T$ for the induced sheaf on T_{et}. An *isocrystal* is a sheaf $\mathscr{F}$ of $\mathscr{K}_{(Y, M_Y)/V}$–modules such that for every (T, M_T) the sheaf $\mathscr{F}_T$ is coherent and such that for any morphism $f : (T', M_{T'}) \to (T, M_T)$ in $\mathrm{Conv}((Y, M_Y)/V)$ the induced morphism $f^*\mathscr{F}_T \to \mathscr{F}_{T'}$ is an isomorphism.

Let W denote the ring of Witt vectors of the residue field k of W. Then we can also consider the convergent topos $((Y, M_Y)/W)_{\mathrm{conv}}$. There is a natural morphism of topoi

$$(13.1.2) \qquad \pi : ((Y, M_Y)/V)_{\mathrm{conv}} \to ((Y, M_Y)/W)_{\mathrm{conv}}.$$

For any isocrystal $\mathscr{F}$ in $((Y, M_Y)/W)_{\mathrm{conv}}$ the induced morphism

$$(13.1.3) \qquad H^*(((Y, M_Y)/W)_{\mathrm{conv}}, \mathscr{F}) \to H^*(((Y, M_Y)/V)_{\mathrm{conv}}, \pi^*\mathscr{F})$$

is an isomorphism (without log structures this is [**40**, 3.2], and the same argument yields the case with log structures).

Note that Frobenius induces a morphism of topoi

(13.1.4) $\sigma : ((Y, M_Y)/W)_{\text{conv}} \to ((Y, M_Y)/W)_{\text{conv}}.$

We can therefore define an F–*isocrystal* to be a pair $(\mathscr{F}, \varphi_{\mathscr{F}})$ where $\mathscr{F}$ is an isocrystal in $((Y, M_Y)/W)$ and $\varphi_{\mathscr{F}} : \sigma^* \mathscr{F} \to \mathscr{F}$ is an isomorphism.

13.2. Let $(\widehat{X}, M_{\widehat{X}})$ denote the p–adic completion of (X, M_X) so that $(Y, M_Y) \hookrightarrow (\widehat{X}, M_{\widehat{X}})$ is an object of $\text{Conv}((Y, M_Y)/V)$. As usual, there is then a morphism of sites
(13.2.1)
$\text{Et}(\widehat{X}) \to \text{Conv}((Y, M_Y)/V), \quad (T \to \widehat{X}) \mapsto ((T \times_X Y, M_Y|_{T \times_X Y}) \hookrightarrow (T, M_{\widehat{X}}|_T))$

inducing a morphism of topoi

(13.2.2) $u_{X/V} : ((Y, M_Y)/V)_{\text{conv}} \to \widehat{X}_{\text{et}}.$

As in the classical case, the category of isocrystals is identified with the category of coherent $\mathscr{K}_{\widehat{X}}$–modules $\mathcal{F}$ with an integrable connection

(13.2.3) $\nabla : \mathcal{F} \to \mathcal{F} \otimes_{\mathscr{O}_{\widehat{X}}} \Omega^1_{(\widehat{X}, M_{\widehat{X}})/V}$

satisfying a p-adic convergence condition. Write $DR(\mathcal{F}, \nabla)$ for the associated de Rham complex. By [**46**, 2.3.5] there is then a canonical quasi–isomorphism $Ru_{X/V*}\mathscr{F} \simeq DR(\mathcal{F}, \nabla)$. Note also that since X/V is proper an isocrystal is induced by a unique coherent $\mathscr{O}_{X_K}$–module with integrable connection satisfying a convergence condition, and the de Rham complex $DR(\mathcal{F}, \nabla)$ computes the de Rham cohomology of this module with integrable connection on the generic fiber by [**11**, III.5.1.2].

13.3. Let $D = E \cup F$ be a decomposition of the boundary of X. We can then define an isocrystal J_E as follows. First let $\overline{M}$ denote the sheaf on $\text{Conv}((Y, M_Y)/W)$ which to any $(U, M_U) \hookrightarrow (T, M_T)$ associates $\Gamma(T, \overline{M}_T) = \Gamma(U, \overline{M}_U)$. For any geometric point $\bar{t} \to U$ the stalk $\overline{M}_U$ is canonically isomorphic to the free monoid with generators indexed by the irreducible components of the inverse image of D in $\text{Spec}(\mathscr{O}_{U, \bar{t}})$. In particular, there is a canonical submonoid $K_{E, \bar{t}} \subset \overline{M}_{\bar{t}}$ generated by the sum of the generators of $\overline{M}_{\bar{t}}$ corresponding to those irreducible components which are in E. One verifies immediately that there is a subsheaf of monoids $K_E \subset \overline{M}$ whose stalks at each geometric point $\bar{t}$ agrees with $K_{E, \bar{t}}$. Define $J'_E \subset \mathscr{O}_{(Y, M_Y)/V}$ to be the sheaf of ideals whose restriction to any (T, M_T) is generated by the images of sections $m \in M_T$ mapping to K_E in $\overline{M}_T$.

If $U \to Y$ is étale and admits a chart 2.16.1 and if $(U, M_U) \hookrightarrow (Z, M_Z)$ is an embedding into a formally log smooth W–scheme, then a chart for (U, M_U) extends to a chart

(13.3.1) $(Z, M_Z) \to \text{Spf}(W[X_1, \ldots, X_d]).$

If E is defined by the inverse image of $\{X_1 \cdots X_s = 0\}$ then $J_{E, Z} \subset \mathscr{O}_Z$ is simply the ideal defined by the image of $X_1 \cdots X_s$. This implies that if $f : (Z', M_{Z'}) \to (Z, M_Z)$ is a morphism of log smooth objects of $\text{Conv}((Y, M_Y)/W)$ then the natural map $f^* J_{E, Z} \to J_{E, Z'}$ is an isomorphism. In particular, applying this to the completion of $(Z, M_Z) \times_V (Z, M_Z)$ (this should be the "strictification" of the diagonal;

see [**29**, 5.8] and [**41**, Appendix A]) we see that $J_{E,Z}$ has a canonical integrable connection defining an isocrystal J_E. Concretely, in local coordinates 13.3.1 this just amounts to saying that the connection

$$(13.3.2) \qquad d: W[X_1, \ldots, X_d] \to W[X_1, \ldots, X_d] \otimes_W (\oplus_i W \cdot d \log X_i)$$

sends the ideal $(X_1 \cdots X_s)$ to $(X_1 \cdots X_s) \otimes_W (\oplus_i W \cdot d \log X_i)$ which is immediate. Let J_E denote the corresponding isocrystal on $\mathrm{Conv}((Y, M_Y)/W)$. There is a natural surjection $J_E \to J'_E$ which is an isomorphism on formally smooth objects of $\mathrm{Conv}((Y, M_Y)/W)$ (the kernel is "parasitic"). If $E = D$ we also sometimes write just J_X for J_E.

LEMMA 13.4. *Let $(U, M_U) \hookrightarrow (T, M_T)$ be an object of $\mathrm{Conv}((Y, M_Y)/W)$ with T affine, and assume given a chart $c : \mathbb{N}^r \to M_U$ such that E is defined by the image f of $(1, \ldots, 1, 0, \ldots, 0)$ (1's in the first s places). Assume that for some lifting $\tilde{c} : \mathbb{N}^r \to M_T$ of the chart the image $\tilde{f}$ of $(1, \ldots, 1, 0, \ldots, 0)$ in $\mathscr{O}_T$ is a non–zero divisor. Then the map $J_E(T) \to J'_E(T)$ is an isomorphism.*

PROOF. This follows immediately from the preceding discussion. □

COROLLARY 13.5. *Let $U = \mathrm{Spec}(R) \to X$ be an étale morphism admitting a morphism as in 2.16.1. Then the natural map $J_E(\mathrm{A_{cris}}(R^\wedge)) \to J'_E(\mathrm{A_{cris}}(R^\wedge))$ is an isomorphism.*

13.6. Note that there is a natural map $\sigma^* J_E \to J_E$, which is not usually an isomorphism. Thus for any F–isocrystal $(\mathscr{F}, \varphi_{\mathscr{F}})$ we can consider the cohomology with compact support along E

$$(13.6.1) \qquad H^*(((Y, M_Y)/W)_{\mathrm{conv}}, \mathscr{F} \otimes J_E)$$

which has a semi-linear Frobenius endomorphism. There is a canonical pairing

$$(13.6.2) \quad H^*(((Y, M_Y)/W)_{\mathrm{conv}}, \mathscr{F} \otimes J_E) \otimes H^{2d-*}(((Y, M_Y)/W)_{\mathrm{conv}}, \mathscr{F}^* \otimes J_F)$$

$$\downarrow$$

$$H^{2d}(((Y, M_Y)/W)_{\mathrm{conv}}, J_X)$$

compatible with Frobenius.

LEMMA 13.7. *This pairing is a perfect pairing.*

PROOF. It suffices to verify this after base changing to K. Thus we may consider the topos $((Y, M_Y)/V)_{\mathrm{conv}}$ instead of $((Y, M_Y)/W)_{\mathrm{conv}}$. In this case it suffices to show the analogous result for de Rham cohomology of the generic fiber. Write also J_E for the ideal in $\mathscr{O}_{X_K}$ defining E_K. Then we wish to show that the natural map

$$(13.7.1) \qquad R\Gamma(DR(\mathcal{F} \otimes J_E, \nabla)) \to \mathrm{RHom}(R\Gamma(DR(\mathcal{F}^* \otimes J_F, \nabla)), \Omega^d_{X_K})$$

is an isomorphism or better the local version that

$$(13.7.2) \qquad DR(\mathcal{F} \otimes J_E, \nabla) \to \mathscr{R}Hom(DR(\mathcal{F}^* \otimes J_F, \nabla), \Omega^d_{X_K})$$

is an isomorphism which is immediate. □

LEMMA 13.8 ([**46**, 2.3.6]). *Let $\mathscr{F}$ be an isocrystal. Associated to the lifting $(\widehat{X}, M_{\widehat{X}})$ is a canonical resolution $\mathscr{F} \to \mathbb{R}^\bullet$ such that each $\mathbb{R}^i$ is acyclic for the projection $u_{X/V*}$. This resolution is functorial in $\mathscr{F}$.*

PROOF. Let T denote the object of the convergent site of (Y, M_Y) given by $(Y, M_Y) \hookrightarrow (\widehat{X}, M_{\widehat{X}})$. There is then a diagram of topoi

