MOTIVIC COHOMOLOGY, LOCALIZED CHERN CLASSES, AND LOCAL TERMS

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ABSTRACT. Let $c : C \rightarrow X \times X$ be a correspondence with C and X quasi-projective schemes over an algebraically closed field k . We show that if $u_\ell : c_1^* \mathbb{Q}_\ell \rightarrow c_2^! \mathbb{Q}_\ell$ is an action defined by the localized Chern classes of a c_2 -perfect complex of vector bundles on C , where ℓ is a prime invertible in k , then the local terms of u_ℓ are given by the class of an algebraic cycle independent of ℓ . We also prove some related results for quasi-finite correspondences. The proofs are based on the work of Cisinski and Deglise on triangulated categories of motives.

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1. INTRODUCTION

The motivation for this work comes from our study of local terms arising from actions of correspondences defined by local Chern classes of complexes of vector bundles in [16]. The purpose of the present paper is to elucidate the motivic nature of these local terms using the machinery developed by Cisinski and Deglise in [4].

The basic problem we wish to address is the following. Fix an algebraically closed field k of characteristic p (possibly 0), and let $\mathcal{S}$ denote the category of finite type separated k -schemes. Let $c : C \rightarrow X \times X$ be a correspondence with $C, X \in \mathcal{S}$. A c_2 -perfect complex E on C defines for any prime ℓ invertible in k an action $u_\ell : c_1^* \mathbb{Q}_\ell \rightarrow c_2^! \mathbb{Q}_\ell$, and therefore by the general machinery of SGA 5 a class $\mathrm{Tr}(u_\ell) \in H^0(\mathrm{Fix}(c), \Omega_{\mathrm{Fix}(c)})$, where $\mathrm{Fix}(c)$ denotes the scheme of fixed points $\mathrm{Fix}(c) := C \times_{c, X \times X, \Delta_X} X$ and $\Omega_{\mathrm{Fix}(c)}$ is the ℓ -adic dualizing complex (see [12, III, §4] for further discussion). Recall from loc. cit. that for any proper connected component

$Z \subset \text{Fix}(c)$ the local term of u_ℓ is given by the proper pushforward of the restriction of $\text{Tr}(u_\ell)$ to Z , and consequently in good situations can be used via the Grothendieck-Lefschetz trace formula [12, III, 4.7] to calculate the trace of the induced action of u_ℓ on global cohomology.

On the other hand, $H^0(\text{Fix}(c), \Omega_{\text{Fix}(c)})$ is the ℓ -adic Borel-Moore homology of $\text{Fix}(c)$ and there is a cycle class map

$$\text{cl}_\ell : A_0(\text{Fix}(c)) \rightarrow H^0(\text{Fix}(c), \Omega_{\text{Fix}(c)}),$$

where $A_0(\text{Fix}(c))$ denotes the group of 0-cycles on $\text{Fix}(c)$ modulo rational equivalence.

The main result about local terms in this paper is the following:

Theorem 1.1 (Theorem 7.4). *There exists a zero-cycle $\Sigma \in A_0(\text{Fix}(c))_{\mathbb{Q}}$ such that for any prime ℓ invertible in k the class $\text{Tr}(u_\ell)$ is equal to $\text{cl}_\ell(\Sigma)$.*

As we explain, this theorem is a fairly formal consequence of a suitable theory of derived categories of motives and six operations for such categories. The fact that such a theory exists is due to Cisinski and Deglise [4]. They developed a notion of triangulated motivic categories with a six operations formalism realizing a vision of Beilinson. Roughly speaking such a category is a fibered category $\mathcal{M}$ over $\mathcal{S}$ such that for every $X \in \mathcal{S}$ the fiber $\mathcal{M}(X)$ is a monoidal triangulated category and for every morphism $f : X \rightarrow Y$ in $\mathcal{S}$ we have functors

$$f_!, f_* : \mathcal{M}(X) \rightarrow \mathcal{M}(Y), \quad f^*, f^! : \mathcal{M}(Y) \rightarrow \mathcal{M}(X)$$

satisfying the usual properties. In addition there should be a suitable notion of Chern classes. Already in this context we can define localized Chern classes of complexes of vector bundles as well as analogous of the classes $\text{Tr}(u_\ell)$, which are functorial in $\mathcal{M}$. In particular, we can consider the category $\mathcal{M}_B$ of Beilinson motives defined in [4, §14]. In this case, the $\mathcal{M}_B$ -version of $H^0(\text{Fix}(c), \Omega_{\text{Fix}(c)})$ is simply the group $A_0(\text{Fix}(c))_{\mathbb{Q}}$ and the $\mathcal{M}_B$ -version of $\text{Tr}(u_\ell)$ is the cycle appearing in 1.1. Using [5], we then get 1.1 by passing to étale realizations.

The proof of 1.1 can essentially be phrased as saying that actions arising from c_2 -perfect complexes are motivic. In general it seems a difficult question to prove that a given action of a correspondence is motivic. There is one other case, however, where one can fairly easily detect if an action is motivic. Namely, for a quasi-finite morphism $f : Y \rightarrow X$ there is a natural necessary condition for a section $u_\ell \in H^0(Y, f^! \mathbb{Q}_\ell)$ to be the étale realization of a morphism $u : 1_Y \rightarrow f^! 1_X$ in the triangulated category of Beilinson motives over Y . In theorem 8.2 we show that this condition is also sufficient. This also has global consequences. In particular, a special case of theorem 8.18 is the following:

Theorem 1.2. *Let k be an algebraically closed field and let X/k be a separated Deligne-Mumford stack. Let $f : X \rightarrow X$ be a finite morphism (as a morphism of stacks). Then the alternating sum of traces*

$$\sum_i (-1)^i \text{tr}(f^* | H^i(X, \mathbb{Q}_\ell))$$

is in $\mathbb{Q}$ and independent of ℓ .

Remark 1.3. Since the trace appearing 1.2 is in $\mathbb{Z}_\ell$ it follows that the alternating sum of traces is in $\mathbb{Z}[1/p]$, where p is the characteristic of k . In fact, notice that since $R\Gamma(X, \mathbb{Z}_\ell)$ is a perfect complex we can define $\text{tr}(u^* | R\Gamma(X, \mathbb{Z}_\ell)) \in \mathbb{Z}_\ell$, which by the above is an element of

$\mathbb{Z}[1/p]$ which reduces mod ℓ to $\mathrm{tr}(u^*|R\Gamma(X, \mathbb{F}_\ell))$, thereby yielding ℓ -independence for mod ℓ traces as well.

Remark 1.4. One might hope more generally to use the techniques of this paper to study motivic local terms with $\mathbb{Z}$ coefficients to obtain cycles in $A_0(\mathrm{Fix}(c))$ before tensoring with $\mathbb{Q}$. However, the theory at present seems restricted to $\mathbb{Q}$ -coefficients as the six operations on a suitable triangulated category of motives is not known to exist integrally. Work in preparation by Cisinski and Deglise on integral motives may, however, lead to integral results.

Remark 1.5. In this paper we discuss étale cohomology and local terms defined in the étale theory. However, with a suitable theory of p -adic local terms and p -adic realization functors one would also get rationality of p -adic local terms and compatibility with the étale local terms.

Remark 1.6. Theorem 1.2 has also been obtained by Bondarko using variant motivic methods [3, Discussion following 8.4.1].

Remark 1.7. Many of the foundational results obtained in this paper hold not just over a field but over more general base schemes and we develop the theory in greater generality. For the applications to local terms, however, it suffices to work over an algebraically closed field.

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2. MOTIVIC CATEGORIES AND THE SIX OPERATIONS

Let B be a regular separated scheme of finite dimension, and let $\mathcal{S}$ denote the category of finite type separated B -schemes.

2.1. Recall from [4, Section 1] that a *triangulated premotivic category* $\mathcal{M}$ is a fibered category over $\mathcal{S}$ satisfying the following five conditions (a good summary is given in [5, A.1.1]):

(PM1) For every $S \in \mathcal{S}$ the fiber category $\mathcal{M}(S)$ is a well-generated (in the sense of [15]) triangulated category with a closed monoidal structure.

(PM2) For every morphism $f : X \rightarrow Y$ in $\mathcal{S}$ the functor (well-defined up to unique isomorphism)

$$f^* : \mathcal{M}(Y) \rightarrow \mathcal{M}(X)$$

is triangulated, monoidal, and admits a right adjoint f_* .

(PM3) For every smooth morphism $f : X \rightarrow Y$ in $\mathcal{S}$ the functor $f^* : \mathcal{M}(Y) \rightarrow \mathcal{M}(X)$ admits a left adjoint $f_!$.

(PM4) For every cartesian square with p smooth

$$\begin{array}{ccc} Y & \xrightarrow{q} & X \\ g \downarrow & \Delta & \downarrow f \\ T & \xrightarrow{p} & S \end{array}$$

there is a canonical isomorphism of functors

$$\mathrm{Ex}(\Delta_\sharp^*) : q_\sharp g^* \simeq f^* p_\sharp.$$

(PM5) For every smooth morphism $p : T \rightarrow S$, $M \in \mathcal{M}(T)$, and $N \in \mathcal{M}(S)$ then there is a canonical isomorphism

$$\mathrm{Ex}(p_{\sharp}^*, \otimes) : p_{\sharp}(M \otimes_T p^* N) \simeq p_{\sharp}(M) \otimes_S N.$$

Remark 2.2. Note that for any category $\mathcal{S}$ we can talk about a *triangulated fibered category over $\mathcal{S}$* . By this we mean a fibered category $p : \mathcal{M} \rightarrow \mathcal{S}$ satisfying axioms (PM1)–(PM3) and (PM4).

2.3. For every $X \in \mathcal{S}$, the monoidal structure on $\mathcal{M}(X)$ gives a unit object $1_X \in \mathcal{M}(X)$. For a smooth morphism $f : X \rightarrow S$ in $\mathcal{S}$ define $M_S(X) \in \mathcal{M}(S)$ to be $f_{\sharp}(1_X)$. Because the pullback functor f^* is monoidal we have $f^*1_S = 1_X$ and therefore by adjunction a morphism

$$M_S(X) = f_{\sharp}f^*1_S \xrightarrow{f_{\sharp}f^* \rightarrow \mathrm{id}} 1_S,$$

which we denote by $a_{X/S}$.

A *Tate motive* for $\mathcal{M}$ is a cartesian section $\tau : \mathcal{S} \rightarrow \mathcal{M}$ with $\tau(S)$ fitting into a distinguished triangle

$$\tau(S) \longrightarrow M_S(\mathbb{P}_S^1) \xrightarrow{a_{\mathbb{P}_S^1/S}} 1_S \longrightarrow \tau(S)$$

functorial in S . We usually write just $1_S(1)$ for $\tau(S)$.

2.4. We can consider various other natural axioms on a triangulated premotivic category with a Tate object:

(Homotopy axiom) For every $S \in \mathcal{S}$ the map

$$a_{\mathbb{A}_S^1/S} : M_S(\mathbb{A}_S^1) \rightarrow 1_S$$

is an isomorphism.

(Stability property) The Tate motive $1_S(1)$ is $\otimes$ -invertible. In this case we get motives $1_S(n)$ for all $n \in \mathbb{Z}$.

2.5. Given a triangulated premotivic category $\mathcal{M}$ with a Tate motive satisfying the stability property we define motivic cohomology, a bigraded cohomology theory on $\mathcal{S}$, by

$$H_{\mathcal{M}}^{i,n}(S) := \mathrm{Hom}_{\mathcal{M}(S)}(1_S, 1_S(n)[i]).$$

2.6. A morphism between two triangulated premotivic categories $\mathcal{M}$ and $\mathcal{M}'$ is a cartesian functor $\varphi^* : \mathcal{M} \rightarrow \mathcal{M}'$ such that the following hold:

- (i) For every $S \in \mathcal{S}$ the functor $\varphi_S^* : \mathcal{M}(S) \rightarrow \mathcal{M}'(S)$ is a triangulated monoidal functor which admits a right adjoint φ_{S*} .
- (ii) For every smooth morphism $p : T \rightarrow S$ in $\mathcal{S}$ there is a canonical isomorphism

$$\mathrm{Ex}(p_{\sharp}, \varphi^*) : p_{\sharp}\varphi_T^* \rightarrow \varphi_S^*p_{\sharp}.$$

In fact triangulated premotivic categories form a 2-category in which the above morphisms are the 1-morphisms, and 2-morphisms are given by morphisms of cartesian functors $\epsilon : \varphi^* \rightarrow \psi^*$ compatible with the structures in (i) and (ii).

