MATH 253: PROBLEM SET 3, DUE MONDAY FEB 21

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(1). Let $\mathscr{C}$ be a category and S a multiplicative system of morphisms in $\mathscr{C}$.

(a) In class we defined a notion of locally small for a multiplicative system using left fractions. Define the analogous notion for right fractions.

(b) Prove that if S is both locally small on the left and on the right, then $S^{-1}\mathscr{C}$ may be constructed using equivalence classes of right fractions

$$X \xleftarrow{f} Y' \xrightarrow{t} Y,$$

with $t \in S$.

(c) Assume that S is locally small on the right. Show that if $f, g : X \to Y$ are two morphisms in $\mathscr{C}$ which become equal in $S^{-1}\mathscr{C}$, then there exists a morphism $t : Y \to Z$ in S such that $tf = tg$.

(2). Let R be a ring. The purpose of this exercise is to show that the category Mod_R of left R-modules has enough injectives.

(a) Show that an abelian group A is injective in the category of abelian groups if and only if A is divisible (i.e. that for every nonzero integer n the multiplication by n map $\cdot n : A \to A$ is surjective). In particular, the group $\mathbb{Q}/\mathbb{Z}$ is injective.

(b) Now let A be any abelian group, and let F_A be the set of functions

$$\mathrm{Hom}_{\mathrm{Ab}}(A, \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}.$$

In more compact notation

$$F_A = (\mathbb{Q}/\mathbb{Z})^{\mathrm{Hom}_{\mathrm{Ab}}(A,\mathbb{Q}/\mathbb{Z})}.$$

Show that F_A is an injective abelian group, and that there is a natural injection

$$e_A : A \to F_A.$$

Conclude that the category of abelian groups has enough injectives.

(c) Now let R be any ring and consider the left R-module

$$I_0 : \mathrm{Hom}_{\mathrm{Ab}}(R, \mathbb{Q}/\mathbb{Z}).$$

Show that I_0 is an injective R-module.

(d) Finally consider an arbitrary R-module M, and let F_M denote the set of functions

$$\mathrm{Hom}_R(M, I_0) \to I_0.$$

Show that F_M is an injective R-module and that there is a natural injection

$$e_M : M \hookrightarrow F_M.$$

In particular, Mod_R has enough injectives.