MATH 254 A: PROBLEM SET 8

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(1) Let $K = \mathbb{Q}(\sqrt{d})$ with d a positive square free integer, and assume d is not congruent to 1 mod 4. View K as a subfield of $\mathbb{R}$ by the standard embedding $K \subset \mathbb{R}$.

(a) Show that the group $\mathcal{O}_K^* \cap \mathbb{R}_{>0}$ is isomorphic to $\mathbb{Z}$. Show that this group has one generator > 1 and one generator < 1. The generator > 1 is called the *fundamental unit.*

(b) If $u \in \mathcal{O}_K^*$, show that $u > 1$ if and only if $u = a + b\sqrt{d}$ with $a > 0$ and $b > 0$.

(c) Let u be the fundamental unit, and write $c = a + b\sqrt{d}$. Also for $n \geq 1$ write $u^n = a_n + b_n\sqrt{d}$. Show that the sequence of integers b_n is strictly increasing. Deduce that, for computing the fundamental unit, it is enough to compute db^2 for $b = 1, 2, 3, ...$ and test if it is 1 plus a square: The first b that meets the test is the b of the fundamental unit.

(d) Using (c) compute the fundamental unit of $\mathbb{Q}(\sqrt{d})$ for $d = 2, 3, 6, 7$.

(2) Find the smallest integral solution $y > 0$ to Pell's equation $x^2 - 61y^2 = 1$.

(3) Show that the three cubic fields that are obtained by adjoining to $\mathbb{Q}$ a root of one of the equations

$$X^3 - 18X - 6, \quad X^3 - 36X - 78, \quad X^3 - 54X - 150$$

all have the same discriminant, but no two of them are isomorphic.

Compare this with Theorem 5 in Chapter V. §4 of Lang's book.