(13.8.1)
$$
\begin{array}{ccc}
((Y, M_Y)/V)_{\mathrm{conv}}|_T & \xrightarrow{\ \phi_T\ } & \widehat{X}_{\mathrm{et}} \\[4pt]
\Big\downarrow{\scriptstyle j_T} & & \\[4pt]
((Y, M_Y)/V)_{\mathrm{conv}}. & &
\end{array}
$$

As explained in [**46**, 2.3.6] (in loc. cit. our $\mathbb{R}^j$ is denoted $\omega_P^j(\mathcal{E})$) the resolution is then given by $\mathbb{R}^j := j_{T*}(\phi_T^* \Omega^j_{(X,M_X)/V} \otimes j_T^* \mathscr{F})$. $\qquad\qquad\qquad\square$

REMARK 13.9. Note that since $\mathscr{F}$ is an isocrystal, there is a canonical isomorphism $j_T^* \mathscr{F} \simeq \phi_T^* \mathcal{F}$. Therefore if $\mathcal{F}$ comes is equipped with a filtration $A^\bullet$ satisfying Griffiths transversality (see 13.11.1 below), then each of the sheaves $\mathbb{R}^i$ is naturally filtered by the subsheaves $j_{T*}(\phi_T^* \Omega^j_{(X,M_X)/V} \otimes \phi_T^* A^i)$ and $\mathbb{R}^\bullet$ is a filtered complex (where $j_{T*}(\phi_T^* \Omega^j_{(X,M_X)/V} \otimes \phi_T^* A^i)$ lies in the $(i+j)$-th step of the filtration). Furthermore, if $(\mathcal{F}, \nabla)$ is the module with integrable connection corresponding to $\mathscr{F}$ then the natural map $(\mathcal{F}, \nabla) \to \mathbb{R}^\bullet|_{\widehat{X}}$ is a filtered quasi–isomorphism.

13.10. Let $\mathscr{F} \to \mathbb{R}^\bullet$ be the resolution corresponding to our lifting $(X, M_X)/V$. The sheaves $\mathbb{R}^i$ are not isocrystals, but still the value $\mathbb{R}^i(Z, M_Z)$ of $\mathbb{R}^i$ on any affine enlargment $(U, M_U) \hookrightarrow (Z, M_Z)$, with (Z, M_Z) formally smooth over V, has a canonical integrable connection

(13.10.1) $$\mathbb{R}^i(Z, M_Z) \to \mathbb{R}^i(Z, M_Z) \otimes \Omega^1_{(Z,M_Z)/V}.$$

This follows from the construction of $\mathbb{R}^i$ (see the proof of [**46**, 2.3.5] and without log structures [**39**, 0.5.4]). Moreover, by the proof of the convergent Poincaré lemma [**46**, 2.3.5 (2)], the complex $\mathbb{R}^\bullet(Z, M_Z)$ is a resolution of $\mathscr{F}(Z, M_Z)$.

13.11. As in [**14**] there is a category $MF_X^\nabla(\Phi)$ defined as follows. Fix a geometric generic point $\eta : \mathrm{Spec}(\Omega) \to X$, and let $\widehat{\Omega}$ denote the p–adic completion. If $\mathscr{F}$ is an isocrystal on $(Y, M_Y)/W$, let $(\mathcal{F}, \nabla_{\mathcal{F}})$ denote the module with logarithmic connection on (X_K, M_{X_K}) obtained by evaluating $\mathscr{F}$ on the enlargement $(Y, M_Y) \hookrightarrow (X^\wedge, M_{X^\wedge})$, where $(X^\wedge, M_{X^\wedge})$ denotes the p–adic completion of $(X, M_X)/V$. The category $MF_X^\nabla(\Phi)$ is defined to be the category of triples $(\mathscr{F}, \varphi_{\mathscr{F}}, \mathrm{Fil}_{\mathcal{F}})$, where $(\mathscr{F}, \varphi_{\mathscr{F}})$ is an F–isocrystal on $(Y, M_Y)/W$ and $\mathrm{Fil}_{\mathcal{F}}$ is a decreasing filtration on $\mathcal{F}$ satisfying Griffiths transversality

(13.11.1) $$\nabla_{\mathcal{F}}(\mathrm{Fil}_{\mathcal{F}}^i) \subset \mathrm{Fil}_{\mathcal{F}}^{i-1} \otimes \Omega^1_{(X_K,M_{X_K})/K}.$$

13.12. If $(\mathscr{F}, \varphi_{\mathscr{F}}, \mathrm{Fil}_{\mathcal{F}}) \in MF_X^\nabla(\Phi)$ and $\mathrm{Spec}(R) \to X$ a disjoint union of open subsets of X admitting morphisms as in 2.16.1, we can evaluate $\mathscr{F}$ on the enlargement 11.17.2 to get a $\mathrm{A}_{\mathrm{cris}}(R^\wedge) \otimes \mathbb{Q}$–module $\mathscr{F}((\mathrm{Spec}(\mathrm{A}_{\mathrm{cris}}(R^\wedge)), M_{\mathrm{A}_{\mathrm{cris}}(R^\wedge)}))$. Inverting $t \in \mathrm{A}_{\mathrm{cris}}(R^\wedge)$, we get a $\mathrm{B}_{\mathrm{cris}}(R^\wedge)$–module which we denote simply by $\mathscr{F}(\mathrm{B}_{\mathrm{cris}}(R^\wedge))$. The F–isocrystal structure $\varphi_{\mathscr{F}}$ induces a semi–linear automorphism of the $\mathrm{B}_{\mathrm{cris}}(R^\wedge)$–module $\mathscr{F}(\mathrm{B}_{\mathrm{cris}}(R^\wedge))$.

The $\mathrm{B}_{\mathrm{cris}}(R^\wedge)$–module $F(\mathrm{B}_{\mathrm{cris}}(R^\wedge))$ also has a natural filtration $\mathrm{Fil}_{F(\mathrm{B}_{\mathrm{cris}}(R^\wedge))}$ defined as follows. Since $(X, M_X)/V$ is smooth, we can choose a morphism

(13.12.1) $$r : (\mathrm{Spec}(\mathrm{A}_{\mathrm{cris}}(R^\wedge)), M_{\mathrm{A}_{\mathrm{cris}}(R^\wedge)}) \to (\mathrm{Spec}(R), M_{\mathrm{Spec}(R)})$$

such that the diagram

(13.12.2)
$$\begin{array}{ccc} (\mathrm{Spec}(\overline{R}^{\wedge}), M_{R^{\wedge}}) & \longrightarrow & (\mathrm{Spec}(\mathrm{A_{cris}}(R^{\wedge})), M_{\mathrm{A_{cris}}(R^{\wedge})}) \\ \downarrow & & \downarrow r \\ (\mathrm{Spec}(R), M_{\mathrm{Spec}(R)}) & \xrightarrow{\ \mathrm{id}\ } & (\mathrm{Spec}(R), M_{\mathrm{Spec}(R)}) \end{array}$$

commutes. The choice of such an r gives an isomorphism

(13.12.3) $$\sigma_r : F(\mathrm{B_{cris}}(R^{\wedge})) \simeq \mathcal{F}(\mathrm{Spec}(R)) \otimes_R \mathrm{B_{cris}}(R^{\wedge}),$$

and we define $\mathrm{Fil}_{F(\mathrm{B_{cris}}(R^{\wedge}))}$ to be the tensor product filtration of $\mathrm{Fil}_{\mathcal{F}(\mathrm{Spec}(R))}$ and the filtration $\mathrm{Fil}_{\mathrm{B_{cris}}(R^{\wedge})}$ on $\mathrm{B_{cris}}(R^{\wedge})$.

LEMMA 13.13. *The filtration* $\mathrm{Fil}_{F(\mathrm{B_{cris}}(R^{\wedge}))}$ *is independent of the choice of* r.

PROOF. See [**50**] or [**41**, 5.8]. $\qquad\square$

13.14. The module $F(\mathrm{B_{cris}}(R^{\wedge}))$ also comes equipped with a continuous action of $\mathrm{Gal}(\overline{R}^{\wedge}/R^{\wedge})$ which commutes with the Frobenius automorphism induced by the F–isocrystal structure as well as the filtration. As in 11.16, this $\mathrm{Gal}(\overline{R}^{\wedge}/R^{\wedge})$–module $F(\mathrm{B_{cris}}(R^{\wedge}))$ is functorial for morphisms $s : \mathrm{Spec}(\Omega') \to \mathrm{Spec}(\Omega)$ of geometric generic points of $\mathrm{Spec}(R)$.

If $U = \mathrm{Spec}(R) \to X$ is a disjoint union of open subsets of X admitting a morphism as in 2.16.1 and $(F, \varphi_F, \mathrm{Fil}_{\mathcal{F}}) \in MF_X^{\nabla}(\Phi)$, we write $F(\mathrm{B_{cris}}(U^{\wedge}))$ (or $F(\mathrm{B_{cris}}(R^{\wedge}))$) for the filtered Galois module with semi–linear automorphisms on $U_K^{\wedge o} := \mathrm{Spec}(R^{\wedge}) \times_X X_K^o$ obtained from the construction 13.12 on each connected component.

13.15. If L is a smooth $\mathbb{Q}_p$–sheaf on X_K^o and $U \to X$ is étale and admitting a morphism 2.16.1, the pullback of L to $U_K^{\wedge o}$ is a Galois module on $U_K^{\wedge o}$ which we denote by $L_{U_K^{\wedge o}}$. Define an *association* ι between $(F, \varphi_F, \mathrm{Fil}_{\mathcal{F}}) \in MF_X^{\nabla}(\Phi)$ and a smooth $\mathbb{Q}_p$–sheaf L on X_K^o to be a collection of isomorphisms of Galois modules, one for each étale $U \to X$ as in 2.16.1,

(13.15.1) $$\iota_U : F(\mathrm{B_{cris}}(U^{\wedge})) \simeq L_{U_K^{\wedge o}} \otimes \mathrm{B_{cris}}(U^{\wedge})$$

compatible with the semi–linear Frobenius automorphisms, and the filtrations. Furthermore, we require that the isomorphisms ι_U be compatible with morphisms over X. A smooth sheaf L on X_K is called *crystalline* if it is associated to some object in $MF_X^{\nabla}(\Phi)$.