Remark 2.7. Similarly we can consider the 2-category of triangulated fibered categories over any base category $\mathcal{S}$.

2.8. Let $\mathcal{S}$ be as above, and let $\text{Ar}(\mathcal{S})$ be the category of morphisms in $\mathcal{S}$. We have two functors

$$s, t : \text{Ar}(\mathcal{S}) \rightarrow \mathcal{S}$$

given by the source and target respectively. For a triangulated premotivic category $\mathcal{M}$ over $\mathcal{S}$ let $\mathcal{M}^s$ (resp. $\mathcal{M}^t$) denote $s^*\mathcal{M}$ (resp. $t^*\mathcal{M}$), a triangulated fibered category over $\text{Ar}(\mathcal{S})$.

A *six functor formalism* for $\mathcal{M}$ consists of the following data:

- (1) 2-functors $f \mapsto f_*$ and $f \mapsto f_!$ from $\mathcal{M}^s \rightarrow \mathcal{M}^t$ and $f \mapsto f^*$ and $f \mapsto f^!$ from $\mathcal{M}^t$ to $\mathcal{M}^s$ such that for every $f : X \rightarrow Y \in \text{Ar}(\mathcal{S})$ the functors f_* and f^* are as previously defined, and $f_!$ is left adjoint to $f^!$.
- (2) There exists a morphism of 2-functors $\alpha : f_! \rightarrow f_*$ which is an isomorphism if f is proper.
- (3) For any smooth morphism $f : X \rightarrow S$ in $\mathcal{S}$ of relative dimension d there are isomorphisms $\mathbf{p}_f : f_{\sharp} \rightarrow f_!(d)[2d]$ and $\mathbf{p}'_f : f^* \simeq f^!(-d)[-2d]$. These are given by isomorphisms of 2-functors on the category of smooth morphisms of relative dimension d .
- (4) For every cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{q} & X' \\ g \downarrow & \Delta & \downarrow f \\ Y & \xrightarrow{p} & X \end{array}$$

there are natural isomorphisms of functors

$$p^* f_! \simeq g_! q^*,$$

$$g_* q^! \simeq p^! f_*.$$

- (5) For every $f : Y \rightarrow X$ there are natural isomorphisms

$$\text{Ex}(f_!^*, \otimes) : (f_! K) \otimes_X L \simeq f_!(K \otimes_Y f^* L),$$

$$\text{Hom}_X(f_! L, K) \simeq f_* \text{Hom}_Y(L, f^! K),$$

and

$$f^! \text{Hom}_X(L, M) \simeq \text{Hom}_Y(f^* L, f_! M).$$

- (Loc) Let $X \in \mathcal{S}$ be an object, $i : Z \hookrightarrow X$ a closed imbedding, and let $j : U \hookrightarrow X$ be the complementary open set. Then there exists a map of functors $\partial : i_* i^* \rightarrow j_! j^![1]$ such for every $F \in M(X)$ the induced triangle

$$j_! j^! F \longrightarrow F \longrightarrow i_* i^* F \xrightarrow{\partial} j_! j^! F[1]$$

is distinguished, where the first two maps are those induced by adjunction.

Finally Deglise and Cisinski consider purity and duality properties:

(*Relative Purity*) For a closed immersion $i : Z \hookrightarrow X$ of smooth separated B -schemes there is a canonical isomorphism

$$1_Z(-c)[-2c] \simeq i^!(1_X),$$

where c is the codimension of Z in X .

(*Duality*) For $X \in \mathcal{S}$ with structure morphism $f : X \rightarrow B$ we write $\Omega_X^{\mathcal{M}}$ (or just Ω_X if no confusion seems likely to arise) for $f^! 1_B \in \mathcal{M}(X)$. Define $D_X : \mathcal{M}(X)^{\text{op}} \rightarrow \mathcal{M}(X)$ to be the functor $M \mapsto \text{Hom}_X(M, \Omega_X^{\mathcal{M}})$.

- (a) For every $M \in \mathcal{M}(X)$ the natural map

$$M \rightarrow D_X(D_X(M))$$

is an isomorphism.

- (b) For every X and $M, N \in \mathcal{M}(X)$ we have a canonical isomorphism

$$D_X(M \otimes D_X(N)) \simeq \text{Hom}_X(M, N).$$

- (c) For every $f : Y \rightarrow X$ in $\mathcal{S}$, $M \in \mathcal{M}(X)$, and $N \in \mathcal{M}(Y)$ we have natural isomorphisms

$$\begin{aligned} D_Y(f^*(M)) &\simeq f^!(D_X(M)), \\ f^* D_X((M)) &\simeq D_Y(f^!(M)), \\ D_X(f_!(N)) &\simeq f_*(D_Y(N)), \\ f_!(D_Y(N)) &\simeq D_X(f_*(N)). \end{aligned}$$

We say that a triangulated premotivic category $\mathcal{M}$ is a *triangulated motivic category* over $\mathcal{S}$ if all of the above conditions hold.

Remark 2.9. This is stronger than what is in [4, 2.4.45] but we will not need their slightly weaker notion.

Remark 2.10. The relative purity property follows from property 2.8 (3), but we state it explicitly for later use.

Remark 2.11. If R is a ring we can also consider a notion of an *R -linear triangulated motivic category* over $\mathcal{S}$. By definition this means that each $\mathcal{M}(X)$ is an R -linear symmetric monoidal triangulated category, and that all the above structure respects this R -linear structure.

3. CHERN CLASSES

As in the previous section let B be a regular separated scheme of finite dimension, and let $\mathcal{S}$ denote the category of finite type separated B -schemes.

3.1. Fix an R -linear triangulated motivic category $\mathcal{M}$ over $\mathcal{S}$. For $n, m \in \mathbb{Z}$ let

$$H_{\mathcal{M}}^{n,m} : \mathcal{S}^{\text{op}} \rightarrow \text{Mod}_R$$

be the functor sending $X \in \mathcal{S}$ to $H_{\mathcal{M}}^{n,m}(X) := \text{Hom}_{\mathcal{M}(X)}(1_X, 1_X(m)[n])$.

Let Pic (resp. Vec , K^0) be the functor on $\mathcal{S}$ sending X to the Picard group $\text{Pic}(X)$ of X (resp. the set of isomorphism classes of finite rank vector bundles on X , the Grothendieck group of vector bundles on X). A *pre-orientation* on $\mathcal{M}$ is a morphism of functors

$$c_1 : \text{Pic} \rightarrow H_{\mathcal{M}}^{2,1}.$$

Let $X \in \mathcal{S}$ be an object smooth over B and let $i : Z \hookrightarrow X$ be a Cartier divisor smooth over B . By relative purity, we get a canonical isomorphism

$$i^! 1_X \simeq 1_Z(-1)[-2].$$

Combining this with the adjunction $i_* i^! \rightarrow \text{id}_X$ we get a morphism

$$H^{0,0}(Z) \rightarrow H^{2,1}(X).$$

We say that a pre-orientation c_1 is an *orientation* if this map sends the identity class in $H^{0,0}(Z)$ to $c_1(\mathcal{O}_X(Z))$ for every such closed imbedding $i : Z \hookrightarrow X$.

For the remainder of this section we fix an orientation c_1 on $\mathcal{M}$.

3.2. A *theory of Chern classes* for $\mathcal{M}$ is a collection of morphisms of functors

$$c_n : \text{Vec} \rightarrow H_{\mathcal{M}}^{2n,n}, \quad n \geq 0$$

such that the following conditions hold:

- (i) c_0 is the constant 1 and c_1 is the given orientation.
- (ii) (*Vanishing*) For a vector bundle E on X of rank r we have $c_i(E) = 0$ for $i > r$.
- (iii) (*Commutativity*) For vector bundles E and F on $X \in \mathcal{S}$ and integers $i, j \in \mathbb{Z}$ we have

$$c_i(E) \cdot c_j(F) = c_j(F) \cdot c_i(E).$$

- (iv) (*Whitney sum*) For a short exact sequence of vector bundles on $X \in \mathcal{S}$

$$0 \rightarrow E'' \rightarrow E \rightarrow E' \rightarrow 0$$

we have

$$c_k(E) = \sum_{i+j=k} c_i(E'') c_j(E').$$

Remark 3.3. When R is a $\mathbb{Q}$ -algebra, we can define as usual the Chern character which defines a morphism of functors

$$\text{ch} : K^0 \rightarrow \prod_n H_{\mathcal{M}}^{2n,n}.$$

as well as Todd classes.

3.4. Assume given a theory of Chern classes for $\mathcal{M}$. The classical computations of cohomology for flag varieties can then be carried out in our cohomology theory as well. Let us briefly recall the statement and construction. For $X \in \mathcal{S}$ define

$$A_{\mathcal{M}}^{n,m}(X) := H_{\mathcal{M}}^{2n,m}(X),$$

and set

$$A_{\mathcal{M}}^{*,*}(X) := \bigoplus_{n,m \in \mathbb{Z}} A_{\mathcal{M}}^{n,m}(X).$$

Then $A_{\mathcal{M}}^{*,*}(X)$ is a bigraded ring. For $F \in \mathcal{M}(X)$ define

$$A_{\mathcal{M}}^{n,m}(X, F) := \text{Ext}_{\mathcal{M}(X)}^{2n}(1_X, F(m)),$$

and set

$$A_{\mathcal{M}}^{*,*}(X, F) := \bigoplus_{n,m} A_{\mathcal{M}}^{n,m}(X, F),$$

a module over $A_{\mathcal{M}}^{*,*}(X)$. The main case of interest is when $F = 1_X(m)[2n]$ for some n and m , in which case $A_{\mathcal{M}}^{*,*}(X, F)$ is a free module of rank 1 over $A_{\mathcal{M}}^{*,*}(X)$ with generator in degree $A_{\mathcal{M}}^{-n,-m}(X, F)$.

3.5. Let $X \in \mathcal{S}$ be a scheme, let E be a vector bundle on X . Fix a sequence of integers $(p_1, \dots, p_m)$ and let $p : \mathcal{F} \rightarrow X$ be the flag variety classifying flags

$$0 = F_0 \subset F_1 \subset \dots \subset F_m = E$$

such that the rank of F_i/F_{i-1} is equal to p_i . Over $\mathcal{F}$ there is a universal flag F^u on p^*E . Set $E_i := F_i^u/F_{i-1}^u$ ($i = 1, \dots, m$), so E_i is a locally free sheaf of rank p_i on $\mathcal{F}$.

Consider the polynomial ring $A_{\mathcal{M}}^{*,*}(X)[T_{i,j_i}]_{1 \leq i \leq m, 1 \leq j_i \leq p_m}$ with variable T_{i,j_i} of bidegree (j_i, j_i) , and let J be the bigraded ideal in this ring generated by the homogeneous elements (note that $c_p(E) \in A_{\mathcal{M}}^{p,p}(X)$)

$$c_p(E) - \sum_{i_1 + \dots + i_m = p} T_{1,i_1} \cdots T_{m,i_m}, \quad p \geq 1.$$

There is a map of bigraded rings

$$\alpha : A_{\mathcal{M}}^{*,*}(X)[T_{i,j_i}] \rightarrow A_{\mathcal{M}}^{*,*}(\mathcal{F}), \quad T_{i,j_i} \mapsto c_{j_i}(E_i).$$

Proposition 3.6. *The map α induces an isomorphism*

$$A_{\mathcal{M}}^{*,*}(X)[T_{i,j_i}]/J \simeq A_{\mathcal{M}}^{*,*}(\mathcal{F}).$$

Proof. This follows from the argument of [12, Exposé VII, §5]. Let us just indicate the necessary modifications here.

Let $f : D \rightarrow X$ be the flag variety classifying flags of type $(1, 1, \dots, 1)$ on E . There is a natural map $g : D \rightarrow \mathcal{F}$ realizing D as the fiber product $D_1 \times_{\mathcal{F}} D_2 \cdots \times_{\mathcal{F}} D_m$, where D_i is the variety over $\mathcal{F}$ classifying full flags in E_i .

