MATH 253: PROBLEM SET 5, DUE WEDNESDAY MARCH 30

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(1) Let R be a commutative ring. Show that the following conditions on an R-module M are equivalent:

(a) M is flat.

(b) $\mathrm{Tor}^i(M, N) = 0$ for all $i > 0$ and all R-modules N.

(c) $\mathrm{Tor}^1(M, N) = 0$ for all R-modules N.

(2) Let $f : R \to S$ be a morphism of commutative rings. Mimic the construction given in class to construct a functor

$$S \otimes^{\mathbb{L}}_R (-) : D^-(\mathrm{Mod}_R) \to D^-(\mathrm{Mod}_S).$$

There is also a functor

$$D(\mathrm{Mod}_S) \to D(\mathrm{Mod}_R), \quad M \mapsto M_0$$

obtained by viewing an S-module as an R-module via the homomorphism f. Then show that for any $M \in D^-(\mathrm{Mod}_R)$ and $N \in D^+(\mathrm{Mod}_S)$ we have a natural isomorphism

$$\mathrm{RHom}_R(M, N_0) \simeq \mathrm{RHom}_S(S \otimes^{\mathbb{L}}_R M, N).$$

(3) Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories and let

$$F : \mathscr{A} \to \mathscr{B}$$

be a left exact functor. Assume that $\mathscr{A}$ has enough injectives. Let $A \in \mathscr{A}$ be an object and suppose given a resolution

$$A \to T^0 \to T^1 \to T^2 \to \cdots$$

in $\mathscr{A}$ such that for every n and $i > 0$ we have $R^iF(T^n) = 0$ (but the T^n need not be injective). Show that there is a natural isomorphism

$$H^n(F(T^\cdot)) \simeq R^nF(A)$$

for every n (Hint: use the spectral sequence of a filtered complex).