REMARK 13.16. Crystalline sheaves arise as follows. Let X and Y be smooth proper V–schemes with normal crossing divisors $D \subset X$ and $E \subset Y$, and let $f : Y \to X$ be a morphism with $f^{-1}(D) \subset E$. Assume that the induced morphism of log schemes $f : (Y, M_Y) \to (X, M_X)$ (log structures defined by E and D) is of Cartier type [**29**, 4.8]. For example, if étale locally on X and Y we can describe the morphism as

(13.16.1)
$$V[X_1, \ldots, X_r, t_1, \ldots, t_s]$$
$$\downarrow$$
$$V[X_1, \ldots, X_r, Y_1, \ldots, Y_p, z_1, \ldots, z_s, w_1, \ldots, w_s]/(t_i = z_i w_i)_{i=1}^s$$

such that D is the divisor defined by $t_1 \cdots t_s = 0$ and E is the divisor defined by $z_1 \ldots, z_s w_1 \cdots w_s = 0$. Then for any crystalline sheaf L on Y associated to some object $(F, \varphi_F, \mathrm{Fil}_{\mathcal{F}}) \in MF_Y^{\nabla}(\Phi)$ and integer i the direct image $R^i f_* L$ on $X_{\overline{K}}^o$ is crystalline and is associated to $R^i f_* F$ (with the filtration and F–isocrystal structure induced by that on F). This is shown in [**14**, 6.3]. In particular we can take $L = \mathbb{Q}_p$ which is associated to the trivial F–isocrystal.

13.17. Let L be a crystalline sheaf associated to $(F, \varphi_F, \mathrm{Fil}_{\mathcal{F}})$. For any decomposition $D = E \cup F$ we now define a transformation
(13.17.1)
$$\alpha : \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V}) \otimes_K H^*(((Y, M_Y)/V)_{\mathrm{conv}}, F \otimes J_E) \to \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V}) \otimes_{\mathbb{Q}_p} H^*_{E,F}(X_{\overline{K},\mathrm{et}}^o, L)$$

compatible with Galois action, filtrations, and Frobenius as well as cup-products.

For this choose an étale hypercover $U. \to X$ with each U_n admitting a morphism 2.16.1. Let $Z. \to Y$ be the reduction of $U.$.

For any étale morphism $Z \to Y$, let h_Z denote the sheaf on $\mathrm{Conv}((Y, M_Y)/V)$ which to any $((U, M_U) \hookrightarrow (T, M_T))$ associates the set of morphisms $U \to Z$ over Y. Then the localized topos $((Y, M_Y)/V)_{\mathrm{conv}}|_{h_Z}$ is canonically equivalent to the topos $((Z, M_Z)/V)_{\mathrm{conv}}$. Write $\mathrm{A}_{\mathrm{cris}}(U.^{\wedge})$ for the simplicial object of $\mathrm{Conv}((Y, M_Y)/V)$ defined by the enlargements $\mathrm{A}_{\mathrm{cris}}(U_n^{\wedge})$. Then there is a natural morphism of sheaves $\mathrm{A}_{\mathrm{cris}}(U.^{\wedge}) \to h_{Z.}$ which induces a morphism of topoi

(13.17.2) $\pi : ((Y, M_Y)/V)_{\mathrm{conv}}|_{\mathrm{A}_{\mathrm{cris}}(U.^{\wedge})} \to ((Y, M_Y)/V)_{\mathrm{conv}}|_{h_{Z.}}.$

This morphism sits naturally in a commutative diagram of topoi

(13.17.3)
$$
\begin{array}{ccc}
((Y, M_Y)/V)_{\mathrm{conv}}|_{h_{Z.}} & \xleftarrow{\;\;\pi\;\;} & ((Y, M_Y)/V)_{\mathrm{conv}}|_{\mathrm{A}_{\mathrm{cris}}(U.^{\wedge})} \\
u_{U.^{\wedge}} \downarrow & & \downarrow u_{\mathrm{A}_{\mathrm{cris}}(U.^{\wedge})} \\
Z_{\cdot,\mathrm{et}} & =\!=\!=\!=\!= & \mathrm{A}_{\mathrm{cris}}(U.^{\wedge})_{\mathrm{et}}
\end{array}
$$

Let $\pi_0(U_{.,\overline{K}})$ denote the simplicial set

(13.17.4) $[n] \mapsto \{\text{the set of connected components of } U_{n,\overline{K}}\}.$

We view $\pi_0(U_{.,\overline{K}})$ as a simplicial topos with the discrete topology. Let $\mathrm{Vec}_K^{\pi_0(U_{.,\overline{K}})}$ denote the category of sheaves of K–vector spaces in this topos. Concretely, $\mathrm{Vec}_K^{\pi_0(U_{.,\overline{K}})}$ assigns in a functorial manner a K–vector space to each connected component of each $U_{n,\overline{K}}$. Taking global section defines a functor

(13.17.5) $\Gamma_{\bullet} : (\mathscr{K}_{X/V}\text{–modules in } ((Y, M_Y)/V)_{\mathrm{conv}}|_{h_{Z.}}) \to \mathrm{Vec}_K^{\pi_0(U_{.,\overline{K}})}.$

We can also consider the evaluation on the enlargements $\mathrm{A}_{\mathrm{cris}}(U_n^{\wedge})$ which gives a functor

(13.17.6) $\mathrm{ev}_{\mathrm{A}_{\mathrm{cris}}(U.^{\wedge})} : (\mathscr{K}_{X/V}\text{–modules in } ((Y, M_Y)/V)_{\mathrm{conv}}|_{h_{Z.}}) \to \mathrm{Vec}_K^{\pi_0(U_{.,\overline{K}})}.$

There is a natural transformation of functors $\Gamma_{\bullet} \to \mathrm{ev}_{\mathrm{A}_{\mathrm{cris}}(U.^{\wedge})}$. Since the functor $\mathrm{ev}_{\mathrm{A}_{\mathrm{cris}}(U.^{\wedge})}$ is exact on quasi–coherent isocrystals this induces a natural morphism

(13.17.7) $R\Gamma_{\bullet}(F \otimes J_E) \to \mathrm{ev}_{\mathrm{A}_{\mathrm{cris}}(U.^{\wedge})}(F \otimes J_E)$

in the derived category $D(\mathrm{Vec}_K^{\pi_0(U_{.,\overline{K}})})$.

We can also consider the topos $((Y, M_Y)/W)_{\mathrm{conv}}|_{h_{Z.}}$ which sits in a diagram
(13.17.8)
$$((Y, M_Y)/W)_{\mathrm{conv}}|_{h_{Z.}} \leftarrow ((Y, M_Y)/V)_{\mathrm{conv}}|_{h_{Z.}} \leftarrow ((Y, M_Y)/V)_{\mathrm{conv}}|_{\mathrm{A}_{\mathrm{cris}}(U^\wedge)}.$$

If

(13.17.9) $\Gamma'_\bullet : (\mathscr{K}_{(Y,M_Y)/W}$–modules in $((Y, M_Y)/W)_{\mathrm{conv}}|_{h_{Z.}}) \to \mathrm{Vec}_K^{\pi_0(U_{.,\overline{K}})}$

denotes the global section functor we then obtain a diagram

(13.17.10) $R\Gamma'_\bullet(F \otimes J_E) \to R\Gamma_\bullet(F \otimes J_E) \to \mathrm{ev}_{\mathrm{A}_{\mathrm{cris}}(U^\wedge)}(F \otimes J_E).$

It follows from the construction that the composite

(13.17.11) $R\Gamma'_\bullet(F \otimes J_E) \to \mathrm{ev}_{\mathrm{A}_{\mathrm{cris}}(U^\wedge)}(F \otimes J_E)$

is compatible with the Frobenius actions. This map induces a morphism in the category $D(\mathrm{Vec}_K^{\pi_0(U_{.,\overline{K}})})$

(13.17.12) $R\Gamma'_\bullet(F \otimes J_E) \to (F \otimes J_E)(\mathrm{B}_{\mathrm{cris}}(U_.^\wedge)).$

13.18. Next we need some facts about Galois representations. Let $\mathrm{Rep}_K^{\mathrm{cts}}(\Delta.)$ denote the simplicial category whose fiber over $[n] \in \underline{\Delta}$ is the category of collections $(L_i)_{i \in \pi_0(U_{n,\overline{K}})}$, where each L_i is a continuous representation of the fundamental group (with respect to the fixed generic base point) of the i–th connected component of $U_{\overline{K}}^o$. The category $\mathrm{Rep}_K^{\mathrm{cts}}(\Delta.)$ is easily seen to be an abelian tensor category with enough injectives. We denote by $D(\mathrm{Rep}_K^{\mathrm{cts}}(\Delta.))$ its derived category.

The functor $\mathrm{ev}_{\mathrm{A}_{\mathrm{cris}}(U^\wedge)}$ is naturally viewed as taking values in $\mathrm{Rep}_K^{\mathrm{cts}}(\Delta.)$, and the transformation of functors $\Gamma_\bullet \to \mathrm{ev}_{\mathrm{A}_{\mathrm{cris}}(U^\wedge)}$ in fact factors through the subfunctor of invariants in $\mathrm{ev}_{\mathrm{A}_{\mathrm{cris}}(U^\wedge)}$.

From this it follows that the morphism 13.17.12 induces a morphism in the derived category $D(\mathrm{Vec}_K^{\pi_0(U_{.,\overline{K}})})$

(13.18.1) $R\Gamma_\bullet(F \otimes J_E) \to GC((F \otimes J_E)(\mathrm{B}_{\mathrm{cris}}(U_.^\wedge)))$

compatible with Frobenius.

LEMMA 13.19. *The induced morphism*

(13.19.1) $R\Gamma_\bullet(F \otimes J_E) \to GC((F \otimes J_E)(\mathrm{B}_{\mathrm{cris}}(U_.^\wedge)))$

extends canonically to a morphism in the filtered derived category.