Lemma 3.7. *Let $S \in \mathcal{S}$ be an object and E a locally free sheaf of rank r on S with associated projective bundle $p : \mathbb{P}E \rightarrow S$. Let $c_1 \in H_{\mathcal{M}}^{2,1}(\mathbb{P}E)$ denote the first Chern class of the universal quotient, which induces a map $1_S(-1)[-2] \rightarrow p_*1_{\mathbb{P}E}$. Then the map induced by summing the maps c_1^i*

$$\oplus_{i=0}^{r-1} 1_S(-i)[-2i] \rightarrow p_*1_{\mathbb{P}E}$$

in $\mathcal{M}(S)$ is an isomorphism.

Proof. Notice that we have $p_{\#} \simeq p_!(r-1)[2(r-1)] \simeq p_*(r-1)[2(r-1)]$, so the desired isomorphism can also be written as an isomorphism $M_S(\mathbb{P}E) \simeq \oplus_{i=0}^{r-1} 1_S(i)[2i]$. This is shown in [6, Theorem 3.2]. $\square$

Consider the algebra $A_{\mathcal{M}}^{*,*}(X)[U_k]_{k=1,\dots,r}$ (with the U_k of bidegree $(1, 1)$) and the ideal J_D generated by elements $c_p(E) - \sigma_p$, where σ_p is the p -th symmetric function in the U_k . We then have a map

$$\alpha_D : A_{\mathcal{M}}^{*,*}(X)[U_k]/J_D \rightarrow A_{\mathcal{M}}^{*,*}(D), \quad U_k \mapsto c_1(L_k),$$

where L_k is the k -th universal quotient on D . Factoring $f : D \rightarrow X$ as a sequence of projective bundles one sees that the map α_D is an isomorphism. Now let

$$\theta : A_{\mathcal{M}}^{*,*}(X)[T_{i,j_i}]/J \rightarrow A_{\mathcal{M}}^{*,*}(X)[U_k]/J_D$$

be the map induced by the map sending T_{i,j_i} to the j_i -th elementary symmetric polynomial in the variables

$$(U_{p_1 + \dots + p_{i-1} + s})_{1 \leq s \leq p_i}.$$

We then have a commutative diagram

$$\begin{array}{ccc} A_{\mathcal{M}}^{*,*}(\mathcal{F}) & \longrightarrow & A_{\mathcal{M}}^{*,*}(D) \\ \uparrow \alpha & & \uparrow \alpha_D \\ A_{\mathcal{M}}^{*,*}(X)[T_{i,j_i}]/J & \xrightarrow{\theta} & A_{\mathcal{M}}^{*,*}(X)[U_k]/J_D. \end{array}$$

Analyzing this as in [12, p. 310] one gets that α is an isomorphism as well. $\square$

Remark 3.8. By the splitting principle and using 3.7, a theory of Chern classes is unique if it exists.

Corollary 3.9. *Let $X \in \mathcal{S}$, let $E_1, \dots, E_s$ be vector bundles on X , and let $v_1, \dots, v_s$ be integers ≥ 0 . Let G_i denote the Grassmanian of v_i -planes in E_i , and let P_i denote the universal v_i -sub-bundle of $E_i|_{G_i}$. Then the $A_{\mathcal{M}}^{*,*}(X)$ -algebra $A_{\mathcal{M}}^{*,*}(\prod_i G_i)$ is generated by the homogeneous components of the elements $\text{pr}_i^* c.(P^i)$.*

Proof. This follows from the above description and factoring $\prod_i G_i \rightarrow X$ through a sequence of Grassman bundles. $\square$

4. LOCAL CHERN CLASSES

We continue with the notation of the preceding section.

4.1. For a closed imbedding $i : X \hookrightarrow M$ in $\mathcal{S}$, define

$$A_{\mathcal{M}}^{n,m}(M \text{ on } X) := \text{Ext}_{\mathcal{M}(X)}^{2n}(1_X, i^! 1_M(m)),$$

and set

$$A_{\mathcal{M}}^{*,*}(M \text{ on } X) := \bigoplus_{n,m \in \mathbb{Z}} A_{\mathcal{M}}^{n,m}(M \text{ on } X).$$

This is a bigraded module over $A_{\mathcal{M}}^{*,*}(X)$. A *theory of local Chern classes* consists of an assignment to every bounded complex $K^\bullet$ of locally free sheaves on M with support in X cohomology classes

$$c_i^{M \text{ on } X}(K^\bullet) \in A_{\mathcal{M}}^{i,i}(M \text{ on } X)$$

satisfying the following properties:

- (i) (*Pullback*) If $f : M' \rightarrow M$ is a morphism and $i' : X' \hookrightarrow M'$ denotes $f^{-1}(X)$ then $f^* c_i^{M \text{ on } X}(K^\bullet) \in A_{\mathcal{M}}^{i,i}(M' \text{ on } X')$ is equal to $c_i^{M' \text{ on } X'}(f^* K^\bullet)$.
- (ii) Let $r : A_{\mathcal{M}}^{*,*}(M \text{ on } X) \rightarrow A_{\mathcal{M}}^{*,*}(M)$ be the map induced by the adjunction $i_* i^! \rightarrow \text{id}$. Then if $c_i^{M \text{ on } X}(K^\bullet) \in \prod_{i \geq 1} A_{\mathcal{M}}^{i,i}(M \text{ on } X)$ denotes the vector of the $c_i^{M \text{ on } X}(K^\bullet)$ then

$$r(c_i^{M \text{ on } X}(K^\bullet)) + 1 = \prod_i c.(K^{2i}) c.(K^{2i-1})^{-1}.$$

Using 3.6 and the argument of Iversen [14] one obtains:

Proposition 4.2. *Suppose given a theory of Chern classes for $\mathcal{M}$. Then a theory of local Chern classes for $\mathcal{M}$ exists and is unique.*

$\square$

4.3. In the case when R is a $\mathbb{Q}$ -algebra one can introduce as in [14, §1] the localized Chern character

$$\mathrm{ch}^{M \text{ on } X}(K^\cdot) \in \prod_i A_{\mathcal{M}}^{i,i}(M \text{ on } X).$$

By the argument of [14] this satisfies the following properties:

- (i) (*Functoriality*) If $f : M' \rightarrow M$ is a morphism and $i' : X' \hookrightarrow M'$ denotes $f^{-1}(X)$ then $f^* \mathrm{ch}^{M \text{ on } X}(K^\cdot) = \mathrm{ch}^{M' \text{ on } X'}(f^* K^\cdot)$.
- (ii) $r(\mathrm{ch}^{M \text{ on } X}(K^\cdot)) = \mathrm{ch}(K^\cdot)$.
- (iii) (*Decalage*) $\mathrm{ch}^{M \text{ on } X}(K^\cdot[1]) = -\mathrm{ch}^{M \text{ on } X}(K^\cdot)$.
- (iv) For complexes $K^\cdot$ and $L^\cdot$ on M supported on X we have

$$\mathrm{ch}^{M \text{ on } X}(K^\cdot \oplus L^\cdot) = \mathrm{ch}^{M \text{ on } X}(K^\cdot) + \mathrm{ch}^{M \text{ on } X}(L^\cdot).$$

- (v) (*Multiplicativity*) Let $K^\cdot$ (resp. $L^\cdot$) be a complex on M supported on Z (resp. V). Then

$$\mathrm{ch}^{M \text{ on } (Z \cap V)}(K^\cdot \otimes L^\cdot) = \mathrm{ch}^{M \text{ on } Z}(K^\cdot) \cdot \mathrm{ch}^{M \text{ on } V}(L^\cdot).$$

4.4. More generally, for a morphism $f : X \rightarrow Y$ in $\mathcal{S}$ we define $A_{\mathcal{M}}^{*,*}(f : X \rightarrow Y)$ by the formula

$$A_{\mathcal{M}}^{n,m}(f : X \rightarrow Y) := \mathrm{Ext}_{\mathcal{M}(X)}^n(1_X, f^! 1_Y(m)).$$

Note that for a factorization of f

$$(4.4.1) \quad \begin{array}{ccc} & f & \\ & \curvearrowright & \\ X & \xrightarrow{i} M & \xrightarrow{g} Y \end{array}$$

with i an imbedding and g smooth of relative dimension d we have $f^! 1_Y(m) \simeq i^! g^! 1_Y(m) \simeq i^! 1_Y(m+d)[2d]$, whence a canonical isomorphism

$$A_{\mathcal{M}}^{n,m}(f : X \rightarrow Y) \simeq A_{\mathcal{M}}^{n+d,m+d}(i : X \hookrightarrow M).$$

4.5. For a quasi-projective morphism $f : X \rightarrow Y$ in $\mathcal{S}$, one has the Grothendieck group of f -perfect complexes defined as in [16, 3.10]. Moreover, the same argument is in loc. cit. shows that there is a transformation

$$\tau_Y^X : K(f\text{-perfect complexes on } X) \rightarrow \oplus_i A_{\mathcal{M}}^{i,i}(f : X \rightarrow Y).$$

This transformation is defined by choosing a factorization of f as in 4.4.1 and sending a complex $K^\cdot$ to

$$\mathrm{td}(i^* T_{M/Y}) \cdot \mathrm{ch}_M^X(K^\cdot) \in A_{\mathcal{M}}^{*,*}(X \hookrightarrow M) \simeq A_{\mathcal{M}}^{*-d,*-d}(X \rightarrow Y).$$

4.6. In particular, for $X \in \mathcal{S}$, quasi-projective over B , we can consider $A_{\mathcal{M}}^{n,m}(X \rightarrow B)$, with $X \rightarrow B$ the structure morphism. In this case we define

$$H_{i,BM}^{\mathcal{M}}(X) := A_{\mathcal{M}}^{-i,-i}(X \rightarrow B),$$

called the i -th $\mathcal{M}$ -valued Borel-Moore homology of X (or just i -th Borel-Moore homology if the reference to $\mathcal{M}$ is clear). In this case the Grothendieck group of f -perfect complexes is simply the Grothendieck group of coherent sheaves on X (since B is regular) so we get a map

$$(4.6.1) \quad \tau_X : K(\mathrm{Coh}(X)) \rightarrow \oplus_i H_{i,BM}^{\mathcal{M}}(X).$$

Remark 4.7. In the case when B is the spectrum of a field, the map (4.6.1) can also be viewed as a cycle class map, using the identification of $K(\text{Coh}(X))$ with Chow groups (tensor $\mathbb{Q}$).

5. LOCAL TERMS FOR MOTIVIC ACTIONS

Let k be a field and let $\mathcal{S}$ denote the category of finite type separated k -schemes. Fix a ring R and let $\mathcal{M}$ be an R -linear triangulated motivic category.

5.1. Let $X, Y \in \mathcal{S}$ be two objects. For $F \in \mathcal{M}(X)$ and $G \in \mathcal{M}(Y)$ let $F \boxtimes G \in \mathcal{M}(X \times Y)$ denote $\text{pr}_1^* F \otimes_{X \times Y} \text{pr}_2^* G$. There is a map

$$\epsilon_{X \times Y} : \Omega_X \boxtimes \Omega_Y \rightarrow \Omega_{X \times Y}$$

defined as follows. We have isomorphisms

$$\begin{aligned} \text{Hom}_{X \times Y}(\Omega_X \boxtimes \Omega_Y, \Omega_{X \times Y}) &\simeq \text{Hom}_{X \times Y}(\text{pr}_1^* \Omega_X, \text{pr}_2^! 1_Y) \\ &\simeq \text{Hom}_Y(\text{pr}_{2!} \text{pr}_1^* \Omega_X, 1_Y) \\ &\simeq \text{Hom}_Y(g^* f_! f^! 1_{\text{Spec}(k)}, g^* 1_{\text{Spec}(k)}), \end{aligned}$$

where $f : X \rightarrow \text{Spec}(k)$ (resp. $g : Y \rightarrow \text{Spec}(k)$) is the structure morphism, the first isomorphism is induced by the isomorphism $\text{Hom}(\text{pr}_2^* \Omega_Y, \Omega_{X \times Y}) \simeq \text{pr}_2^! 1_Y$, the second isomorphism is by adjunction, and the third isomorphism is by base change. The map $\epsilon_{X \times Y}$ is the map corresponding under these isomorphisms to the adjunction map $f_! f^! 1_{\text{Spec}(k)} \rightarrow 1_{\text{Spec}(k)}$.

