COMPATIBLE SYSTEMS AND DUALITY

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1. Introduction

1.1. Our aim in this paper is to develop a general formalism for dealing with some questions on independence of ℓ for actions of correspondences on ℓ-adic cohomology.

To get a sense for the kind of question we are interested in, consider the following. Let k be the algebraic closure of a finite field of characteristic $p > 0$, and let X/k be a separated scheme of finite type. Let $u : X \to X$ be a finite morphism. Then one expects, and in fact it is a consequence of the results of this paper, that the trace

$$\mathrm{tr}(u|R\Gamma(X, \mathbb{Q}_\ell)) = \sum_{i\in\mathbb{Z}}(-1)^i \mathrm{tr}(u|H^i(X, \mathbb{Q}_\ell))$$

is in $\mathbb{Q}$, and is independent of ℓ, where u acts on the cohomology of X by functoriality.

The standard way to prove such independence of ℓ and rationality statements, is to use a trace formula to relate the trace to the calculation of certain local terms. For the cohomology $R\Gamma(X, \mathbb{Q}_\ell)$, however, there is no such trace formula without further assumptions (such as for X smooth).

1.2. For compactly supported cohomology, one does indeed have such a trace formula thanks to Fujiwara [4]. Since it plays a central role in this paper let us recall this result.

Let $\mathbb{F}_q$ be a finite field of characteristic $p > 0$, and let k be an algebraic closure of $\mathbb{F}_q$. Let $X_0/\mathbb{F}_q$ be a separated scheme of finite type, and let

$$c = (c_1, c_2) : C_0 \to X_0 \times X_0$$

be a morphism of $\mathbb{F}_q$-schemes, with C_0 also separated and of finite type. Assume that c_1 is proper and that c_2 is quasi-finite, and let $\ell \neq p$ be a prime. Let $K \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$ be a complex (where X denotes $X_0 \otimes_{\mathbb{F}_q} k$), and let

$$u : c_1^*K \to c_2^!K$$

be a morphism in $D^b_c(C, \overline{\mathbb{Q}}_\ell)$ (which by adjunction corresponds to a morphism $c_{2!}c_1^*K \to K$ in $D^b_c(X, \overline{\mathbb{Q}}_\ell)$). We then get an endomorphism

$$R\Gamma_c(u) : R\Gamma_c(X, K) \to R\Gamma_c(X, K),$$

defined as the composite

$$R\Gamma_c(X, K) \longrightarrow R\Gamma_c(X, c_{1*}c_1^*K) \xrightarrow{c_{1*}\simeq c_{1!}} R\Gamma(C, c_1^*K) \xrightarrow{\simeq} R\Gamma_c(X, c_{2!}c_1^*K) \xrightarrow{u} R\Gamma_c(X, K).$$

Suppose given further an isomorphism

$$\epsilon : F_X^*K \to K,$$

where $F_X : X \to X$ is the relative Frobenius of X/k (which equals the base change to k of the q-power Frobenius on X_0). For an integer $n \geq 0$ define

$$c^{(n)} : C \to X \times X$$

to be the composite map

$$C \xrightarrow{c} X \times X \xrightarrow{F_X^n \times \mathrm{id}} X \times X.$$

Let

$$u^{(n)} : (c^{(n)})_1^* K \to (c^{(n)})_2^! K$$

be the composite map

$$(c^{(n)})_1^* K \xrightarrow{\simeq} c_1^* F_X^{n*} K \xrightarrow{\epsilon^n} c_1^* K \xrightarrow{u} c_2^! K = (c^{(n)})_2^! K.$$

Also define $\mathrm{Fix}(c^{(n)})$ to be the fiber product of the diagram

$$\begin{tikzcd} & C \arrow[d, "c^{(n)}"] \\ X \arrow[r, "\Delta"] & X \times X. \end{tikzcd}$$

Fujiwara's theorem is then the following:

Theorem 1.3 (Fujiwara [4, 5.4.5])**.** *There exists an integer n_0 such that for every $n \geq n_0$ the set $\mathrm{Fix}(c^{(n)})(k)$ is finite and*

$$\mathrm{tr}(R\Gamma(u^{(n)})|R\Gamma_c(X, K)) = \sum_{z \in \mathrm{Fix}(c^{(n)})(k)} \mathrm{lt}_z(K, u^{(n)}),$$

where $\mathrm{lt}_z(K, u^{(n)})$ denotes the naive local term at z (see section 2 for precise definitions).

Remark 1.4. Fujiwara's theorem also holds for algebraic spaces, and Deligne-Mumford stacks. The case of algebraic spaces follows from Varshavsky's argument [15] together with the compactification theorem of Raoult [14]. The case of Deligne-Mumford stacks can be deduced from this by passing to the coarse moduli spaces (see [13]).