PROOF. The map can be represented on the level of filtered complexes as follows. Let $F \otimes J_E \to \mathbb{R}^\bullet$ be the resolution given by 13.8. By 13.9 the complex $R\Gamma_\bullet(F \otimes J_E)$ can then be represented, as a filtered complex, by $\mathbb{R}^\bullet(U_.^\wedge)^\nabla$ with the filtration induced by the filtration on the $\mathbb{R}^i$. Then 13.19.1 is given by the map in the filtered derived category
(13.19.2)
$$\mathbb{R}^\bullet(U_.^\wedge)^\nabla \longrightarrow GC(\mathbb{R}^\bullet(\mathrm{B}_{\mathrm{cris}}(U_.^\wedge))) \xleftarrow{\ \simeq\ } GC((F \otimes J_E)(\mathrm{B}_{\mathrm{cris}}(U_.^\wedge))).$$
$\square$

13.20. Combining all of this with 12.5 we obtain the transformation 13.17.1. By functoriality of the above construction and the fact that any two hypercovers as in 13.17 can be refined by a third, the map α is independent of the choices. The main comparison theorem, whose proof occupies the next section, is then the following:

THEOREM 13.21. *The transformation 13.17.1 is an isomorphism compatible with Frobenius, Galois action, and strictly compatible with the filtrations. In addition 13.17.1 is compatible with Chern classes of vector bundles on X.*

14. Proof of the comparison theorem

14.1. Let us begin by proving that the transformation α in 13.17.1 is compatible with Chern classes of vector bundles.

First consider the case of a line bundle $\mathscr{L}$ on Y. For the crystalline Chern class it is now necessary to recall the definition of the crystalline topos (as opposed to the convergent). For this let $\mathrm{Cris}((Y, M_Y)/W)$ denote the site whose objects are strict closed immersions $((U, M_U) \hookrightarrow (T, M_T))$, where U is an étale Y–scheme, together with a divided power structure γ on the ideal of U in T such that the ideal of U in T is a nil–ideal. Morphisms and coverings are defined as in [**29**, 5.2], and the associated topos is denoted $((Y, M_Y)/W)_{\mathrm{cris}}$. Let $\mathscr{O}_{(Y, M_Y)\mathrm{cris}}$ denote the structure sheaf sending $(U, M_U) \hookrightarrow (T, M_T)$ to $\Gamma(T, \mathscr{O}_T)$, and let $\mathscr{O}_{(Y, M_Y)}$ denote the sheaf $((U, M_U) \hookrightarrow (T, M_T)) \mapsto \Gamma(U, \mathscr{O}_U)$. There is a natural surjection $\mathscr{O}_{(Y, M_Y)\mathrm{cris}} \to \mathscr{O}_{(Y, M_Y)}$ whose kernel F^1 is a PD–ideal. Looking at the units we get an exact sequence

$$(14.1.1) \qquad 0 \to 1 + F^1 \to \mathscr{O}^*_{(Y, M_Y)\mathrm{cris}} \to \mathscr{O}^*_{(Y, M_Y)} \to 0.$$

Since F^1 has divided powers there is also a logarithm map

$$(14.1.2) \qquad \log : 1 + F^1 \to \mathscr{O}_{(Y, M_Y)\mathrm{cris}}, \quad 1 + t \mapsto \sum_{m \geq 1} (-1)^{m-1}(m-1)! t^{[m]}.$$

Passing to cohomology we therefore get a map

$$(14.1.3) \qquad
\begin{array}{ccl}
H^1(Y, \mathscr{O}_Y^*) & \underset{\log \otimes \mathbb{Q}}{\overset{\to}{\to}} & H^2(((Y, M_Y)/W)_{\mathrm{cris}}, 1 + F^1) \\
 & & H^2(((Y, M_Y)/W)_{\mathrm{cris}}, \mathscr{O}_{(Y, M_Y)\mathrm{cris}}) \otimes \mathbb{Q} \\
 & \simeq & H^2(((Y, M_Y)/W)_{\mathrm{conv}}, \mathscr{K}_{(Y, M_Y)/W}),
\end{array}$$

where the last isomorphism is by [**46**, 3.1.1]. This defines the crystalline Chern class $c_1^{\mathrm{cr}}(\mathscr{L})$.

To identify the image of $c_1^{\mathrm{cr}}(\mathscr{L})$ under α, assume that $\mathscr{L}$ is trivialized on each U_n (after further refinement we may assume this is the case). Let R_n denote the coordinate ring of U_n so we have an exact sequence of simplicial groups

$$(14.1.4) \qquad 0 \to 1 + F_{\mathrm{A_{cris}}(U_\cdot^\wedge)} \to \mathrm{A_{cris}}(U_\cdot^\wedge)^* \to (\overline{R}_\cdot^\wedge / p \overline{R}_\cdot^\wedge)^* \to 0.$$

Choose trivializations of $\mathscr{L}$ on each U_n so that the gluing data is given by an element $u \in (R_1^\wedge / p R_1^\wedge)^*$. By the method of 13.18 this defines a class in $H^1(GC((\overline{R}_\cdot^\wedge / p \overline{R}_\cdot^\wedge)^*))$ and hence by composing with the boundary map and the logarithm map we get $\alpha(c_1^{\mathrm{cr}}(\mathscr{L}))$ from the composite

$$(14.1.5) \qquad
\begin{array}{ccl}
H^1(GC(\overline{R}_\cdot^\wedge / p \overline{R}_\cdot^\wedge)^*) & \to & H^2(GC(1 + F_{\mathrm{A_{cris}}(U_\cdot^\wedge)})) \\
 & \overset{\log}{\to} & H^2(GC(\mathrm{A_{cris}}(U_\cdot^\wedge))) \\
 & \to & H^2(GC(\mathrm{B_{cris}}(U_\cdot^\wedge))).
\end{array}$$

14.2. The image of the étale Chern class $c_1^{\mathrm{et}}(\mathscr{L})$ in $H^2(GC(\mathrm{B_{cris}}U_\cdot^\wedge))(1)$ under the isomorphism 12.5 is obtained as follows. Consider the sequence

$$(14.2.1) \qquad 0 \to \mathbb{Z}_p(1) \to \varprojlim \overline{R}_\cdot^{\wedge *} \to \overline{R}_\cdot^{\wedge *} \to 0,$$

where the projective limit in the middle is given by the maps $\overline{R}_{\cdot}^{\wedge *} \to \overline{R}_{\cdot}^{\wedge *}$ sending f to f^p. Then $c_1^{\mathrm{et}}(\mathscr{L})$ is given by the map

$$
(14.2.2) \quad
\begin{aligned}
H^1(R_{\cdot}^{\wedge *}) &\to & H^1(GC(\overline{R}_{\cdot}^{\wedge *})) \\
&\to & H^2(GC(\mathbb{Z}_p(1))) \\
&\to & H^2(GC(\mathrm{A}_{\mathrm{cris}}(R_{\cdot}^{\wedge})))(1).
\end{aligned}
$$

Here the first map is induced by the maps $R_m^{\wedge *} \to GC(\overline{R}_m^{\wedge *})$ obtained from the natural inclusion of $R_m^{\wedge *}$ into the Galois invariants in $\overline{R}_m^{\wedge *}$.

Let $\alpha : \mathbb{Z}_p(1) \to 1 + F^1$ be the map defined in 11.1.8, and let $\sigma : \varprojlim \overline{R}_{\cdot}^{\wedge *} \to \mathrm{A}_{\mathrm{cris}}(R_{\cdot}^{\wedge})$ be the composite

$$
(14.2.3) \quad
\begin{aligned}
\varprojlim \overline{R}_{\cdot}^{\wedge *} &\underset{[-]}{\to} & S_{\cdot}^* := \varprojlim (\overline{R}_{\cdot}^{\wedge}/p\overline{R}_{\cdot}^{\wedge})^* \\
&\mapsto & W(S_{\cdot})^* \\
&\to & \mathrm{A}_{\mathrm{cris}}(R_{\cdot}^{\wedge})^*,
\end{aligned}
$$

where $[-] : S_{\cdot}^* \to W(S_{\cdot})^*$ denote the Teichmuller lifting. Then it follows from the definitions that the diagram

(14.2.4)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}_p(1) & \longrightarrow & \varprojlim \overline{R}_{\cdot}^{\wedge *} & \longrightarrow & \overline{R}_{\cdot}^{\wedge *} & \longrightarrow & 0 \\
& & \alpha \downarrow & & \sigma \downarrow & & \downarrow & & \\
0 & \longrightarrow & 1 + F_{\mathrm{A}_{\mathrm{cris}}(U_{\cdot}^{\wedge})} & \longrightarrow & \mathrm{A}_{\mathrm{cris}}(U_{\cdot}^{\wedge})^* & \longrightarrow & (\overline{R}_{\cdot}^{\wedge}/p\overline{R}_{\cdot}^{\wedge})^* & \longrightarrow & 0
\end{array}
$$

commutes, and from this one obtains the following:

PROPOSITION 14.3. *The diagram*
(14.3.1)

$$
\begin{array}{ccc}
H^1(X, \mathscr{O}_X^*) & \xrightarrow{\;c_1^{\mathrm{cr}}(-)\;} & H^2(((Y,M_Y)/V)_{\mathrm{conv}}, \mathscr{K}_{(Y,M_Y)/V} \otimes \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V}) \\
\downarrow{\scriptstyle c_1^{\mathrm{et}}(-)} & & \downarrow{\scriptstyle \alpha} \\
H^2(X_{\overline{K},\mathrm{et}}^o, \mathbb{Q}_p(1)) \otimes_{\mathbb{Q}_p} \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V}) & \xrightarrow{\quad\beta\quad} & H^2(X_{\overline{K},\mathrm{et}}^o, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V})
\end{array}
$$

commutes, where the map β is obtained from the map 11.1.9.

14.4. It is now an essentially formal consequence that α is also compatible with higher Chern classes of vector bundles. For this let H^* denote either étale cohomology (with appropriate Tate twists) or crystalline cohomology and let $c_1 : H^1(X, \mathscr{O}_X^*) \to H^2(X)$ denote the Chern class map. Then for a vector bundle $\mathscr{E}$ on X the higher Chern classes $c_i(\mathscr{E}) \in H^{2i}(X)$ are defined as follows. Let $Y = \mathbb{P}_X(\mathscr{E}^*)$ denote the projective bundle associated to the dual of $\mathscr{E}$. Let $\xi \in H^2(Y)$ denote the first Chern class of the tautological line bundle on Y. Then the cohomology ring $H^*(Y)$ is a free $H^*(X)$–module with basis $1, \xi, \ldots, \xi^{r-1}$. In particular, there exists a unique relation

$$
(14.4.1) \qquad \xi^r + c_1(\mathscr{E})\xi^{r-1} + \cdots + c_{r-1}(\mathscr{E})\xi + c_r(\mathscr{E}) = 0
$$

in $H^{2r}(Y)$ with $c_i(\mathscr{E}) \in H^{2i}(X)$. This defines the higher Chern classes of a vector bundle. Since the transformation α is functorial with respect to the map $Y \to X$ the following proposition follows from the case of a line bundle.