Assumption 5.2. Assume that the map $\epsilon_{X \times Y}$ is an isomorphism.

5.3. Note that there is a natural map

$$1_X \boxtimes \Omega_Y \rightarrow \text{Hom}_{X \times Y}(\Omega_X \boxtimes 1_Y, \Omega_X \boxtimes \Omega_Y)$$

which, with the above identification of $\Omega_X \boxtimes \Omega_Y$ with $\Omega_{X \times Y}$, gives a map

$$\rho_{X \times Y} : 1_X \boxtimes \Omega_Y \rightarrow D(\Omega_X \boxtimes 1_Y).$$

Lemma 5.4. The map $\rho_{X \times Y}$ is an isomorphism.

Proof. Consider a closed imbedding $i : Z \hookrightarrow Y$ with complement $j : U \hookrightarrow Y$, and let $\tilde{i} : X \times Z \hookrightarrow X \times Y$ and $\tilde{j} : X \times U \hookrightarrow X \times Y$ be the inclusions defined by base change. We then have a distinguished triangle (using assumption 5.2)

$$\tilde{i}_*(\Omega_X \boxtimes \Omega_Z) \rightarrow \Omega_X \boxtimes \Omega_Y \rightarrow \tilde{j}_*(\Omega_X \boxtimes \Omega_U) \rightarrow \tilde{i}_*(\Omega_X \boxtimes \Omega_Z)[1].$$

Applying $\text{Hom}_{X \times Y}(\Omega_X \boxtimes 1_Y, -)$ to this triangle we get a distinguished triangle

$$\tilde{i}_* D(\Omega_X \boxtimes 1_Z) \rightarrow D(\Omega_X \boxtimes 1_Y) \rightarrow \tilde{j}_* D(\Omega_X \boxtimes 1_U) \rightarrow \tilde{i}_* D(\Omega_X \boxtimes 1_Z)[1].$$

Our map $\rho_{X \times Y}$ is compatible with this triangle, in the sense that we get a commutative diagram

$$\begin{array}{ccccccc} \tilde{i}_*(1_X \boxtimes \Omega_Z) & \longrightarrow & 1_X \boxtimes \Omega_Y & \longrightarrow & \tilde{j}_*(1_X \boxtimes \Omega_U) & \longrightarrow & \tilde{i}_*(1_X \boxtimes \Omega_Z)[1] \\ \downarrow \rho_{X \times Z} & & \downarrow \rho_{X \times Y} & & \downarrow \rho_{X \times U} & & \downarrow \rho_{X \times Z} \\ \tilde{i}_*D(\Omega_X \boxtimes 1_Z) & \longrightarrow & D(\Omega_X \boxtimes 1_Y) & \longrightarrow & \tilde{j}_*D(\Omega_X \boxtimes 1_U) & \longrightarrow & \tilde{i}_*D(\Omega_X \boxtimes 1_Z)[1]. \end{array}$$

From this and induction on the dimension of Y it follows that it suffices to consider the case when Y is smooth of some dimension d , in which case the result is immediate. $\square$

5.5. Fix a correspondence $c : C \rightarrow X \times X$ with $C, X \in \mathcal{S}$ and fixed point scheme $\text{Fix}(c) := C \times_{c, X \times X, \Delta_X} X$. For an action (a morphism in $\mathcal{M}(C)$)

$$u : c_1^* 1_X \simeq 1_C \rightarrow c_2^! 1_X,$$

we define $\text{Tr}_c(u) \in H_{0, BM}^{\mathcal{M}}(\text{Fix}(c))$ as follows. Note that we have

$$c^!(\Omega_X \boxtimes 1_X) \simeq D_C c^*(1_X \boxtimes \Omega_X) \simeq D_C c_2^* \Omega_X \simeq c_2^! 1_X.$$

Therefore we can also view u as an element of $H_{\mathcal{M}}^0(C, c^!(\Omega_X \boxtimes 1_X))$. Let $\Delta : X \rightarrow X \times X$ be the diagonal morphism and note that there is a natural map

$$\Omega_X \boxtimes 1_X \rightarrow \Delta_* \Omega_X.$$

Applying this map and using the base change isomorphism for the cartesian diagram

$$\begin{array}{ccc} C & \xleftarrow{\delta} & \text{Fix}(c) \\ \downarrow c & & \downarrow c' \\ X \times X & \xleftarrow{\Delta_X} & X \end{array}$$

we get a morphism

$$c^!(\Omega_X \boxtimes 1_X) \rightarrow c^! \Delta_* \Omega_X \simeq \delta_* c^! \Omega_X \simeq \delta_* \Omega_{\text{Fix}(c)}.$$

Applying $\text{Hom}_{\mathcal{M}(C)}(1_C, -)$ to this map we get a map

$$\text{Tr}^{\mathcal{M}} : \text{Hom}_{\mathcal{M}(C)}(c_1^* 1_X, c_2^! 1_X) \rightarrow H_{0, BM}^{\mathcal{M}}(\text{Fix}(c)).$$

If no confusion seems likely we write simply Tr for $\text{Tr}^{\mathcal{M}}$.

5.6. Let $\mathcal{N}$ be a second motivic category satisfying the assumption 5.2, and let

$$R : \mathcal{M} \rightarrow \mathcal{N}$$

be a morphism of fibered categories such that for every $X \in \mathcal{S}$ the morphism on fibers

$$R_X : \mathcal{M}(X) \rightarrow \mathcal{N}(X)$$

is a triangulated monoidal functor, and assume further that R is compatible with the operations $f_*, f^*, f_!, f^!$ for morphisms $f : X \rightarrow Y$ in $\mathcal{S}$, Tate twists, and internal Hom . Note that then R induces for every $X \in \mathcal{S}$ a map (which we abusively also denote simply by R)

$$R : H_{*, BM}^{\mathcal{M}}(X) \rightarrow H_{*, BM}^{\mathcal{N}}(X).$$

Let $c : C \rightarrow X \times X$ be a correspondence as in 5.5, and let $u : c_1^* 1_{X, \mathcal{M}} \rightarrow c_2^! 1_{X, \mathcal{M}}$ be an action in $\mathcal{M}(C)$, where we write $1_{X, \mathcal{M}}$ for the unit object in $\mathcal{M}(X)$. Then it follows from the construction and the fact that R is compatible with the six operations that

$$R(\mathrm{Tr}^{\mathcal{M}}(u)) = \mathrm{Tr}^{\mathcal{N}}(R(u))$$

in $H_{0, BM}^{\mathcal{N}}(\mathrm{Fix}(c))$.

6. BEILINSON MOTIVES

In this section B is a regular excellent scheme of finite Krull dimension δ , and $\mathcal{S}$ denotes the category of finite type separated B -schemes.

6.1. For $R = \mathbb{Q}$, there are several equivalent constructions of triangulated motivic categories. The one most convenient for us in this paper is the category of constructible Beilinson motives defined in [4, §14] which we will denote by $\mathcal{M}_B$.

The main properties of this category that we will need are the following 6.2 and 6.5 (which in particular implies that 5.2 holds).

Proposition 6.2. *For $X \in \mathcal{S}$ quasi-projective over B the map 4.6.1 induces for every i an isomorphism*

$$A_{\delta+i}(X)_{\mathbb{Q}} \simeq H_{i, BM}^{\mathcal{M}_B}(X),$$

where the left side refers to Chow homology groups, tensor $\mathbb{Q}$, as defined in [11, §1.8] (or in the case when B is the spectrum of a field [10, §1.3]).

Proof. To properly defined $\mathcal{M}_B$ requires a lot of preparatory material, for which we refer to [4, §14]. With notation as in [4, 14.1.2], we have a map of ring spectra

$$(6.2.1) \quad H_B \rightarrow KGL_{\mathbb{Q}}.$$

One definition of the category of Beilinson motives (see [4, 14.2.9]) is as the homotopy category $\mathrm{Ho}(H_B - \mathrm{mod})$. The map 6.2.1 therefore induces a morphism of motivic categories

$$\mathcal{M}_B \subset \mathrm{Ho}(H_B - \mathrm{mod}) \rightarrow \mathrm{Ho}(KGL_{\mathbb{Q}} - \mathrm{mod}).$$

Fix a closed imbedding $i : X \hookrightarrow M$, with M smooth of constant dimension d over B , and let $j : U \hookrightarrow M$ be the complement of X . Taking cohomology of the distinguished triangle

$$i_* i^! 1_M(d+a)[2d] \rightarrow 1_M(d+a)[2d] \rightarrow j_* 1_U(d+a)[2d]$$

we get a long exact sequence

$$(6.2.2) \quad \cdots \rightarrow H_{\mathcal{M}_B}^s(X, \Omega_X(a)) \rightarrow H_{\mathcal{M}_B}^{s+2d, d+a}(M) \rightarrow H_{\mathcal{M}_B}^{s+2d, d+a}(U) \rightarrow \cdots$$

To compare this with K-theory we following the argument of [17, Proof of Théorème 8]. Recall from [17, 7.2 and Théorème 8 (v)] that for any $X \in \mathcal{S}$ the filtration F on $K_0(X)_{\mathbb{Q}}$ defined by the Riemann-Roch isomorphism $K_0(X)_{\mathbb{Q}} \simeq A_*(X)_{\mathbb{Q}}$ (so $F_j/F_{j+1} \simeq A_j(X)_{\mathbb{Q}}$) is induced by operations ϕ^k on $K_0(X)_{\mathbb{Q}}$. Following the proof of [17, Théorème 8 (ii)] we get the long exact sequence

$$\cdots \rightarrow K_m(X) \rightarrow K_m(M) \rightarrow K_m(U) \rightarrow K_{m-1}(X) \rightarrow \cdots,$$

and as in loc. cit. this sequence is compatible with the Adams operations and induces an exact sequence on the associated graded pieces

$$(6.2.3) \quad \cdots \rightarrow \mathrm{gr}_j K_m(X) \rightarrow \mathrm{gr}_j K_m(M) \rightarrow \mathrm{gr}_j K_m(U) \rightarrow \cdots$$

The fact that the map from K-theory to the cohomology of Beilinson motives is defined by the map of ring spectra 6.2.1 and the description of the Adams's operations as coming from the decomposition in [4, 14.1.1 (K5)] implies that we get an induced map from the long exact sequence 6.2.3 to the long exact sequence 6.2.2. We therefore obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc} \mathrm{gr}_{\delta+i} K_1(M) & \longrightarrow & \mathrm{gr}_{\delta+i} K_1(U) & \longrightarrow & \mathrm{gr}_{\delta+i} K_0(X) & \longrightarrow & \mathrm{gr}_{\delta+i} K_0(M) & \longrightarrow & \mathrm{gr}_{\delta+i} K_0(U) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{\mathcal{M}_B}^{2(d-i)-1, d-i}(M) & \longrightarrow & H_{\mathcal{M}_B}^{2(d-i)-1, d-i}(U) & \longrightarrow & H_{i, BM}^{\mathcal{M}_B}(X) & \longrightarrow & H_{\mathcal{M}_B}^{2(d-i), d-i}(M) & \longrightarrow & H_{\mathcal{M}_B}^{2(d-i), d-i}(U). \end{array}$$

It therefore suffices to show that if $M \rightarrow B$ is smooth of relative dimension d then for any integer $s \geq 0$ the map

$$\mathrm{gr}_{\delta+i} K_s(M) \rightarrow H_{\mathcal{M}_B}^{2(d-i)-s, d-i}(M)$$

is an isomorphism. This follows from [4, 14.2.14], which identifies $H_{\mathcal{M}_B}^{2(d-i)-s, d-i}(M)$ with $\mathrm{gr}_{\gamma}^{d-i} K_s(M)$ (associated graded of the γ -filtration), and [17, 7.2 (vi)] which shows that $\mathrm{gr}_{\gamma}^{d-i} K_s(M) = \mathrm{gr}_{i+\delta} K_s(M)$.

