MATH 254A: PROBLEM SET 1

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Due Monday Sep. 8

(1) Let A be an integrally closed domain with field of fractions K, and let $K \subset L$ be a finite separable extension. Let $x \in L$ be an element integral over A. Show that every coefficient of the characteristic polynomial of x over K lies in A (note that already we saw that the trace and norm lie in A).

(2) Let K be a number field and let $x \in \mathcal{O}_K$ be an element. Show that x is a unit in $\mathcal{O}_K$ if and only if the norm of x is ± 1. If K is Galois over $\mathbb{Q}$ show that the same is true for any subring $R \subset \mathcal{O}_K$ which is Galois invariant.

(3) Let D is be a square free integer. *Pell's equation* is the equation

$$X^2 - DY^2 = 1.$$

Show that $a + b\sqrt{D}$ is a unit in $\mathbb{Z}[\sqrt{D}]$ if and only if (a, b) either satisfies Pell's equation or the equation

$$X^2 - DY^2 = -1.$$

(4) Prove that $\mathbb{Z}[\sqrt{-5}]$ is not a principal ideal domain (hint: consider the ideal generated by 2 and $1 + \sqrt{-5}$).

(5) Determine the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt[3]{2})$ and the field

$$\mathbb{Q}[x]/(x^3 - x - 4).$$