PROPOSITION 14.5. *For any vector bundle $\mathscr{E}$ on X, we have*

$$(14.5.1) \qquad\qquad \alpha(c_i^{\mathrm{cr}}(\mathscr{E})) = \beta^{\otimes i}(c_i^{\mathrm{et}}(\mathscr{E})),$$

where $\beta^{\otimes i}$ is defined as in 11.1.10.

14.6. By construction the transformation α is functorial with respect to morphisms $f : (Y, D_Y) \to (X, D_X)$, where X and Y are smooth V–schemes of dimension d_Y and d_X respectively, $D_Y \subset Y$ and $D_X \subset X$ are divisors with normal crossings relative to V, and $f^{-1}(D_X) \subset D_Y$. For such a morphism f the pullback functor f^* has an adjoint f_* with respect to the pairings provided by Poincaré duality. Precisely, for any decompositions $D_X = E \cup F$ and $D_Y = E_Y \cup F_Y$ such that $f^{-1}(E) \subset E_Y$ we get as in 8.20 maps
(14.6.1)
$$f_*^{\mathrm{cr}} : H^i(((Y, M_Y)/V)_{\mathrm{conv}}, f^*M \otimes J_{E_Y}) \to H^{2d_X - 2d_Y + i}(((X, M_X)/V)_{\mathrm{conv}}, M \otimes J_E)$$

and

$$(14.6.2) \qquad f_*^{\mathrm{et}} : H_{E_Y, F_Y}^i(Y_{\overline{K}}^o, f^*L) \to H_{E,F}^{2d_X - 2d_Y + i}(X_{\overline{K}}^o, L)(d_X - d_Y).$$

PROPOSITION 14.7. *Assume that f is a closed immersion of pure codimension e. Then $\alpha \circ f_*^{\mathrm{cr}} = \beta^{\otimes e} \circ f_*^{\mathrm{et}} \circ \alpha$.*

PROOF. To ease notation write just

$$(14.7.1) \qquad \begin{array}{c} H_{\mathrm{cris}}^i(X) \text{ for } H^i(((X, M_X)/V)_{\mathrm{conv}}, M \otimes J_E), \\ H_{\mathrm{cris}}^i(Y) \text{ for } H^i(((Y, M_Y)/V)_{\mathrm{conv}}, f^*M \otimes J_{E_Y}), \\ H_{\mathrm{et}}^i(X) \text{ for } H_{E,F}^i(X_{\overline{K}}^o, L) \text{ etc.} \end{array}$$

Let Z denote the blow-up of $X \times \mathbb{P}^1$ along $Y \times \{\infty\}$. A calculation in local coordinates shows that the strict transform $W \subset Z$ of $Y \times \mathbb{P}^1$ maps isomorphically to $Y \times \mathbb{P}^1$ so we obtain a lifting $\tilde{f} : Y \times \mathbb{P}^1 \hookrightarrow Z$. For any point $t \in \mathbb{P}^1$, let Z_t denote the fiber of Z over t and let $f_t : Y \hookrightarrow Z_t$ be the inclusion induced by $\tilde{f}$. Let $\mathscr{N}$ denote the normal bundle of Y in X. A calculation shows that Z_∞ has two irreducible components Z_∞' and Z_∞'', where Z_∞' is the projective bundle $\mathbb{P}_Y(\mathscr{N}^* \oplus \mathscr{O}_Y)$ and Z_∞'' is the blow-up of X along Y. Furthermore, the intersection $Z_\infty' \cap Z_\infty''$ is the projective bundle $\mathbb{P}_Y(\mathscr{N}^*)$ sitting in $\mathbb{P}_Y(\mathscr{N}^* \oplus \mathscr{O}_Y)$ by the embedding given by the projection $\mathscr{N}^* \oplus \mathscr{O}_Y \to \mathscr{N}^*$ to the first factor. Finally the image of Y in Z_∞ is contained in Z_∞' and does not meet $Z_\infty' \cap Z_\infty''$.

Let $H^*(Z)$ denote the cohomology of Z with partial compact support along the strict transform $\overline{D}$ of D, and let $H^*(Z, Z_\infty)$ denote the cohomology with compact support along $\overline{D} \cup Z_\infty$. If $j : Z - (\overline{D} \cup Z_\infty) \to Z - D$ and $i : Z_\infty - (\overline{D} \cap Z_\infty) \hookrightarrow Z - D$ are the inclusions, then for any sheaf L on Z_{et} there is an exact sequence of étale sheaves

$$(14.7.2) \qquad\qquad 0 \to j_! j^* L \to L \to i_* i^* L \to 0.$$

It follows that the sequence

$$(14.7.3) \qquad H_{\mathrm{et}}^*(Z, Z_\infty) \xrightarrow{\ p\ } H_{\mathrm{et}}^*(Z) \xrightarrow{\ q\ } H_{\mathrm{et}}^*(Z_\infty)$$

is exact. We also have a pullback map $j_0^* : H_{\mathrm{et}}^*(Z) \to H_{\mathrm{et}}^*(Z_0)$.

LEMMA 14.8. *The composite*

$$(14.8.1) \qquad H_{\mathrm{et}}^*(Z, Z_\infty) \xrightarrow{\ p\ } H_{\mathrm{et}}^*(Z) \xrightarrow{\ j_0^*\ } H_{\mathrm{et}}^*(Z_0)$$

is zero.

PROOF. The scheme $Z - (D \cup Z_\infty)$ is isomorphic to $(X - D) \times \mathbb{A}^1$ and hence by Künneth we have $H^*_{\mathrm{et}}(Z, Z_\infty) \simeq H^*_{\mathrm{et}}(X - D) \otimes H^2_c(\mathbb{A}^1) \simeq H^{*-2}_{\mathrm{et}}(X - D)(1)$. With this identification the map p is given by the pushforward map

(14.8.2) $$j_{0*} : H^{*-2}_{\mathrm{et}}(X - D)(1) \to H^*_{\mathrm{et}}(Z).$$

Thus we wish to show that the map

(14.8.3) $$j_0^* j_{0*} : H^{*-2}_{\mathrm{et}}(X - D)(1) \to H^*_{\mathrm{et}}(X - D)$$

is zero. This follows from the self–intersection formula [**24**, VII.4.1] which shows that this map is given by cup-product by the first Chern class of the conormal bundle of Z_0 in Z which is trivial. $\square$

Note also that the map

(14.8.4) $$H^*_{\mathrm{et}}(Z'_\infty) \to H^*_{\mathrm{et}}(Z'_\infty \cap Z''_\infty)$$

is surjective since both are generated over $H^*_{\mathrm{et}}(Y)$ by the first Chern class of the tautological line bundle. From the excision sequence we deduce that the map

(14.8.5) $$H^*_{\mathrm{et}}(Z_\infty) \to H^*_{\mathrm{et}}(Z'_\infty) \oplus H^*_{\mathrm{et}}(Z''_\infty)$$

is injective so we obtain an exact sequence

(14.8.6) $$H^*_{\mathrm{et}}(Z, Z_\infty) \xrightarrow{\ p\ } H^*_{\mathrm{et}}(Z) \longrightarrow H^*_{\mathrm{et}}(Z'_\infty) \oplus H^*_{\mathrm{et}}(Z''_\infty).$$

In either theory, the composite

(14.8.7) $$H^*(Y) \xrightarrow{\mathrm{pr}_1^*} H^*(Y \times \mathbb{P}^1) \xrightarrow{\tilde{f}_*} H^*(Z) \xrightarrow{j_0^*} H^*(Z_0) = H^*(X)$$

is equal to f_*. Combining this with 14.8 and 14.8.6 it follows that it suffices to show that for any element $a \in H^*_{\mathrm{cris}}(Y)$ the images of the classes $\alpha(\tilde{f}_*^{\mathrm{cr}}(a))$ and $\tilde{f}_*^{\mathrm{et}}(\alpha(a))$ in $H^*_{\mathrm{et}}(Z'_\infty)$ and in $H^*_{\mathrm{et}}(Z''_\infty)$ are equal. In either theory the formation of pushforwards commutes with pullbacks along smooth subvarieties transverse to the boundary. In the case of a divisor transverse to the boundary, this follows from the description of the pushforward as a boundary map 8.23 and its crystalline analog (which we leave to the reader). The general case is deduced from this using the trick following 8.54 in the proof of 8.21. This reduces the proof of 14.7 to the case of $Y \hookrightarrow Z'_\infty$ (note that the case of Z''_∞ is trivial since $Y \times \mathbb{P}^1 \cap Z''_\infty$ is empty).

The proof is now reduced to the case of the inclusion $f : Y \hookrightarrow \mathbb{P}_Y(\mathscr{N}^* \oplus \mathscr{O}_Y)$. For any class $\beta \in H^*(Y)$ we have $f_*(\beta) = f_*(f^* p^* \beta) = f_*(1) \cup p^* \beta$, and hence it suffices to consider the case of constant coefficients and $f_*(1)$. But in this case it is well–known that $f_*(1)$ in either theory is given by $c_r(\mathscr{N}^*)$ (this can for instance be verified in the rational Chow ring where it is standard). By 14.5 this completes the proof of 14.7 $\square$

COROLLARY 14.9. *Let* $\mathrm{tr}^{\mathrm{cr}} : H^{2d}(((X, M_X)/V)_{conv}, M \otimes J_D) \otimes_K \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V}) \to \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V})$ *denote the crystalline trace map and* $\mathrm{tr}^{\mathrm{et}} : H^{2d}_c(X^o_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V}) \to \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V})(d)$ *the étale trace map. Then the diagram*

(14.9.1)
$$
\begin{array}{ccc}
H^{2d}(((X, M_X)/V)_{\mathrm{conv}}, M \otimes J_D) \otimes_K \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V}) & \xrightarrow{\ \mathrm{tr}^{\mathrm{cr}}\ } & \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V}) \\
{\scriptstyle \alpha} \downarrow & & \uparrow {\scriptstyle \beta^{\otimes d}} \\
H^{2d}_c(X^o_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V}) & \xrightarrow{\ \mathrm{tr}^{\mathrm{et}}\ } & \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V})(d)
\end{array}
$$

commutes. In particular, the map

$$\alpha : H^{2d}(((X, M_X)/V)_{\mathrm{conv}}, M \otimes J_D) \otimes_K \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V}) \to H_c^{2d}(X_{\overline{K}}^o, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V})$$

is an isomorphism.

