MATH 254A: PROBLEM SET 2

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Due Monday Sep. 15

(1) Let A be a Dedekind domain with field of fractions K, and let $\mathfrak{p} \subset A$ be a prime. For an ideal $\mathfrak{a}$ define (as in class)

$$\mathfrak{a}^{-1} := \{x \in K | x\mathfrak{a} \subset A\}.$$

Show that

$$\mathfrak{a}^{-1} \cdot A_{\mathfrak{p}} = \{x \in K | x(\mathfrak{a} \cdot A_{\mathfrak{p}}) \subset A_{\mathfrak{p}}\}.$$

(2) Show that in $\mathbb{Z}[\sqrt{-3}]$ (the subring of $\mathbb{Q}(\sqrt{-3})$ generated by $\sqrt{-3}$), the ideal generated by 2 cannot be written as a product of prime ideals. Show that the ideal $(2, 1+\sqrt{-3})$ does not have any fraction ideal inverse.

(3) Let D be a square-free integer, and let p be a prime not dividing $2D$. Let $\mathcal{O}_K$ be the ring of integers in $\mathbb{Q}(\sqrt{D})$. Show that $(p) = p \cdot \mathcal{O}_K$ is a prime ideal in $\mathcal{O}_K$ if and only if the congruence

$$x^2 \equiv D \pmod{p}$$

has no solution in rational integers.

(4) Show that $\mathbb{Z}[\sqrt{-5}]$ has exactly two ideal classes.

(5) Let K be a field, and let L be a finite separable extension of the field of rational functions $K(x)$. Prove that the integral closure of $K[x]$ in L is a Dedekind domain.