1.5. This theorem can be used to deduce independence of ℓ results for compactly supported cohomology. For example, let X/k be a separated finite type scheme, and let $\alpha : X \to X$ be a proper endomorphism. Then by standard limit arguments there exists a finite subfield $\mathbb{F}_q \subset k$ and a finite type separated $\mathbb{F}_q$-scheme X_0 with an endomorphism $\alpha_0 : X_0 \to X_0$ inducing (X, α) by base change to k. Let $C_0 = X_0$, and let

$$c : C_0 \to X_0 \times X_0$$

be the map given by $\alpha_0 \times \mathrm{id}$. Since c_2 is the identity, the canonical isomorphism

$$\alpha^* \mathbb{Q}_\ell \to \mathbb{Q}_\ell$$

defines a map

$$u : c_1^* \mathbb{Q}_\ell \to c_2^! \mathbb{Q}_\ell.$$

In this case the induced action $R\Gamma_c(u)$ on $R\Gamma_c(X, \mathbb{Q}_\ell)$ is the natural action induced by functoriality of compactly supported cohomology. Moreover, it follows from the definitions that all the local terms are 1. If

$$f : R\Gamma_c(X, \mathbb{Q}_\ell) \to R\Gamma_c(X, \mathbb{Q}_\ell)$$

denotes the Frobenius automorphism, then f commutes with $R\Gamma_c(u)$ and Fujiwara's theorem gives that there exists an integer n_0 such that for all $n \geq n_0$ we have

$$\mathrm{tr}(R\Gamma_c(u) \circ f^n | R\Gamma_c(X, \mathbb{Q}_\ell)) = |X^{\alpha^{(n)}}(k)|,$$

where $X^{\alpha^{(n)}}$ denotes the fixed points of $F_X^n \circ \alpha$. In particular, for $n \geq n_0$ the trace

$$\mathrm{tr}(R\Gamma_c(u) \circ f^n | R\Gamma_c(X, \mathbb{Q}_\ell))$$

is in $\mathbb{Q}$ (even $\mathbb{Z}$) and independent of ℓ. By some linear algebra (see section 9), this implies that in fact

$$\mathrm{tr}(R\Gamma_c(u) | R\Gamma_c(X, \mathbb{Q}_\ell))$$

is in $\mathbb{Q}$ and independent of ℓ.

1.6. Following a suggestion of Illusie [7, 3.8] we take Fujiwara's theorem as a starting point for developing a theory of 'compatible systems'. Let us describe such systems in the simplest case (in section 9 we consider a more general notion involving a field of definition $E/\mathbb{Q}$). Fix a finite field $\mathbb{F}_q$ of characteristic p and an algebraic closure k of of $\mathbb{F}_q$. Let $\{\ell_\alpha\}_{\alpha \in I}$ be a set of primes indexed by a set I, with each $\ell_\alpha \neq p$ (but we allow $\ell_\alpha = \ell_\beta$ for $\alpha \neq \beta$).

Fix a correspondence over $\mathbb{F}_q$

$$c : C_0 \to X_0 \times X_0,$$

with both c_1 and c_2 quasi-finite. A *compatible system of Weil complexes with c-structure on X* (or *I-compatible system* to make clear the reference to I) is a collection of data $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}_{\alpha \in I}$ as follows (see the text for precise definitions):

(1) $K_\alpha \in D_c^b(X, \overline{\mathbb{Q}}_{\ell_\alpha})$.
(2) $\epsilon_\alpha : F_X^* K_\alpha \to K_\alpha$ is an isomorphism.
(3) $u_\alpha : c_1^* K_\alpha \to c_2^! K_\alpha$ is a morphism which commutes with ϵ_α in a suitable sense (we refer to the triple $(K_\alpha, \epsilon_\alpha, u_\alpha)$ as a *Weil complex with c-structure*). For any $n \geq 0$ we then get a map

$$u_\alpha^{(n)} : c_1^{(n)*} K_\alpha \to c_2^! K_\alpha,$$

as discussed in 1.2.
(4) This data is further required that for every $\alpha \in I$, $n \geq 0$, and $z \in \mathrm{Fix}(c^{(n)})(k)$ the local term $\mathrm{lt}_z(K_\alpha, u_\alpha^{(n)})$ is in $\mathbb{Q}$, and for every $\alpha, \beta \in I$ and $n \geq 0$ we have for every $z \in \mathrm{Fix}(c^{(n)})(k)$ an equality of rational numbers

$$\mathrm{lt}_z(K_\alpha, u_\alpha^{(n)}) = \mathrm{lt}_z(K_\beta, u_\beta^{(n)}).$$

It is not hard to show (see section 12), using Fujiwara's theorem, that for suitable morphisms of correspondences

$$\begin{tikzcd} & C_0 \arrow[dl, "c_1"'] \arrow[ddd, "g"] \arrow[dr, "c_2"] & \\ X_0 \arrow[dddd, "f"] & & X_0 \arrow[dddd, "f"] \\ & & \\ & & \\ & D_0 \arrow[dl, "d_1"'] \arrow[dr, "d_2"] & \\ Y_0 & & Y_0, \end{tikzcd}$$

the pushforwards $f_! K_\alpha$ have a natural structure of Weil complexes with d-structure, and that the resulting collection of data $\{(f_! K_\alpha, f_! \epsilon_\alpha, f_! u_\alpha)\}$ is a compatible system.