We therefore get an induced isomorphism $A_i(X)_{\mathbb{Q}} \simeq H_{i, BM}^{\mathcal{M}_B}(X)$. The fact that this isomorphism is induced by the map 4.6.1 follows from the uniqueness in 4.2 $\square$

Remark 6.3. If $X \in \mathcal{S}$ is regular we have, by [4, 14.2.14], an isomorphism

$$H_{\mathcal{M}_B}^{q, p}(X) \simeq \mathrm{gr}_{\gamma}^p K_{2p-q}(X)_{\mathbb{Q}}.$$

This isomorphism implies various vanishing results for motivic cohomology. We will need two cases in what follows:

- (i) ($p < 0$) If d is the dimension of X then by [17, Théorème 7.2 (vi)] we have

$$\mathrm{gr}_{\gamma}^p K_{2p-q}(X)_{\mathbb{Q}} = \mathrm{gr}^{d-p} K_{2p-q}(X)_{\mathbb{Q}},$$

which in the notation of [17, 7.4] is equal to $H_{2d-q}(X, d-p)$. If $p < 0$ then $d-p > d$ in which case this group is 0 by [17, Théorème 8 (i)].

- (ii) (X affine $p = 0$ and $q < 0$) In this case we have

$$H_{\mathcal{M}_B}^{q, p}(X) \simeq \mathrm{gr}_{\gamma}^0 K_{-q}(X).$$

The vanishing of this group is a known special case of Beilinson-Soulé vanishing [17, 2.9].

The second property of the category of Beilinson motives that we will need is the following result, which will enable us to use de Jong's results on equivariant alterations [7].

6.4. Let Y be a quasi-projective k -scheme and G a finite group acting on Y . Let X' denote the coarse moduli space of the quotient stack $[Y/G]$ and let $\pi : X' \rightarrow X$ be a finite surjective radicial morphism. Let $p : Y \rightarrow X$ be the projection and define $(p_* \Omega_Y)^G$ as in [4, 3.3.21].

There is a natural morphism $p_*\Omega_Y \rightarrow \Omega_X$ (dual to the adjunction morphism $1_X \rightarrow p_*1_Y$) which induces a morphism

$$(6.4.1) \quad h : (p_*\Omega_Y)^G \rightarrow \Omega_X.$$

Proposition 6.5. *The map 6.4.1 is an isomorphism.*

Proof. Let $i : Z \hookrightarrow X$ be a closed imbedding with complement $j : U \hookrightarrow X$ such that the following hold:

- (1) U is everywhere dense in X .
- (2) If Y_U denotes $U \times_X Y$ then U_{red} and $Y_{U,\text{red}}$ are regular, and the map $Y_{U,\text{red}} \rightarrow U_{\text{red}}$ is flat.

Let Y_Z denote $Y \times_X Z$ and let X'_Z denote the coarse moduli space of $[Y_Z/G]$. The formation of the coarse moduli space does not in general commute with base change. It is still true, however, that $Y_Z \rightarrow X'_Z \rightarrow Z$ satisfies the assumptions in 6.4. There is an induced map of distinguished triangles

$$\begin{array}{ccccccc} (i_*p_{Z*}\Omega_{Y_Z})^G & \longrightarrow & (p_*\Omega_Y)^G & \longrightarrow & (j_*p_{U*}\Omega_{Y_U})^G & \longrightarrow & (i_*p_{Z*}\Omega_Z)^G[1] \\ \downarrow h_Z & & \downarrow h & & \downarrow h_U & & \downarrow h_Z \\ i_*\Omega_Z & \longrightarrow & \Omega_X & \longrightarrow & j_*\Omega_U & \longrightarrow & i_*\Omega_Z, \end{array}$$

where we use the fact that the formation of homotopy fixed points commutes with push-forward. By induction we may assume that h_Z is an isomorphism. It therefore suffices to consider the case when X and Y are regular of the same dimension, where it follows from [4, 3.3.35 and 14.3.3] and the following lemma. $\square$

Lemma 6.6. *Let $X \in \mathcal{S}$ be regular of dimension d and quasi-projective over B . Then $\Omega_X \simeq 1_X(d - \delta)[2(d - \delta)]$.*

Proof. Since X is quasi-projective over B we can find a locally free sheaf E of finite rank $r + 1$ on B and an imbedding $i : X \hookrightarrow \mathbb{P}E$ over B . Then i is a regular imbedding of codimension $r + \delta - d$. We have $\Omega_X \simeq i^!1_{\mathbb{P}E}(r)[2r]$ so it suffices to show that $i^!1_{\mathbb{P}E} \simeq 1_X(d - \delta - r)[2(d - \delta - r)]$. This follows from absolute purity for Beilinson motives [4, 14.4.1] (see also [5, A.2.8]). $\square$

An immediate corollary is the following:

Corollary 6.7. *Let $Y \in \mathcal{S}$ be a regular quasi-projective B -scheme of dimension d , and let G be a finite group acting on Y . Let $X := Y/G$ be the coarse moduli space of the corresponding Deligne-Mumford stack $[Y/G]$. Then $\Omega_X \simeq 1_X(d - \delta)[2(d - \delta)]$.*

In particular, if $f : Z \rightarrow X$ is a morphism then we have $A_{\mathcal{M}_B}^{i,i}(Z \rightarrow X) \simeq A_{i-d+\delta}(Z)_{\mathbb{Q}}$.

Proof. The first statement follows immediately from 6.5. The second statement follows from this and 6.2. $\square$

6.8. For the remainder of this section we assume that B is the spectrum of a field. In this case, proposition 6.5 also implies that assumption 5.2 holds for $\mathcal{M}_B$. The proof of this is a bit more intricate and it is useful to consider the following variant statements and intermediate results.

6.9. Fix $X, Y \in \mathcal{S}$ and a closed imbedding $i : Z \hookrightarrow X$. Let $j : U \hookrightarrow X$ be the complement of Z and let

$$Z \times Y \xrightarrow{\tilde{i}} X \times Y \xleftarrow{\tilde{j}} U \times Y$$

denote the base changes to Y . For $F \in \mathcal{M}_B(X)$ let F_U denote the restriction to U , and consider the following conditions:

$(A_X^Z(F))$ The natural map $(j_* F_U) \boxtimes \Omega_Y \rightarrow \tilde{j}_*(F_U \boxtimes \Omega_Y)$ is an isomorphism.

$(B_X^Z(F))$ The natural map $(i^! F) \boxtimes \Omega_Y \rightarrow \tilde{i}^!(F \boxtimes \Omega_Y)$ is an isomorphism.

(C_X) The map $\epsilon_{X \times Y} : \Omega_X \boxtimes \Omega_Y \rightarrow \Omega_{X \times Y}$ is an isomorphism.

Lemma 6.10. *Properties $A_X^Z(F)$ and $B_X^Z(F)$ are equivalent.*

Proof. Indeed the maps in question fit into a morphism of distinguished triangles

$$\begin{array}{ccccccc} \tilde{i}_*(i^! F \boxtimes \Omega_Y) & \longrightarrow & F \boxtimes \Omega_Y & \longrightarrow & (j_* F) \boxtimes \Omega_Y & \longrightarrow & \tilde{i}_*(i^! F \boxtimes \Omega_Y)[1] \\ \downarrow & & \parallel & & \downarrow & & \downarrow \\ \tilde{i}_* \tilde{i}^!(F \boxtimes \Omega_Y) & \longrightarrow & F \boxtimes \Omega_Y & \longrightarrow & \tilde{j}_*(F \boxtimes \Omega_Y) & \longrightarrow & \tilde{i}_* \tilde{i}^!(F \boxtimes \Omega_Y)[1]. \end{array}$$

□

Lemma 6.11. *Let $f : X' \rightarrow X$ be a proper morphism, let $i' : Z' \hookrightarrow X'$ be the preimage of Z , and let $F' \in \mathcal{M}_B(X')$ be an object. Then $B_{X'}^{Z'}(F')$ implies $B_X^Z(f_* F')$.*

Proof. Let $\tilde{f} : X' \times Y \rightarrow X \times Y$ (resp. $f_Z : Z' \rightarrow Z$, $\tilde{f}_Z : Z' \times Y \rightarrow Z \times Y$) denote the morphism induced by f . Since f is proper we have isomorphisms

$$i^! f_* \simeq f_{Z*} i'^!, \quad \tilde{i}^! \tilde{f}_* \simeq \tilde{f}_{Z*} \tilde{i}'^!.$$

From this and the projection formula for proper morphisms 2.8 (4) we get isomorphisms

$$(i^! f_* F') \boxtimes \Omega_Y \simeq \tilde{f}_{Z*}((i'^! F') \boxtimes \Omega_Y)$$

and

$$\tilde{i}^!(f_* F' \boxtimes \Omega_Y) \simeq \tilde{f}_{Z*}(\tilde{i}'^!(F' \boxtimes \Omega_Y))$$

which identifies the map in $B_X^Z(f_* F')$ with the pushforward of the map in $B_{X'}^{Z'}(F')$. From this the lemma follows. □

Lemma 6.12. *Let*

$$F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow F_1[1]$$

be a distinguished triangle in $\mathcal{M}_B(X)$. If $B_X^Z(F_1)$ and $B_X^Z(F_2)$ hold then so does $B_X^Z(F_3)$.

Proof. This follows from noting that the morphisms in the properties $B_X^Z(F_i)$ fit into a morphism of distinguished triangles

$$\begin{array}{ccccccc} i^! F_1 \boxtimes \Omega_Y & \longrightarrow & i^! F_2 \boxtimes \Omega_Y & \longrightarrow & i^! F_3 \boxtimes \Omega_Y & \longrightarrow & i^! F_1 \boxtimes \Omega_Y[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \tilde{i}^!(F_1 \boxtimes \Omega_Y) & \longrightarrow & \tilde{i}^!(F_2 \boxtimes \Omega_Y) & \longrightarrow & \tilde{i}^!(F_3 \boxtimes \Omega_Y) & \longrightarrow & \tilde{i}^!(F_1 \boxtimes \Omega_Y)[1]. \end{array}$$

□

Lemma 6.13. *Let $X \in \mathcal{S}$ be a scheme, and suppose that for every nowhere dense closed subscheme $i : Z \hookrightarrow X$ the properties C_Z and $B_X^Z(\Omega_X)$ hold. Then property C_X also holds.*

Proof. Let $j : U \hookrightarrow X$ be an everywhere dense open subscheme with U_{red} smooth over k , and let $i : Z \hookrightarrow X$ be the complementary closed subscheme (with the reduced structure). From the distinguished triangle

$$i_*\Omega_Z \rightarrow \Omega_X \rightarrow j_*\Omega_U \rightarrow i_*\Omega_Z[1]$$

and its variant for $X \times Y$ we get a commutative diagram

$$\begin{array}{ccccc} i_*\Omega_Z \boxtimes \Omega_Y & \longrightarrow & \Omega_X \boxtimes \Omega_Y & \longrightarrow & j_*\Omega_U \boxtimes \Omega_Y \\ \downarrow a & & \parallel & & \downarrow b \\ \tilde{i}_*(\tilde{i}^!(\Omega_X \boxtimes \Omega_Y)) & \longrightarrow & \Omega_X \boxtimes \Omega_Y & \longrightarrow & \tilde{j}_*(\Omega_U \boxtimes \Omega_Y) \\ \downarrow d & & \downarrow f & & \downarrow e \\ \tilde{i}_*\Omega_{Z \times Y} & \longrightarrow & \Omega_{X \times Y} & \longrightarrow & \tilde{j}_*\Omega_{U \times Y} \end{array}$$

By property $B_X^Z(\Omega_X)$ the map labelled a is an isomorphism. Since U_{red} is smooth the map e is trivially an isomorphism. Now by property C_Z the map c is an isomorphism whence the map d is also an isomorphism. From this it follows that the map f is an isomorphism as well. $\square$

Lemma 6.14. *Let $i : Z \hookrightarrow X$ be a closed imbedding. The properties C_Z and C_X imply $B_X^Z(\Omega_X)$.*

Proof. Indeed we have

$$(i^!\Omega_X) \boxtimes \Omega_Y \simeq \Omega_Z \boxtimes \Omega_Y \simeq \Omega_{Z \times Y},$$

where the second isomorphism is by property C_Z . Similarly we have

$$\tilde{i}^!(\Omega_X \boxtimes \Omega_Y) \simeq \tilde{i}^!(\Omega_{X \times Y}) \simeq \Omega_{Z \times Y},$$

where the first isomorphism is by property C_X . Under these identifications the map occurring in property $B_X^Z(\Omega_X)$ is identified with the identity map on $\Omega_{Z \times Y}$. $\square$

6.15. Let $p : E \rightarrow X$ be a proper morphism, and fix a distinguished triangle in $\mathcal{M}_B(X)$

$$1_X \rightarrow p_*1_E \rightarrow \mathcal{F} \rightarrow 1_X[1].$$

Assume there exists a closed imbedding $i : Z \hookrightarrow X$ with everywhere dense complement $j : U \rightarrow X$ such that the restriction $p_U : E_U \rightarrow U$ of p to U is finite radicial and surjective. Let $p_Z : E_Z \rightarrow Z$ be the restriction of p to Z . Let $\mathcal{F}_Z$ denote a cone of the morphism $1_Z \rightarrow p_{Z*}1_{E_Z}$ so we can find a morphism of distinguished triangles in $\mathcal{M}_B(X)$

$$\begin{array}{ccccccc} 1_X & \longrightarrow & p_*1_E & \longrightarrow & \mathcal{F} & \longrightarrow & 1_X[1] \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow a \\ i_*1_Z & \longrightarrow & i_*p_{Z*}1_{E_Z} & \longrightarrow & i_*\mathcal{F}_Z & \longrightarrow & i_*1_Z[1], \end{array}$$

where the maps labelled a and b are the adjunction maps.