PROOF. It suffices to show this after making an extension of V when $X^o \to$ $\mathrm{Spec}(V)$ has a section. In this case the corollary follows from observing that the trace map is characterized by the fact that it sends the class of a point in X^o to 1. $\qquad\square$

14.10. It follows that there exists a unique map

$$\alpha^t : H_{E,F}^*(X_{\overline{K}}^o, L) \otimes_{\mathbb{Q}_p} \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V}) \to H^*(((X, M_X)/V)_{\mathrm{conv}}, M \otimes J_E) \otimes_K \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V})$$

such that for any element

$$a \in H^{2d-*}(((X, M_X)/V)_{\mathrm{conv}}, M \otimes J_F)$$

we have

$$(14.10.1) \qquad\qquad \mathrm{tr}^{\mathrm{cr}}(\alpha^t(b) \cup a) = \beta^{\otimes d}(\mathrm{tr}^{\mathrm{et}}(b \cup \alpha(a))).$$

Using the same argument used in 8.60 and 8.61 one shows that α^t is also an isometry. It follows that α and α^t are both isomorphisms. This completes the proof that 13.17.1 is an isomorphism.

14.11. Finally let us verify that the isomorphism α is strictly compatible with the filtrations. For this it suffices to show that the morphism on the associated graded is an inclusion, for then the inverse for α also preserves the filtrations.

Let $D = E \cup F$ be a decomposition of the boundary, and let $F^\bullet$ denote the filtration on $H^*(X/K, M \otimes J_E)$ induced by the filtration on M and let $\widehat{F}^\bullet$ denote the filtration on $H^*(X/K, M^* \otimes J_F)$ induced by the dual filtration on M^*. It then suffices to show that the induced pairing

$$(14.11.1) \quad \mathrm{gr}_F^s H^k(X/K, M \otimes J_E) \otimes \mathrm{gr}_{\widehat{F}}^{-s} H^{2d-k}(X/K, M^* \otimes J_F) \to H^d(X, \Omega_X^d)$$

is a perfect pairing for all k and s. To see that this suffices let G denote the filtration on $H_{E,F}^*(X_{\overline{K}}^o, L) \otimes_{\mathbb{Q}_p} \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V})$ induced by the filtration on $\widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V})$, and to ease the notation write

$$\mathrm{gr}_F^{s,k} := \mathrm{gr}_F^s H^k(X/K, M \otimes J_E) \otimes_K \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V}),$$

$$\mathrm{gr}_{\widehat{F}}^{-s,2d-k} := \mathrm{gr}_{\widehat{F}}^{-s} H^{2d-k}(X/K, M^* \otimes J_F) \otimes_K \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V}),$$

and

$$\mathrm{gr}_G^{s,k} := \mathrm{gr}_G^s H_{E,F}^k(X_{\overline{K}}^o, L) \otimes_{\mathbb{Q}_p} \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V}).$$

We then have a commutative diagram

$$(14.11.2) \qquad\qquad \begin{array}{ccc} \mathrm{gr}_F^{s,k} \times \mathrm{gr}_{\widehat{F}}^{-s,2d-k} & \longrightarrow & \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V}) \\ \downarrow{\scriptstyle \alpha \times \alpha} & & \| \\ \mathrm{gr}_G^{s,k} \times \mathrm{gr}_G^{-s,2d-k} & \longrightarrow & \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V}) \end{array}$$

and since both the top and bottom lines are perfect pairings it follows that the map

$$(14.11.3) \quad \mathrm{gr}_F^s H^k(X/K, M \otimes J_E) \otimes_K \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V}) \to \mathrm{gr}_G^s H_{E,F}^k(X_{\overline{K}}^o, L) \otimes_{\mathbb{Q}_p} \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V})$$

is an inclusion.

The following therefore completes the proof of 13.21.

LEMMA 14.12. *The pairing 14.11.1 is perfect.*

PROOF. For every integer i there is a natural inclusion
(14.12.1)
$$\widehat{F}^{-i} \subset \{c \in H^{2d-k}(X/K, M^* \otimes J_F)|c \cup \beta = 0 \text{ for all } \beta \in F^i H^k(X/K, M \otimes J_E)\}.$$

To prove the lemma it suffices to show that these inclusions are isomorphisms. Let $\mathcal{F}^i \otimes J_E$ denote the complex

$$(14.12.2) \qquad \text{Fil}^i_{\mathcal{M}} \otimes J_E \to \text{Fil}^{i-1}_{\mathcal{M}} \otimes J_E \otimes \Omega^1_{(X_K, M_{X_K})/K} \to \cdots,$$

and let $\text{DR}(\mathcal{M} \otimes J_E)/\mathcal{F}^i$ denote the quotient of the de Rham complex of $(\mathcal{M}, \nabla)$ tensored with the ideal J_E by the subcomplex $\mathcal{F}_i \otimes J_E$. The subspace F^i of $H^k(X/K, M \otimes J_E)$ is defined to be the image of the natural map

$$(14.12.3) \qquad H^k(X_K, \mathcal{F}^i \otimes J_E) \to H^k(X_K, DR(\mathcal{M} \otimes J_E)),$$

and $\widehat{F}^{-i} \subset H^{2d-k}(X_K, DR(\mathcal{M}^* \otimes J_F))$ is the image of the natural map

$$(14.12.4) \qquad H^{2d-k}(X_K, \widehat{\mathcal{F}}^{-i} \otimes J_F) \to H^{2d-k}(X_K, DR(\mathcal{M}^* \otimes J_F)),$$

One shows as in 13.7 that the natural pairing
(14.12.5)
$$H^k(X_K, \mathcal{F}^s \otimes J_E) \times H^{2d-k}(X_K, DR(\mathcal{M}^* \otimes J_F)/\widehat{F}^{-s} \otimes J_F) \to H^d(X_K, \Omega^d_{X_K})$$

is a perfect pairing, and from this it follows that the image of 14.12.4 is via Poincaré duality identified with the elements annihilating the image of 14.12.3. □

15. From $\widetilde{\text{B}}_{\text{cris}}(\overline{V})$ to $\text{B}_{\text{cris}}(\overline{V})$

In this section we explain how to lift the isomorphism 13.17.1 to an isomorphism over $\text{B}_{\text{cris}}(\overline{V})$. This will be done using some general observations about the rings $\text{B}_{\text{dR}}, \widetilde{\text{B}}_{\text{cris}}(\overline{V})$, and $\text{B}_{\text{cris}}(\overline{V})$. Let us start by recalling the definitions of these rings.

15.1. Let V be a complete discrete valuation ring of mixed characteristic $(0, p)$, field of fractions K, and perfect residue field k. Let W be the ring of Witt vectors of k, and let $K_0 \subset K$ be the field of fractions of W. Let $K \hookrightarrow \overline{K}$ be an algebraic closure, and let $S_{\overline{V}}$ (resp. $\text{A}_{\text{cris}}(\overline{V}), \text{B}_{\text{cris}}(\overline{V}), \widetilde{\text{B}}_{\text{cris}}(\overline{V}))$ be the rings defined in 11.1. As in 11.1.2 there is a surjection

$$(15.1.1) \qquad\qquad \theta : W(S_{\overline{V}}) \to \overline{V}^{\wedge},$$

where $\overline{V}^{\wedge}$ denotes the p-adic completion of $\overline{V}$. If J denotes the kernel of θ then $\text{A}_{\text{cris}}(\overline{V})$ is defined to be the p-adic completion of the divided power envelope $D_J(W(S_{\overline{V}}))$.

Let $t \in \text{A}_{\text{cris}}(\overline{V})$ be the element 11.1.5 so that $\text{B}_{\text{cris}}(\overline{V})$ is equal to the localization $\text{A}_{\text{cris}}(\overline{V})[1/t]$. Also for $\alpha \in \Lambda_+ := \mathbb{Z}[1/p] \cap \mathbb{Q}_{>0}$ define $\delta_\alpha \in \text{A}_{\text{cris}}(\overline{V})$ as in 11.12 so that $\widetilde{\text{B}}_{\text{cris}}(\overline{V}) = \text{B}_{\text{cris}}(\overline{V})[1/\delta_\alpha]_{\alpha \in \Lambda_+}$. Note that by construction $\delta^{p^n}_{1/p^n} = \delta_1$ so we also have $\widetilde{\text{B}}_{\text{cris}}(\overline{V}) = \text{B}_{\text{cris}}(\overline{V})[1/\delta_1]$.

15.2. Let us also recall the rings B_{dR}^+ and its field of fractions B_{dR}. Let $(B_n, J_n, [\ ])$ be the divided power ring which is the reduction of $A_{cris}(\overline{V})$ modulo p^n (for a cohomological interpretation of this ring see [**49**, A1.6 and paragraph following A2.10]). Then we set

$$(15.2.1) \qquad\qquad B_{dR}^+ := \varprojlim_{r} (\mathbb{Q} \otimes \varprojlim_{n} B_n / J_n^{[r]}).$$

There is a natural map $A_{cris}(\overline{V}) \otimes \mathbb{Q} \to B_{dR}^+$, and one can show (see for example [**16**, 1.5.2]) that B_{dR}^+ is a complete discrete valuation ring and that the image of $t \in A_{cris}(\overline{V})$ in B_{dR}^+ is a uniformizer. The field B_{dR} is defined to be the field of fractions of B_{dR}^+. There is a natural inclusion $B_{cris}(\overline{V}) \hookrightarrow B_{dR}$, which induces an inclusion $\widetilde{B}_{cris}(\overline{V}) \hookrightarrow B_{dR}$. Note that the action of G_K on $A_{cris}(\overline{V})$ induces an action of G_K on B_{dR}^+ and B_{dR} and the inclusion $\widetilde{B}_{cris}(\overline{V}) \hookrightarrow B_{dR}$ is compatible with the G_K-actions.

LEMMA 15.3. (i) *The natural map*

$$(15.3.1) \qquad\qquad K_0 \simeq B_{cris}(\overline{V})^{G_K} \to \widetilde{B}_{cris}(\overline{V})^{G_K}$$

is an isomorphism, where the first isomorphism is by [**17**, 5.1.2].
(ii) *For any G_K-representation W the natural map*

$$(15.3.2) \qquad\qquad K \otimes_{K_0} \widetilde{B}_{cris}(\overline{V}) \otimes_{\mathbb{Q}_p} W \to B_{dR} \otimes_{\mathbb{Q}_p} W$$

is injective.

