MATH 253: PROBLEM SET 1, DUE FRIDAY FEB 4

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(1). (An example of an additive category which is not abelian) Let k be a field. A filtration $F^\cdot$ on a k-vector space V is a decreasing sequence of subspaces (indexed by the integers)

$$\cdots \subset F^i \subset F^{i-1} \subset \cdots V.$$

A morphism of filtered vector spaces

$$\varphi : (V, F^\cdot) \to (W, G^\cdot)$$

is a map of vector spaces $\varphi : V \to W$ such that $\varphi(F^i) \subset G^i$ for all i. Let $\mathscr{A}$ be the category of finite dimensional filtered k-vector spaces $(V, F^\cdot)$. Show that $\mathscr{A}$ is additive but not abelian.

(2). Consider the complex of abelian groups

$$\cdots \mathbb{Z}/(4) \xrightarrow{\cdot 2} \mathbb{Z}/(4) \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \cdots .$$

Show that this complex is acyclic but that there does not exist a homotopy between the zero map and the identity.

(3). Let $\mathscr{A}$ be an abelian category. Show that the category $C(\mathscr{A})$ of complexes in $\mathscr{A}$ is an abelian category. How about the category $C^{\geq 0}(\mathscr{A})$ of complexes $K^\cdot$ with $K^i = 0$ for $i < 0$?

(4). Let $\mathscr{C}$ be any small category, and let $\mathscr{A}$ be an abelian category. Show that the category $\mathscr{C}^{\mathscr{A}}$ of functors

$$F : \mathscr{C} \to \mathscr{A}$$

is also an abelian category.