Lemma 6.16. *The map $c : \mathcal{F} \rightarrow i_*\mathcal{F}_Z$ is an isomorphism.*

Proof. Considering the distinguished triangle

$$j_! j^* \mathcal{F} \rightarrow \mathcal{F} \rightarrow i_* i^* \mathcal{F} \rightarrow j_! j^* \mathcal{F}[1]$$

it suffices to show that $j^* \mathcal{F} = 0$ and that the map $i^* \mathcal{F} \rightarrow \mathcal{F}_Z$ is an isomorphism. The first statement follows from the fact that the morphism $1_X \rightarrow p_* 1_E$ is an isomorphism over U , and the second statement follows from the fact that the base change map $i^* p_* 1_E \rightarrow p_{Z*} 1_{E_Z}$ is an isomorphism. $\square$

Theorem 6.17. *Let $i : Z \hookrightarrow X$ be a closed imbedding in $\mathcal{S}$. Then properties $A_X^Z(1_X)$, $B_X^Z(1_X)$, $A_X^Z(\Omega_X)$, $B_X^Z(\Omega_X)$, and C_X hold.*

Proof. By induction on the dimension d of X we may assume that we have $X \in \mathcal{S}$ of dimension d and that the theorem is true for every $E \in \mathcal{S}$ of dimension $< d$. To verify the theorem for X it then suffices by 6.10 and 6.13 to show that for every $i : Z \hookrightarrow X$ the properties $B_X^Z(1_X)$ and $B_X^Z(\Omega_X)$ hold.

By [7, 7.3] we can find a proper morphism $\tilde{p} : \tilde{E} \rightarrow X$ with $\tilde{E}$ smooth and equipped with an action of finite group G over X , such that if $p : E \rightarrow X$ is the coarse moduli space of the stack $[\tilde{E}/G]$ then p is generically on X finite surjective and radicial. By 6.5 this implies that $\Omega_E \simeq 1_E(d)[2d]$, where d is the dimension of $\tilde{E}$ (a locally constant function). This also implies that $\Omega_{E \times Y} \simeq 1_E(d)[2d] \boxtimes \Omega_Y$ since by 6.5 we have an isomorphism

$$(\tilde{p}_* \Omega_{\tilde{E} \times Y})^G \simeq (\tilde{p}_*(1_{\tilde{E}}(d)[2d] \boxtimes \Omega_Y))^G \simeq (\tilde{p}_* \tilde{p}^*(1_E(d)[2d] \boxtimes \Omega_Y))^G \simeq 1_E(d)[2d] \boxtimes \Omega_Y,$$

where the last isomorphism is by [4, 3.3.35]. In particular, property C_E holds.

This implies that for every closed $t : T \hookrightarrow E$ properties $B_E^T(\Omega_E)$ and $B_E^T(1_E)$ hold. Indeed since E is obtained as the coarse moduli space of a smooth Deligne-Mumford stack, for any connected component E_i of E the intersection $T \cap E_i$ is either all of E_i or of dimension $< d$. We can therefore apply 6.14.

Let Q be a cone of the morphism $1_X \rightarrow p_* 1_E$. By 6.11 we then have properties $B_X^Z(p_* 1_E)$ and $B_X^Z(p_* \Omega_E)$. Lemma 6.12 then implies that to verify property $B_X^Z(1_X)$ it suffices to verify the property $B_X^Z(Q)$. Dualizing we also have a distinguished triangle

$$D_X(Q) \rightarrow p_* \Omega_E \rightarrow \Omega_X \rightarrow D_X(Q)[1],$$

and to verify property $B_X^Z(\Omega_X)$ it suffices to verify property $B_X^Z(D_X(Q))$.

Let $\alpha : T \hookrightarrow X$ be a nowhere dense closed subscheme such that the restriction of p to the complement of T is finite and radicial. Let Z_T (resp. E_T , Z'_T) denote $Z \cap T$ (resp. $T \times_X E$, $Z_T \times_X E$). Then by 6.16 we have $Q = \alpha_* Q_T$ for some $Q_T \in \mathcal{M}_B(T)$ fitting into a distinguished triangle

$$1_T \rightarrow p_{T*} 1_{E_T} \rightarrow Q_T \rightarrow 1_T[1].$$

Dualizing we also get a distinguished triangle

$$D_T(Q_T) \rightarrow p_{T*} \Omega_{E_T} \rightarrow \Omega_T \rightarrow D_T(Q_T)[1].$$

By the induction hypothesis and applying 6.12 we conclude that $B_T^{Z_T}(Q_T)$ and $B_T^{Z_T}(D_T(Q_T))$ hold, and therefore by 6.11 properties $B_X^Z(Q)$ and $B_X^Z(D_X(Q))$ also hold. $\square$

7. APPLICATION: LOCAL TERMS FOR ACTIONS GIVEN BY LOCALIZED CHERN CLASSES

Let k be an algebraically closed field, and let $\mathcal{M}_B$ denote the motivic category of Beilinson motives over k .

7.1. For a prime ℓ invertible in k there is constructed in [5, 5.9.21] an étale realization functor

$$R_\ell : \mathcal{M}_B \rightarrow DM_{c,\ell}$$

where for $X \in \mathcal{S}$ the fiber $DM_{c,\ell}(X)$ is isomorphic to the idempotent completion $D_c^b(X, \mathbb{Q}_\ell)^\#$ of the triangulated category $D_c^b(X, \mathbb{Q}_\ell)$. Here the idempotent completion is defined as in [2]. This realization functor is compatible with the six operations and Chern classes. Note also that by [2, 1.4] the functor

$$D_c^b(X, \mathbb{Q}_\ell) \rightarrow D_c^b(X, \mathbb{Q}_\ell)^\#$$

is fully faithful. Combining this with 7.2 we obtain the following:

Corollary 7.2. *Let $c : C \rightarrow X \times X$ be a correspondence and let $u : c_1^* 1_X \rightarrow c_2^! 1_X$ be an action in $\mathcal{M}_B(C)$. Then there exists an algebraic cycle $\Sigma \in A_0(\text{Fix}(c))_{\mathbb{Q}}$ such that for any prime ℓ invertible in k we have $\text{Tr}(u_\ell) = \text{cl}(\Sigma)$, where $u_\ell : c_1^* \mathbb{Q}_\ell \rightarrow c_2^! \mathbb{Q}_\ell$ is the ℓ -adic realization of u .*

Proof. In fact the algebraic cycle Σ is given by $\text{Tr}^{\mathcal{M}_B}(u) \in H_{0,BM}^{\mathcal{M}_B}(\text{Fix}(c)) \simeq A_0(\text{Fix}(c))_{\mathbb{Q}}$. $\square$

7.3. We apply this to correspondences as follows. Let $c : C \rightarrow X \times X$ be a correspondence with C and X quasi-projective schemes, and let E be a c_2 -perfect complex on C . We then get an action $u_\ell : c_1^* \mathbb{Q}_\ell \rightarrow c_2^! \mathbb{Q}_\ell$ from the class $\tau_X^C(E) \in H^0(C, c_2^! \mathbb{Q}_\ell)$ defined in [16, 4.2].

Theorem 7.4. *There exists a cycle $\Sigma \in A_0(\text{Fix}(c))_{\mathbb{Q}}$, independent of ℓ , such that $\text{Tr}_c(u_\ell) \in H^0(\text{Fix}(c), \Omega_{\text{Fix}(c)})$ is equal to $\text{cl}(\Sigma)$.*

Proof. Since the ℓ -adic realization functor is compatible with Chern classes 7.1, it is also compatible with localized Chern classes, by 4.2. Therefore the action u_ℓ is induced by a morphism in $\mathcal{M}_B(C)$ and the result follows from 7.2. $\square$

8. APPLICATION: QUASI-FINITE MORPHISMS AND CORRESPONDENCES

In this section B denotes a regular excellent scheme of dimension ≤ 2 , and $\mathcal{S}$ is the category of finite type separated B -schemes.

8.1. Let ℓ be a prime invertible on B , and let $f : Y \rightarrow X$ be a quasi-finite morphism between quasi-projective B -schemes. Let $u_\ell \in H^0(Y, f^! \mathbb{Q}_\ell)$ be a section. We say that u_ℓ is *motivic* if there exists a morphism $u : 1_Y \rightarrow f^! 1_X$ in $\mathcal{M}_B(Y)$ such that u_ℓ is the ℓ -adic realization of u .

The condition that u_ℓ be motivic has the following more concrete characterization. Since f is quasi-finite, $f_! \mathbb{Q}_\ell$ is a sheaf. For any dense open subscheme $j : U \hookrightarrow X$ the adjunction map $\mathbb{Q}_{\ell,X} \rightarrow R^0 j_* \mathbb{Q}_{\ell,U}$ is injective, so the map u_ℓ is determined by its restriction to $f^{-1}(U)$. In particular, let $\{Y_i\}_{i \in I}$ be the irreducible components of Y which dominate an irreducible component of X via f , and choose a dense open subscheme $U \subset X$ such that U_{red} is regular and

$$f^{-1}(U) = \coprod_{i \in I} V_i,$$

where $V_i \subset Y_i$ is a dense open and $V_{i,\text{red}}$ is regular of the same dimension of its image in U . We then have a canonical isomorphism

$$H^0(f^{-1}(U), f^! \mathbb{Q}_{\ell,U}) \simeq \mathbb{Q}_{\ell}^I,$$

and therefore we obtain an inclusion

$$(8.1.1) \quad H^0(Y, f^! \mathbb{Q}_{\ell,X}) \hookrightarrow \mathbb{Q}_{\ell}^I.$$

It follows immediately from the construction that this is independent of the choice of U . The image of u_{ℓ} in $\mathbb{Q}_{\ell}^I$ will be called the *weight vector* of u_{ℓ} , and will be denoted $w(u_{\ell})$.

Theorem 8.2. (i) *The section u_{ℓ} is motivic if and only if the weight vector $w(u_{\ell})$ lies in $\mathbb{Q}^I \subset \mathbb{Q}_{\ell}^I$.*

(ii) *If u_{ℓ} is the ℓ -adic realization of $u : 1_Y \rightarrow f^! 1_X$, then for any other prime ℓ' invertible in k the ℓ' -adic realization $u_{\ell'}$ of u has $w(u_{\ell'}) = w(u_{\ell})$ in $\mathbb{Q}^I$.*

Remark 8.3. If the weight vector $w(u_{\ell})$ lies in $\mathbb{Q}^I$ we say that u_{ℓ} has *rational weight vector*.

The proof of 8.2 occupies the following (8.4)–(8.13).

8.4. Fix a prime ℓ and an element $u_{\ell} \in H^0(Y, f^! \mathbb{Q}_{\ell})$ with weight vector $w \in \mathbb{Q}^I$. We show that u_{ℓ} is motivic as follows.