PROOF. By [**16**, 1.5.7] we have $K \simeq B_{dR}^{G_K}$, so $\widetilde{B}_{cris}(\overline{V})^{G_K}$ is a subring of K containing K_0. Also by [**16**, 4.2.4] the natural map

$$(15.3.3) \qquad\qquad K \otimes_{K_0} B_{cris}(\overline{V}) \to B_{dR}$$

is injective. Since localization is an exact functor this implies that the natural map

$$(15.3.4) \qquad\qquad K \otimes_{K_0} \widetilde{B}_{cris}(\overline{V}) \to B_{dR}$$

is also injective, which implies that the map

$$(15.3.5) \qquad\qquad K \otimes_{K_0} \widetilde{B}_{cris}(\overline{V})^{G_K} \to B_{dR}^{G_K} = K$$

is an inclusion. This is only possible if $\widetilde{B}_{cris}(\overline{V})^{G_K} = K_0$ and we obtain (i).

For (ii) note that it suffices to consider $W = \mathbb{Q}_p$, in which case we already remarked the result (namely that 15.3.4 is injective). $\qquad\square$

15.4. For any finite dimensional continuous G_K-representation W set

$$(15.4.1) \qquad\qquad D_{\widetilde{cris}}(W) := (W \otimes_{\mathbb{Q}_p} \widetilde{B}_{cris}(\overline{V}))^{G_K},$$

$$(15.4.2) \qquad\qquad D_{cris}(W) := (W \otimes_{\mathbb{Q}_p} B_{cris}(\overline{V}))^{G_K},$$

$$(15.4.3) \qquad\qquad D_{dR}(W) := (W \otimes_{\mathbb{Q}_p} B_{dR})^{G_K}.$$

There are natural inclusions

$$(15.4.4) \qquad\qquad D_{cris}(W) \subset D_{\widetilde{cris}}(W) \subset D_{dR}(W).$$

Since $D_{dR}(W)$ is a finite dimensional K-vector space of dimension $\leq \dim_{\mathbb{Q}_p} W$ [**17**, 1.4.2 Proposition] and the natural map

$$(15.4.5) \qquad\qquad K \otimes_{K_0} D_{\widetilde{cris}}(W) \to D_{dR}(W)$$

is injective (this follows from 15.3 (ii) by taking G_K-invariants), the K_0-vector space $D_{\widetilde{\mathrm{cris}}}(W)$ has dimension $\leq \dim_{\mathbb{Q}_p}(W)$. Moreover if

(15.4.6) $$\dim_{K_0} D_{\widetilde{\mathrm{cris}}}(W) = \dim_{\mathbb{Q}_p} W$$

then the map 15.4.5 is an isomorphism.

There is also a commutative diagram

(15.4.7)
$$
\begin{array}{ccc}
D_{\widetilde{\mathrm{cris}}}(W) \otimes_{K_0} \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V}) & \xrightarrow{\ a\ } & D_{\mathrm{dR}}(W) \otimes_K \mathrm{B}_{\mathrm{dR}} \\
\downarrow & & \downarrow{\scriptstyle b} \\
W \otimes_{\mathbb{Q}_p} \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V}) & \longrightarrow & W \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}.
\end{array}
$$

The map a is an inclusion as it is equal to the composite of the inclusion

(15.4.8) $D_{\widetilde{\mathrm{cris}}}(W) \otimes_{K_0} \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V}) \hookrightarrow D_{\widetilde{\mathrm{cris}}}(W) \otimes_{K_0} \mathrm{B}_{\mathrm{dR}} = (D_{\widetilde{\mathrm{cris}}}(W) \otimes_{K_0} K) \otimes_K \mathrm{B}_{\mathrm{dR}}$

and the inclusion

(15.4.9) $$(D_{\widetilde{\mathrm{cris}}}(W) \otimes_{K_0} K) \otimes_K \mathrm{B}_{\mathrm{dR}} \hookrightarrow D_{\mathrm{dR}}(W) \otimes_K \mathrm{B}_{\mathrm{dR}}$$

obtained from 15.4.5. The map b is an injection by [**17**, 3.6 Proposition]. It follows that the natural map

(15.4.10) $$D_{\widetilde{\mathrm{cris}}}(W) \otimes_{K_0} \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V}) \to W \otimes_{\mathbb{Q}_p} \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V})$$

is also an injection. We say that W is $\widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V})$-*admissible* if this map 15.4.10 is an isomorphism.

Similarly one obtains a notion of $\mathrm{B}_{\mathrm{cris}}(\overline{V})$-admissible (resp. B_{dR}-admissible) representations. Following standard terminology we call a $\mathrm{B}_{\mathrm{cris}}(\overline{V})$-admissible (resp. B_{dR}-admissible) representation a *crystalline* (resp. *de Rham*) representation. Note that we have the following implications:

(15.4.11) $$\text{crystalline} \implies \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V})\text{-admissible} \implies \text{de Rham}.$$

The main result of this section is the following:

THEOREM 15.5. *Any $\widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V})$-admissible representation is crystalline.*

PROOF. Let W be a $\widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V})$-admissible representation and let M_0 denote $D_{\widetilde{\mathrm{cris}}}(W)$ so we have a G_K-equivariant isomorphism

(15.5.1) $$W \otimes_{\mathbb{Q}_p} \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V}) \simeq M_0 \otimes_{K_0} \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V}).$$

Note that then the natural map

(15.5.2) $$M_0 \to W \otimes_{\mathbb{Q}_p} \widetilde{\mathrm{B}}_{\mathrm{cris}}(\overline{V}) \hookrightarrow W \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}$$

induces an isomorphism

(15.5.3) $$M := M_0 \otimes_{K_0} K \to (W \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}})^{G_K}.$$

Let $\pi \in V$ be a uniformizer and fix a sequence of elements $s = \{s_n\}_{n \geq 0}$ of $\overline{V}$ such that $s_0 = \pi$, $s_{n+1}^p = s_n$. Denote by $\epsilon(s) \in W(S_{\overline{V}})$ the Teichmuller lifting of the element of $S_{\overline{V}}$ defined by the reductions of the s_n. Then one can show (see for example [**30**, 2.2]) that the series $\log(\epsilon(s)\pi^{-1})$ converges to an element $u_s \in \mathrm{B}_{\mathrm{dR}}^+$. Let $\mathrm{B}_{\mathrm{st}} \subset \mathrm{B}_{\mathrm{dR}}$ denote the subalgebra over $\mathrm{B}_{\mathrm{cris}}(\overline{V})$ generated by u_s. Then as explained in [**16**, 3.1.5] the element u_s is transcendental over $\mathrm{B}_{\mathrm{cris}}(\overline{V})$ so

B_{st} is isomorphic to a polynomial algebra over $B_{cris}(\overline{V})$. Also the G_K-action on B_{dR} restricts to a G_K-action on B_{st}.

LEMMA 15.6. *We have*

$$(15.6.1) \qquad\qquad B_{cris}(\overline{V}) = \widetilde{B}_{cris}(\overline{V}) \cap B_{st} \subset B_{dR}.$$

PROOF. Since localization is an exact functor, the subalgebra of B_{dR} generated by $\widetilde{B}_{cris}(\overline{V})$ and u_s is isomorphic to a polynomial algebra $\widetilde{B}_{cris}(\overline{V})[u]$ such that a polynomial $\sum_i a_i u^i$ is in $B_{st} = B_{cris}(\overline{V})[u]$ if and only if all the coefficients a_i are in $B_{cris}(\overline{V})$. This implies the lemma. $\qquad\square$

From this we deduce 15.5 as follows. By [**3**, 0.7] any de Rham representation is potentially semistable which implies that there exists a finite extension $K \subset K' \subset \overline{K}$ such that

$$(15.6.2) \qquad K' \otimes_{K'_0} (W \otimes_{\mathbb{Q}_p} B_{st})^{G_{K'}} \to (W \otimes_{\mathbb{Q}_p} B_{dR})^{G_{K'}}$$

is an isomorphism, where K'_0 denotes the field of fractions of the ring of Witt vectors of the residue field of K'. Let $\widetilde{B}_{st}$ denote $B_{st} \otimes_{B_{cris}(\overline{V})} \widetilde{B}_{cris}(\overline{V})$. By a similar reasoning as in 15.3 we have $\widetilde{B}_{st}^{G_{K'}} = K'_0$. It follows that we have a diagram of injections of K'_0-vector spaces of dimension equal to $\dim_{\mathbb{Q}_p} W$

$$(15.6.3) \qquad (W \otimes_{\mathbb{Q}_p} B_{st})^{G_{K'}} \lhook\joinrel\longrightarrow (W \otimes_{\mathbb{Q}_p} \widetilde{B}_{st})^{G_{K'}}$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Bigg\uparrow$$
$$\qquad\qquad\qquad\qquad (W \otimes_{\mathbb{Q}_p} \widetilde{B}_{cris}(\overline{V}))^{G_{K'}}$$

It follows that all these maps are isomorphisms. Therefore the image of the map

$$(15.6.4) \qquad\qquad M_0 \to W \otimes_{\mathbb{Q}_p} \widetilde{B}_{st}$$

is contained in

$$(15.6.5) \qquad (W \otimes_{\mathbb{Q}_p} \widetilde{B}_{cris}(\overline{V})) \cap (W \otimes_{\mathbb{Q}_p} B_{st})$$

which by 15.6 is equal to $W \otimes_{\mathbb{Q}_p} B_{cris}(\overline{V})$. Therefore $M_0 \subset D_{cris}(W)$ which implies that

$$(15.6.6) \qquad\qquad \dim_{K_0} D_{cris}(W) = \dim_{\mathbb{Q}_p} W.$$

Therefore W is crystalline [**17**, 1.4.2 Proposition]. $\qquad\square$

REMARK 15.7. It should be remarked that Berger's theorem that de Rham representations are potentially semistable was not available at the time of Tsuji's original proof of the comparison theorem.

