MATH 253: PROBLEM SET 4, DUE MONDAY MARCH 7

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(1). Let A be a finite abelian group, and let $\mathscr{C}$ denote the category of finite dimensional representations of A over $\mathbb{C}$.

(a) Show that every irreducible representation is 1-dimensional and therefore given by a character

$$\chi : A \to \mathbb{C}^*.$$

(b) Let

$$\omega : \mathscr{C} \to \mathrm{Vec}_{\mathbb{C}}$$

be the forgetful functor. An automorphism λ of ω is called a *tensor automorphism* if for any two representations V and W the diagram

$$\begin{array}{ccc} \omega(V) \otimes_{\mathbb{C}} \omega(W) & \longrightarrow & \omega(V \otimes W) \\ \downarrow{\scriptstyle \lambda_V \otimes \lambda_W} & & \downarrow{\scriptstyle \lambda_{V\otimes W}} \\ \omega(V) \otimes_{\mathbb{C}} \omega(W) & \longrightarrow & \omega(V \otimes W) \end{array}$$

commutes. Let $\mathrm{Aut}^{\otimes}(\omega)$ denote the tensor automorphisms of ω. Show that there is a natural isomorphism

$$A \to \mathrm{Aut}^{\otimes}(\omega).$$

Remark: The conclusion of part (b) is also valid for any finite group G (not necessarily abelian). This is a harder result known as the Tannaka-Krein theorem.

(2). View the natural numbers $\mathbb{N}$ as a category with

$$\mathrm{Hom}(n, m) = \begin{cases} * & \text{if } n \geq m \\ \emptyset & \text{if } n < m. \end{cases}$$

Let $\mathscr{C}$ be the category of functors

$$F : \mathbb{N} \to \mathrm{Ab}$$

from $\mathbb{N}$ to the category of abelian groups. So $\mathscr{C}$ is the category of projective systems of abelian groups

$$A_\cdot : \cdots \to A_n \to A_{n-1} \to \cdots \to A_0.$$

(a) Show that $\mathscr{C}$ is an abelian category (this is a special case of an earlier exercise).

(b) Show that for every $n \in \mathbb{N}$ the forgetful functor

$$\epsilon_n : \mathscr{C} \to \mathrm{Ab}$$

sending a projective system $A_\cdot$ to A_n is exact, and admits a right adjoint. Conclude that the category $\mathscr{C}$ has enough injectives.

(c) Show that taking the projective limit of a projective system defines a functor

$$\varprojlim : \mathscr{C} \to \mathrm{Ab}$$

which is left exact. Let

$$R^i \varprojlim : \mathscr{C} \to \mathrm{Ab}$$

be the corresponding derived functors.

(d) A system $A. \in \mathscr{C}$ is said to satisfy the *Mittag-Leffler condition* if for every n there exists an integer m (depending on n) such that for all $r \geq m$ the images of the maps

$$A_{n+r+1} \to A_n, \quad A_{n+r} \to A_n$$

are equal. Show that if $A.$ satisfies the Mittag-Leffler condition then

$$R^1 \varprojlim A. = 0.$$

(e) Show that for any object $A. \in \mathscr{C}$ we have

$$R^i \varprojlim A. = 0$$

for $i \neq 0, 1.$.