8.5. By [8, 5.15] (in the case when B is the spectrum of a field one can also use [7, 7.3]) we can find a proper morphism $\tilde{p} : \tilde{E} \rightarrow X$ with $\tilde{E}$ regular and equipped with an action of finite group G over X , such that if $p : E \rightarrow X$ is the coarse moduli space of the stack $[\tilde{E}/G]$ then p is generically on X finite surjective and radicial. Next choose a proper surjective generically finite morphism $\kappa : F \rightarrow Y \times_{f,X} E$, with F regular, which fits into a commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{g} & E \\ \downarrow q & & \downarrow p \\ Y & \xrightarrow{f} & X. \end{array}$$

Let $\nu_{\ell} : \mathbb{Q}_{\ell} \rightarrow g^! \mathbb{Q}_{\ell}$ be the map which on a connected component F dominating an irreducible component Y_i of Y is given by $w(u)_i$ divided by the number of irreducible components of F which dominate Y_i . Here we are using the fact that since E is the coarse moduli space of a regular Deligne-Mumford stack we have $\Omega_E \simeq \mathbb{Q}_{\ell}(d - \delta)[2(d - \delta)]$, where d (resp. δ) is the dimension of E (resp. B), as in 6.7.

Lemma 8.6. *The diagram*

$$(8.6.1) \quad \begin{array}{ccc} \mathbb{Q}_{\ell,Y} & \longrightarrow & q_* \mathbb{Q}_{\ell,F} \\ \downarrow u_{\ell} & & \downarrow p_* \nu_{\ell} \\ f^! \mathbb{Q}_{\ell,X} & \longrightarrow & f^! p_* \mathbb{Q}_{\ell,E} \end{array}$$

commutes.

Proof. It suffices to verify this at the generic point of each irreducible component of X , where the result is immediate from the construction. $\square$

8.7. Fix a distinguished triangle in $\mathcal{M}_B(X)$

$$1_X \rightarrow p_* 1_E \rightarrow \mathcal{F} \rightarrow 1_X[1],$$

and let $\mathcal{F}_\ell$ denote the ℓ -adic realization of $\mathcal{F}$, so we have a distinguished triangle in $D_c^b(X, \mathbb{Q}_\ell)$

$$\mathbb{Q}_{\ell,X} \rightarrow p_* \mathbb{Q}_{\ell,E} \rightarrow \mathcal{F}_\ell \rightarrow \mathbb{Q}_{\ell,X}[1].$$

Let $i : Z \hookrightarrow X$ be a closed imbedding with everywhere dense complement $j : U \rightarrow X$ such that the restriction $p_U : E_U \rightarrow U$ of p to U is finite radicial and surjective. Let $p_Z : E_Z \rightarrow Z$ be the restriction of p to Z . Let $\mathcal{F}_Z$ denote a cone of the morphism $1_Z \rightarrow p_{Z*} 1_{E_Z}$ so we can find a morphism of distinguished triangles in $\mathcal{M}_B(X)$

$$\begin{array}{ccccccc} 1_X & \longrightarrow & p_* 1_E & \longrightarrow & \mathcal{F} & \longrightarrow & 1_X[1] \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow a \\ i_* 1_Z & \longrightarrow & i_* p_{Z*} 1_{E_Z} & \longrightarrow & i_* \mathcal{F}_Z & \longrightarrow & i_* 1_Z[1], \end{array}$$

where the maps labelled a and b are the adjunction maps. By 6.16 the map $c : \mathcal{F} \rightarrow i_* \mathcal{F}_Z$ is an isomorphism.

Lemma 8.8. *Let $f : Y \rightarrow X$ be a quasi-finite morphism of quasi-projective B -schemes. Then the ℓ -adic realization map*

$$(8.8.1) \quad H_{\mathcal{M}_B}^i(Y, f^! 1_X) \rightarrow H^i(Y, f^! \mathbb{Q}_\ell)$$

is injective for $i \leq 0$, and $H_{\mathcal{M}_B}^i(Y, f^! 1_X) = 0$ for $i < 0$.

Proof. The second statement follows from the first and the fact that the functor $f^! : D_c^b(X, \mathbb{Q}_\ell) \rightarrow D_c^b(Y, \mathbb{Q}_\ell)$ takes $D_c^{\geq 0}(X, \mathbb{Q}_\ell)$ to $D_c^{\geq 0}(Y, \mathbb{Q}_\ell)$ by [1, XVIII, 3.1.7].

Consider first the case when X is the coarse moduli space of a stack of the form $[M/G]$ with M regular of some dimension d and G a finite group acting on M . In this case we have

$$H_{\mathcal{M}_B}^i(Y, f^! 1_X) \simeq H_{\mathcal{M}_B}^i(Y, \Omega_Y(\delta - d)[2(\delta - d)]).$$

By 6.2 we therefore have

$$H_{\mathcal{M}_B}^0(Y, f^! 1_X) \simeq H_{d-\delta, BM}^{\mathcal{M}}(Y) \simeq A_d(Y)_{\mathbb{Q}}.$$

Since f is quasi-finite this is canonically isomorphic to the $\mathbb{Q}$ -vector space with basis the irreducible components of Y of dimension d . This implies the injectivity of 8.8.1 for $i = 0$, and also shows that if $j : V \subset Y$ is the preimage of a dense open subset in X then the restriction map

$$H_{\mathcal{M}_B}^0(Y, f^! 1_X) \rightarrow H_{\mathcal{M}_B}^0(V, f_V^! 1_X)$$

is injective, where $f_V : V \rightarrow X$ is the restriction of f . Let $r : Z \hookrightarrow Y$ be the complement of V and let $f_Z : Z \rightarrow X$ be the restriction of f . Choose V such that V_{red} is affine and regular of some dimension $e \leq d$. In this case we have $f_V^! 1_X \simeq 1_V(e - d)[2(e - d)]$ so

$$H^i(V, f_V^! 1_X) \simeq H^{i+2(e-d)}(V, 1_V(e - d)).$$

By 6.3 these groups are zero if $i < 0$. Now from the distinguished triangle

$$r_* f_Z^! 1_X \rightarrow f^! 1_X \rightarrow j_* f_V^! 1_X \rightarrow r_* f_Z^! 1_X[1]$$

we get a long exact sequence

$$\cdots \rightarrow H^i(Z, f_Z^! 1_X) \rightarrow H^i(Y, f^! 1_X) \rightarrow H^i(V, f_V^! 1_X) \rightarrow \cdots.$$

By induction on the dimension of Y (with base case handled by the case when Y is smooth) we have $H^i(Z, f_Z^! 1_X) = 0$ for $i < 0$, and as discussed above we also have $H^i(V, f_V^! 1_X) = 0$ for $i < 0$. This therefore completes the proof in the case when X is the coarse space of a stack $[M/G]$ as above.

For the general case we proceed by induction on the dimension of X . Let $p : E \rightarrow X$ be as in 8.5, and consider the resulting distinguished triangle

$$1_X \rightarrow p_* 1_E \rightarrow \mathcal{F} \rightarrow 1_X[1].$$

Applying $f^!$ we get a distinguished triangle

$$f^! 1_X \rightarrow f^! p_* 1_E \rightarrow f^! \mathcal{F} \rightarrow f^! 1_X[1].$$

Let E_Y denote the fiber product $Y \times_X E$ so we have a cartesian square

$$\begin{array}{ccc} E_Y & \xrightarrow{g} & E \\ \downarrow q & & \downarrow p \\ Y & \xrightarrow{f} & X. \end{array}$$

By base change, we have $f^! p_* 1_E \simeq q_* g^! 1_E$ and therefore

$$H^i_{\mathcal{M}_B}(Y, f^! p_* 1_E) \simeq H^i_{\mathcal{M}_B}(E_Y, g^! 1_E).$$

By the regular case, it follows that the ℓ -adic realization map

$$H^i_{\mathcal{M}_B}(Y, f^! p_* 1_E) \rightarrow H^i(Y, f^! p_* \mathbb{Q}_\ell)$$

is injective for $i \leq 0$. To prove the lemma it therefore suffices to show that $H^i(Y, \mathcal{F}) = 0$ for $i < 0$. Let $i : Z \hookrightarrow X$, Y_Z and $\mathcal{F}_Z$ be as in 8.7 so we have

$$H^i(Y, f^! \mathcal{F}) \simeq H^i(Y_Z, f_Z^! \mathcal{F}_Z).$$

Now consider the distinguished triangle on Z

$$1_Z \rightarrow p_{Z*} 1_{E_Z} \rightarrow \mathcal{F}_Z \rightarrow 1_Z[1],$$

and the resulting distinguished triangle

$$f_Z^! 1_Z \rightarrow f_Z^! p_{Z*} 1_{E_Z} \rightarrow f_Z^! \mathcal{F}_Z \rightarrow f_Z^! 1_Z[1]$$

on Y_Z . By induction the lemma holds for the quasi-finite morphisms $f_Z : Y_Z \rightarrow Z$ and $g_Z : E_{Y_Z} \rightarrow E_Z$. To prove that $H^i(Y_Z, f_Z^! \mathcal{F}_Z) = 0$ for $i < 0$ it therefore suffices to show that the map on étale cohomology

$$H^0(Y_Z, f_Z^! \mathbb{Q}_\ell) \rightarrow H^0(E_{Y_Z}, g_Z^! \mathbb{Q}_\ell)$$

is injective, which can be seen by restricting to a regular dense open subset. $\square$

8.9. Returning to the setting of 8.5 and 8.7, fix also a distinguished triangle in $\mathcal{M}_B(Y)$

$$1_Y \rightarrow q_* 1_F \rightarrow \mathcal{G} \rightarrow 1_Y[1],$$

and let $\mathcal{G}_\ell \in D_c^b(Y, \mathbb{Q}_\ell)$ be the ℓ -adic realization of $\mathcal{G}$. Let $\tilde{i} : Y_Z \hookrightarrow Y$ denote $f^{-1}(Z)$, let $q_Z : F_Z \rightarrow Y_Z$ denote the pullback of q , and let $\mathcal{G}_Z$ denote a cone of $1_{Y_Z} \rightarrow q_{Z*} 1_{F_Z}$. Then we have $\mathcal{G} \simeq \tilde{i}_* \mathcal{G}_Z$.

8.10. Applying i^* to the diagram 8.6.1 we obtain a commutative diagram

$$(8.10.1) \quad \begin{array}{ccc} \mathbb{Q}_{\ell, Y_Z} & \longrightarrow & q_{Z*} \mathbb{Q}_{\ell, F_Z} \\ \downarrow u_{Z, \ell} & & \downarrow p_{Z*} \nu_{Z, \ell} \\ f_Z^! \mathbb{Q}_{\ell, Z} & \longrightarrow & f_Z^! p_{Z*} \mathbb{Q}_{\ell, E_Z}. \end{array}$$

The map ν_ℓ is motivic being given by the local Chern classes of a g -perfect complex of sheaves on F . The pullback ν_Z is therefore also motivic, whence the map $p_{Z*} \nu_{Z, \ell}$ is motivic as well.