15.8. Note that if W is a crystalline representation, then the natural map $D_{cris}(W) \to D_{\widetilde{cris}}(W)$ is an isomorphism and the isomorphism

$$(15.8.1) \qquad\qquad D_{\widetilde{cris}}(W) \otimes_{K_0} \widetilde{B}_{cris}(\overline{V}) \simeq W \otimes_{\mathbb{Q}_p} \widetilde{B}_{cris}(\overline{V})$$

is induced by the isomorphism

$$(15.8.2) \qquad D_{\widetilde{cris}}(W) \otimes_{K_0} B_{cris}(\overline{V}) = D_{cris}(W) \otimes_{K_0} B_{cris}(\overline{V}) \simeq W \otimes_{\mathbb{Q}_p} B_{cris}(\overline{V}).$$

COROLLARY 15.9. *The map α in 13.17.1 induces an isomorphism*

$$(15.9.1) \quad \mathrm{B}_{\mathrm{cris}}(\overline{V}) \otimes_K H^*(((Y, M_Y)/V)_{\mathrm{conv}}, F \otimes J_E) \to \mathrm{B}_{\mathrm{cris}}(\overline{V}) \otimes_{\mathbb{Q}_p} H^*_{E,F}(X_{\overline{J},\mathrm{et}}, L)$$

compatible with Galois actions, Frobenius, and filtrations.

Appendix A. Künneth formula in group cohomology

The results of this appendix are standard, but we were unable to locate a suitable reference so we include them here.

A.1. Let R be a ring and let G and H be profinite groups. For a continuous representation M (resp. N) of G (resp. H) over R, let $M \boxtimes N$ be the continuous representation of $G \times H$ over R with underlying R-module $M \otimes_R N$ and action of $(g, h) \in G \times H$ given by

$$m \otimes n \mapsto (g \cdot m) \otimes (h \cdot n).$$

A.2. Let Δ denote the standard simplicial category with objects the finite ordered sets $[n] = \{0, \ldots, n\}$ and morphisms the order preserving maps. Let $G^{[n]}$ denote the set of functions $[n] \to G^{[n]}$, and let

$$\mathrm{Hom}_G^{\mathrm{cts}}(G^{[n]}, M)$$

denote the set of continuous functions $\varphi : G^{[n]} \to M$ (where M has the discrete topology) such that for every $g \in G$ and $(g_0, \ldots, g_n) \in G^{[n]}$ we have

$$\varphi(gg_0, \ldots, gg_n) = g \cdot \varphi(g_0, \ldots, g_n).$$

For any morphism $\delta : [m] \to [n]$ in Δ there is an induced morphism

$$\mathrm{Hom}_G^{\mathrm{cts}}(G^{[m]}, M) \to \mathrm{Hom}_G^{\mathrm{cts}}(G^{[n]}, M)$$

sending $\varphi : G^{[m]} \to M$ to the function $G^{[n]} \to M$ sending

$$(g_0, \ldots, g_n) \mapsto \varphi(g_{\delta(0)}, \ldots, g_{\delta(m)}).$$

In this way we get a cosimplicial R-module

$$A_G := \mathrm{Hom}_G^{\mathrm{cts}}(G^{[-]}, M),$$

and as is well-known the corresponding total complex $\widetilde{A}_G$ computes $R\Gamma(G, M)$ (see for example [**44**, I §2.2]).

Similarly, $R\Gamma(H, N)$ and $R\Gamma(G \times H, M \boxtimes N)$ are computed by the total complexes $\widetilde{A}_H$ and $\widetilde{A}_{G \times H}$ respectively of the cosimplicial R-modules

$$A_H := \mathrm{Hom}_H^{\mathrm{cts}}(H^{[-]}, N) \quad \text{and} \quad A_{G \times H} := \mathrm{Hom}_{G \times H}^{\mathrm{cts}}((G \times H)^{[-]}, M \boxtimes N).$$

A.3. Now in general (see for example [**28**, §I.1.2]) if A and B are cosimplicial R-modules, then we obtain a co-bisimplicial R-module $A \otimes B$ (i.e. functor $\Delta^2 \to \mathrm{Mod}_R$) by the formula

$$([n], [m]) \mapsto A_n \otimes_R B_m,$$

whose associated total complex $\widetilde{A \otimes B}$ is equal to $\widetilde{A} \otimes_R \widetilde{B}$.

Let $A \otimes^\Delta B$ denote the restriction of $A \otimes B$ to the diagonal $\Delta \subset \Delta^2$, so $A \otimes^\Delta B$ is the cosimplicial R-module

$$[n] \mapsto A_n \otimes B_n.$$

By the Eilenberg-Zilber theorem [**28**, I.1.2.2] there is a natural quasi-isomorphism

$$\widetilde{A \otimes B} \to \widetilde{A \otimes^\Delta B}.$$

A.4. For any $[n] \in \Delta$ there is a natural map

(A.4.1) $\operatorname{Hom}_G^{\mathrm{cts}}(G^{[n]}, M) \otimes_R \operatorname{Hom}_H^{\mathrm{cts}}(H^{[n]}, N) \to \operatorname{Hom}_{G \times H}^{\mathrm{cts}}((G \times H)^{[n]}, M \boxtimes N),$

sending

$$\varphi \otimes \psi \mapsto ((g_0, h_0), \dots, (g_n, h_n)) \mapsto \varphi(g_0, \dots, g_n) \otimes \psi(h_0, \dots, h_n).$$

A straightforward verification shows that this is compatible with the cosimplicial structures and therefore defines a morphism of cosimplicial R-modules

$$A_G \otimes^\Delta A_H \to A_{G \times H}.$$

This then defines a morphism

(A.4.2) $\widetilde{A}_G \otimes \widetilde{A}_H \simeq \widetilde{(A_G \otimes A_H)} \to \widetilde{(A_G \otimes^\Delta A_H)} \to \widetilde{A}_{G \times H}.$

A.5. Now assume that both M and N are flat R-modules and that $R\Gamma(G, M)$ and $R\Gamma(H, N)$ lie in $D^b(R)$ (the bounded derived category of complexes of R-modules). Then $\widetilde{A}_G$ and $\widetilde{A}_H$ are also complexes of flat R-modules, so the map A.4.2 defines a morphism in $D(R)$

(A.5.1) $R\Gamma(G, M) \otimes^{\mathbb{L}} R\Gamma(H, N) \to R\Gamma(G \times H, M \boxtimes N).$

THEOREM A.6 (Künneth formula). *The map A.5.1 is an isomorphism.*

PROOF. It suffices to show that for any $[n] \in \Delta$, the map A.4.1 is an isomorphism.

For this note that the inclusion $G^n \hookrightarrow G^{[n]}$ sending

$$(g_1, \dots, g_n) \mapsto (1, g_1, \dots, g_n) \in G^{[n]}$$

defines an isomorphism

$$\operatorname{Hom}_G^{\mathrm{cts}}(G^{[n]}, M) \simeq \operatorname{Hom}^{\mathrm{cts}}(G^n, M),$$

and similarly for H and $G \times H$. It therefore suffices to show that the natural map

(A.6.1) $\operatorname{Hom}^{\mathrm{cts}}(G^n, M) \otimes_R \operatorname{Hom}^{\mathrm{cts}}(H^n, N) \to \operatorname{Hom}^{\mathrm{cts}}((G \times H)^n, M \boxtimes N)$

is an isomorphism. Since M (resp. N) is a continuous representation of G (resp. H) we have

$$M = \cup_U M^U, \quad (\text{resp. } N = \cup_V N^V),$$

where U (resp. V) runs over open normal subgroups of G (resp. H). Since the formation of tensor product commutes with direct limits, it follows that we also have

$$M \boxtimes N = \varinjlim_{(U,V)} M^U \boxtimes N^V.$$

We therefore have

$$\operatorname{Hom}^{\mathrm{cts}}(G^n, M) \otimes \operatorname{Hom}^{\mathrm{cts}}(H^n, N) = \varinjlim_{U,V} \operatorname{Hom}^{\mathrm{cts}}(G^n, M^U) \otimes \operatorname{Hom}^{\mathrm{cts}}(H^n, N^V),$$

and

$$\operatorname{Hom}^{\mathrm{cts}}((G \times H)^n, M \boxtimes N) = \varinjlim_{U,V} \operatorname{Hom}^{\mathrm{cts}}((G \times H)^n, M^U \boxtimes N^V).$$

It therefore suffices to prove that the map A.6.1 is an isomorphism in the case when $M = M^U$ and $N = N^V$ for some open normal subgroups $U \subset G$ and $V \subset H$.

In this case, let $(U_i)_{i \in I}$ (resp. $(V_j)_{j \in J}$) be the set of open normal subgroups in G (resp. H) contained in U (resp. V). We then have

$$\mathrm{Hom}^{\mathrm{cts}}(G^n, M) = \varinjlim_i \mathrm{Hom}((G/U_i)^n, M),$$

$$\mathrm{Hom}^{\mathrm{cts}}(H^n, N) = \varinjlim_j \mathrm{Hom}((H/V_j)^n, N),$$

and

$$\mathrm{Hom}^{\mathrm{cts}}((G \times H)^n, M \boxtimes N) = \varinjlim_{i,j} \mathrm{Hom}^{\mathrm{cts}}((G/U_i \times H/V_j)^n, M \boxtimes N).$$

This reduces the proof to showing that the maps

$$\mathrm{Hom}((G/U_i)^n, M) \otimes \mathrm{Hom}((H/V_j)^n, N) \to \mathrm{Hom}^{\mathrm{cts}}((G/U_i \times H/V_j)^n, M \boxtimes N)$$

are isomorphisms, which is immediate. $\square$

COROLLARY A.7. *Suppose in addition to the hypotheses of A.5 that the groups $H^i(G, M)$ and $H^i(H, N)$ are projective R-modules for all i. Then the natural map of graded R-modules*

$$H^*(G, M) \otimes_R H^*(H, N) \to H^*(G \times H, M \boxtimes N)$$

is an isomorphism.

PROOF. Since $R\Gamma(G, M)$ and $R\Gamma(H, N)$ are bounded complexes of R-modules with cohomology groups projective R-modules, we have isomorphisms in $D(R)$

$$R\Gamma(G, M) \simeq H^*(G, M), \quad R\Gamma(H, N) \simeq H^*(H, N),$$

where $H^*(G, M)$ and $H^*(H, N)$ are viewed as complexes with zero differentials. From A.6 we conclude that

$$R\Gamma(G \times H, M \boxtimes N) \simeq H^*(G, M) \otimes_R H^*(H, N),$$

and in particular for every i the R-module $H^i(G \times H, M \boxtimes N)$ is a projective R-module. Therefore we also have

$$R\Gamma(G \times H, M \boxtimes N) \simeq H^*(G \times H, M \boxtimes N)$$

which implies the corollary. $\square$

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