Lemma 8.11. *The weight vector of $u_{Z, \ell}$ is rational.*

Proof. Let $W \subset Y_Z$ be an irreducible component which dominates an irreducible component $Z' \subset Z$ via f_Z , and let $U' \subset W$ be a nonempty regular open subset mapping to a regular open subset $U \subset Z'$. Let $V' \subset U' \times_Z E_Z$ be a nonempty smooth open subset mapping to a smooth open subset $V \subset E_Z$. Note that since U' is quasi-finite over Z , V' is also quasi-finite over E_Z , whence V' and V have the same dimension. Let $\alpha : U' \rightarrow U$ (resp. $\beta : V' \rightarrow V$) be the map induced by f_Z (resp. g_Z). Then $\alpha^! \simeq \alpha^*$. By the commutativity of (8.10.1), we then have a commutative diagram

$$(8.11.1) \quad \begin{array}{ccccc} H^0(U', \mathbb{Q}_\ell) & \xrightarrow{t_{Z*} \bar{u}} & H^0(U', \alpha^! \mathbb{Q}_\ell) & \xrightarrow{\simeq} & H^0(U', \alpha^* \mathbb{Q}_\ell) \\ \downarrow & & \downarrow & & \downarrow \text{can} \\ H^0(V', \mathbb{Q}_\ell) & \xrightarrow{\epsilon} & H^0(V', \beta^! \mathbb{Q}_\ell) & \xrightarrow{\simeq} & H^0(V', \beta^* \mathbb{Q}_\ell), \end{array}$$

where if d denotes the dimension of V then ϵ is induced by an element of $A_d(V')_{\mathbb{Q}} = \mathbb{Q} \cdot [V']$ independent of ℓ . It follows that the bottom horizontal composite in (8.11.1) is given by multiplication by a rational number independent of ℓ , whence the top horizontal composite is also given by multiplication by a rational number independent of ℓ . $\square$

8.12. By induction we can find $u_Z : 1_{Y_Z} \rightarrow f_Z^! 1_Z$ in $\mathcal{M}_B(Y_Z)$ with ℓ -adic realization $u_{Z, \ell}$. Let $\nu_Z : 1_{F_Z} \rightarrow g_Z^! 1_{E_Z}$ denote a morphism in $\mathcal{M}_B(F_Z)$ inducing $\nu_{Z, \ell}$. By 8.8 the induced diagram

$$\begin{array}{ccc} 1_{Y_Z} & \longrightarrow & q_* 1_{F_Z} \\ \downarrow u & & \downarrow q_* \nu_Z \\ f_Z^! 1_Z & \longrightarrow & f_Z^! p_{Z*} 1_{E_Z} \end{array}$$

commutes since this holds for the ℓ -adic realizations. Let $\rho_Z : \mathcal{G}_Z \rightarrow f_Z^! \mathcal{F}_Z$ be a morphism filling in the diagram

$$\begin{array}{ccccccc} 1_{Y_Z} & \longrightarrow & q_* 1_{F_Z} & \longrightarrow & \mathcal{G}_Z & \longrightarrow & 1_{Y_Z}[1] \\ \downarrow u_Z & & \downarrow q_* \nu_Z & & \downarrow \rho_Z & & \downarrow u_Z \\ f_Z^! 1_Z & \longrightarrow & f_Z^! p_{Z*} 1_{E_Z} & \longrightarrow & f_Z^! \mathcal{F}_Z & \longrightarrow & f_Z^! 1_Z[1], \end{array}$$

and let $\rho : \mathcal{G} \rightarrow f^! \mathcal{F}$ be the morphism obtained by applying i_* to ρ_Z . Then the diagram

$$\begin{array}{ccc} q_* 1_F & \longrightarrow & \mathcal{G} \\ \downarrow q_* \nu & & \downarrow \rho \\ f^! p_* 1_E & \longrightarrow & f^! \mathcal{F} \end{array}$$

commutes. Indeed this can be verified after applying i^* where the result follows from the construction. We can therefore find a morphism $\lambda : 1_Y \rightarrow f^! 1_X$ so that we have a morphism of distinguished triangles

$$\begin{array}{ccccccc} 1_Y & \longrightarrow & q_* 1_F & \longrightarrow & \mathcal{G} & \longrightarrow & 1_Y[1] \\ \downarrow \lambda & & \downarrow \nu & & \downarrow \rho & & \downarrow \lambda \\ f^! 1_X & \longrightarrow & f^! p_* 1_E & \longrightarrow & f^! \mathcal{F} & \longrightarrow & f^! 1_X[1]. \end{array}$$

The ℓ -adic realization of λ is then equal to u_ℓ , as this can be verified over a regular dense open of X . This completes the proof of the “if” part of statement (i).

8.13. To see the “only if” part of statement (i) as well as statement (ii) in 8.2 it suffices to define the weight vector of u without passing to realizations. For this choose $U \subset X$ as in 8.15 so that $f^{-1}(U) = \coprod_i V_i$ with each $V_{i,\text{red}}$ regular. We then get a map

$$H^0_{\mathcal{M}_B}(Y, f^! 1_X) \rightarrow \prod_i H^0_{\mathcal{M}_B}(V_i, f^! 1_X) \simeq \mathbb{Q}^I,$$

where the last isomorphism uses purity. The image of u under this map is the weight vector. This completes the proof of 8.2. $\square$

Remark 8.14. The proof (in particular 8.8) shows that if u_ℓ is motivic, then the morphism $u : 1_Y \rightarrow f^! 1_X$ in $\mathcal{M}_B(X)$ inducing u_ℓ is unique.

8.15. We apply this to correspondences as follows. Assume now that B is the spectrum of an algebraically closed field k , and let $c : C \rightarrow X \times X$ be a correspondence with X and C quasi-projective schemes, and c_2 quasi-finite. Let $u : c_1^* \mathbb{Q}_\ell \rightarrow c_2^! \mathbb{Q}_\ell$ be an action of c on $\mathbb{Q}_\ell$, which we can also view as a global section $u \in H^0(C, c_2^! \mathbb{Q}_{\ell,X})$.

Theorem 8.16. *Fix a weight vector $w \in \mathbb{Q}^I$, and assume that for some prime ℓ invertible in k there exists an action $u_\ell : c_1^* \mathbb{Q}_\ell \rightarrow c_2^! \mathbb{Q}_\ell$ with $w(u) = w$.*

(i) *There exists an algebraic cycle $\Sigma \in A_0(\text{Fix}(c))_{\mathbb{Q}}$ such that for any prime ℓ invertible in k and action $u_\ell : c_1^* \mathbb{Q}_\ell \rightarrow c_2^! \mathbb{Q}_\ell$ with weight vector w we have $\text{Tr}_c(u_\ell) = \text{cl}(\Sigma)$ in $H^0(\text{Fix}(c), \Omega_{\text{Fix}(c)})$.*

(ii) *If u has rational weight vector, then for every proper component $\Gamma \subset \text{Fix}(c)$ the local term $\text{lt}_\Gamma(\mathbb{Q}_{\ell,X}, u)$ is in $\mathbb{Q}$.*

(iii) *If ℓ and ℓ' are two primes different from p , and $u \in H^0(C, c_2^! \mathbb{Q}_{\ell,X})$ and $u' \in H^0(C, c_2^! \mathbb{Q}_{\ell',X})$ are sections with weight vectors $w(u)$ and $w(u')$ in $\mathbb{Q}^I$ and equal, then for every proper component $\Gamma \subset \text{Fix}(c)$, we have equality of rational numbers*

$$\text{lt}_\Gamma(\mathbb{Q}_\ell, u) = \text{lt}_\Gamma(\mathbb{Q}_{\ell'}, u').$$

Proof. Statements (ii) and (iii) follow from (i). Statement (i) follows from 8.2 which implies that there exists a morphism $u : c_1^* 1_X \rightarrow c_2^! 1_X$ in $\mathcal{M}_B(C)$ such that for any prime ℓ invertible in k the ℓ -adic realization $u_\ell : c_1^* \mathbb{Q}_\ell \rightarrow c_2^! \mathbb{Q}_\ell$ of u has weight vector w . $\square$

8.17. Global consequences.

Theorem 8.18. *Let $c : C \rightarrow X \times X$ be a correspondence over the algebraic closure k of a finite field with C and X Deligne-Mumford stacks, and assume c_2 is finite and c_1 is quasi-finite.*

(i) *If $u : c_1^* \mathbb{Q}_\ell \rightarrow c_2^! \mathbb{Q}_\ell$ is an action with rational weight vector $w(u)$, then $\text{tr}(u|R\Gamma(X, \mathbb{Q}_\ell))$ is in $\mathbb{Q}$.*

(ii) *If ℓ and ℓ' are two primes and $u : c_1^* \mathbb{Q}_\ell \rightarrow c_2^! \mathbb{Q}_\ell$ and $u' : c_1^* \mathbb{Q}_{\ell'} \rightarrow c_2^! \mathbb{Q}_{\ell'}$ are actions with rational weight vectors and $w(u) = w(u')$, then $\text{tr}(u|R\Gamma(X, \mathbb{Q}_\ell)) = \text{tr}(u'|R\Gamma(X, \mathbb{Q}_{\ell'}))$.*

Proof. Fix a model $c : C_0 \rightarrow X_0 \times X_0$ for c over a finite field $\mathbb{F}_q \subset k$ such that all irreducible components of C are defined over $\mathbb{F}_q$. Then any map $c_1^* \mathbb{Q}_\ell \rightarrow c_2^! \mathbb{Q}_\ell$ is defined over $\mathbb{F}_q$, and in particular commutes with Frobenius. For $n \geq 0$ let

$$c^{(n)} : C \rightarrow X \times X$$

be the correspondence given by $(c_1, F_X^n \circ c_2)$, where $F_X : X \rightarrow X$ is the base change to k of the relative Frobenius on X_0 .

If $u : c_1^* \mathbb{Q}_\ell \rightarrow c_2^! \mathbb{Q}_\ell$ is an action, let $u^{(n)} : c_1^{(n)*} \mathbb{Q}_\ell \rightarrow c_2^{(n)!} \mathbb{Q}_\ell$ be the action obtained by composing u with the n iterates of the canonical isomorphism $F_X^! \mathbb{Q}_\ell \rightarrow \mathbb{Q}_\ell$. Then as in [13, 3.5 (c)] to prove (i) it suffices to show that there exists n_0 such that for $n \geq n_0$ we have $\text{tr}(u^{(n)}|R\Gamma(X, \mathbb{Q}_\ell)) \in \mathbb{Q}$ for all $n \geq n_0$, and to prove (ii) it suffices to show that there exists n_0 such that for all $n \geq n_0$ we have an equality of rational numbers

$$\text{tr}(u^{(n)}|R\Gamma(X, \mathbb{Q}_\ell)) = \text{tr}(u'^{(n)}|R\Gamma(X, \mathbb{Q}_{\ell'})).$$

Let $d : C \rightarrow X \times X$ be the transpose of c given by (c_2, c_1) . For $n \geq 0$ let ${}^{(n)}d : C \rightarrow X \times X$ denote the correspondence $(F_X^n c_2, c_1)$, so ${}^{(n)}d$ is the transpose of $c^{(n)}$. Let $v : d_1^* \Omega_X \rightarrow d_2^! \Omega_X$ denote the transpose of u , and for $n \geq n_0$ let ${}^{(n)}v : {}^{(n)}d_1^* \Omega_X \rightarrow {}^{(n)}d_2^! \Omega_X$ denote the map obtained by n iterates of the isomorphism $F_X^* \Omega_X \rightarrow \Omega_X$.

By Fujiwara's theorem [9, 5.4.5] there exists an integer n_0 such that for all $n \geq n_0$ the following hold:

- (i) The fixed points $\text{Fix}({}^{(n)}d) = \text{Fix}(c^{(n)})$ consists of a finite set of isolated points.
- (ii) We have

$$\text{tr}({}^{(n)}v|R\Gamma_c(X, \Omega_X)) = \sum_{y \in \text{Fix}({}^{(n)}d)} \text{lt}_y(\Omega_X, {}^{(n)}v).$$

- (iii) If $U \rightarrow X$ is an étale morphism and $d_U : C_U \rightarrow U \times U$ denotes the pullback of c along $U \times U \rightarrow X \times X$, and if $v_U : d_{U1}^* \Omega_U \rightarrow d_{U2}^! \Omega_U$ denotes the pullback of v , then for every $y' \in \text{Fix}({}^{(n)}d_U) = U \times_X \text{Fix}({}^{(n)}d)$ mapping to $y \in \text{Fix}({}^{(n)}d)$ we have

$$\text{lt}_{y'}(\Omega_U, {}^{(n)}d_U) = \text{lt}_y(\Omega_X, {}^{(n)}d).$$

Now since ${}^{(n)}v$ is adjoint to $u^{(n)}$ we have

$$\text{tr}({}^{(n)}v|R\Gamma_c(X, \Omega_X)) = \text{tr}(u^{(n)}|R\Gamma(X, \mathbb{Q}_\ell)),$$

and by [12, III, 5.1.6] we have

$$\mathrm{lt}_y(\Omega_X, {}^{(n)}v) = \mathrm{lt}_y(\mathbb{Q}_\ell, u^{(n)}).$$

It follows that for $n \geq n_0$

$$\mathrm{tr}(u^{(n)} | R\Gamma(X, \mathbb{Q}_\ell)) = \sum_{y \in \mathrm{Fix}(c^{(n)})} \mathrm{lt}_y(\mathbb{Q}_\ell, u^{(n)}).$$

Now by (iii), the local term $\mathrm{lt}_y(\mathbb{Q}_\ell, u^{(n)})$ can be computed after replacing X by an étale covering, which reduces the computation to the case when X is quasi-projective. Combining this with 8.16 we get the theorem (note that if $w(u)_i$ denotes the component of the weight vector corresponding to an irreducible component $C_i \subset C$ then $q^{\dim(C_i)} w(u)_i = w(u^{(n)})_i$). $\square$

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