To get a handle on the ordinary pushforwards $f_* K_\alpha$, we use Verdier duality to write this as

$$f_* K_\alpha = D_Y f_! D_X K_\alpha,$$

where $D_X : D^b_c(X, \overline{\mathbb{Q}}_{\ell_\alpha}) \to D^b_c(X, \overline{\mathbb{Q}}_{\ell_\alpha})$ is the Verdier duality functor. This is the motivation for the following result (stated here informally but see 9.6 for the precise statement), which is our main result on compatible systems:

Theorem 1.7. *Let $c^t : C_0 \to X_0 \times X_0$ be the transpose of c obtained by composing c with the automorphism of $X_0 \times X_0$ interchanging the factors. If $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}$ is a compatible system of Weil complexes with c-structure, then the Verdier duals DK_α are naturally part of a compatible system $\{(DK_\alpha, \epsilon^t_\alpha, u^t_\alpha)\}$ of Weil complexes with c^t-structure.*

With this theorem in hand, we show using the above strategy that applying the ordinary pushforward f_* to a compatible system yields a compatible system (see section 13).

1.8. In fact theorem 1.7 is a consequence of a more general result on (naive) local terms and duality. Let k be an algebraically closed field, and let

$$c = (c_1, c_2) : C \to X \times X$$

be a correspondence, where C and X are separated algebraic spaces of finite type over k, with the maps c_1 and c_2 quasi-finite. Let $K \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$ be a complex with a c-structure

$$u : c_1^* K \to c_2^! K.$$

Let

$$c^t : C \to X \times X$$

be the dual correspondence and let

$$u^t : (c^t)_1^* DK \to (c^t)_2^! DK$$

be the natural c^t-structure on the Verdier dual DK of K (see 1.15). Note that there is a natural isomorphism of fixed point spaces

$$\mathrm{Fix}(c) \simeq \mathrm{Fix}(c^t).$$

The main result of the paper is the following:

Theorem 1.9. *For any $y \in \mathrm{Fix}(c)(k)$ we have*

$$\mathrm{lt}_y(K, u) = \mathrm{lt}_y(DK, u^t),$$

where $\mathrm{lt}_y(-,-)$ *denotes the naive local term at y as defined in 2.1.*

Remark 1.10. The analogous result for the true local terms is shown in [6, III.5.1].

Remark 1.11. Theorem 1.9 can be used to prove a trace formula for actions of correspondences acting on cohomology $H^*(X, K)$ in the case when the correspondence c is *expanding* (which by definition means that c^t is contracting in the sense of [15, 2.1.1]). Precisely, suppose X is proper (but not necessarily smooth), and that c_2 is finite. In this case u induces an endomorphism $H(u)$ of the cohomology $H^*(X, K)$ given by the composite

$$R\Gamma(X, K) \xrightarrow{c_1^*} R\Gamma(C, c_1^*K) \xrightarrow{\simeq} R\Gamma(X, c_{2*}c_1^*K) \xrightarrow{c_{2*} \simeq c_{2!}} R\Gamma(X, c_{2!}c_1^*K) \xrightarrow{u} R\Gamma(X, K).$$

By [15, 2.1.3], the set $\mathrm{Fix}(c)(k)$ consists of a finite set of isolated points, and we have the formula

$$\mathrm{tr}(H(u)|H^*(X, K)) = \sum_{y \in \mathrm{Fix}(c)(k)} \mathrm{lt}_y(K, u). \tag{1.11.1}$$

This follows from the corresponding formula for compactly supported cohomology. Indeed we have

$$\mathrm{tr}(H(u)|H^*(X, K)) = \mathrm{tr}(R\Gamma_c(u)|H^*_c(X, DK)),$$

and by 1.9 we have

$$\sum_{y \in \mathrm{Fix}(c)(k)} \mathrm{lt}_y(K, u) = \sum_{y \in \mathrm{Fix}(c^t)(k)} \mathrm{lt}_y(DK, u^t).$$

The formula (1.11.1) is therefore equivalent to the formula

$$\mathrm{tr}(R\Gamma_c(u)|H^*_c(X, DK)) = \sum_{y \in \mathrm{Fix}(c^t)(k)} \mathrm{lt}_y(DK, u^t),$$

which follows from [15, 2.3.2]. More generally, in the case when X is not proper the formula (1.11.1) still holds with some additional conditions 'at infinity' as in [15, 2.3.2].

In the topological setting, trace formulas for expanding correspondences had previously been considered by Goresky and MacPherson [5].

Remark 1.12. Even for the identity map $X \to X$ (so C is the diagonal in $X \times X$) theorem 1.9 is not obvious (even for constant coefficients!). In this case the assertion amounts to the statement that for any point $x \in X(k)$ giving an inclusion $i : \mathrm{Spec}(k) \hookrightarrow X$ we have an equality of Euler characteristics

$$\chi(i^*K) = \chi(i^!K).$$

This follows from a theorem of Laumon [11].

The paper is organized into two parts.

The aim of part I is to prove 1.9. The proof involves is a rather eloborate devissage to the case of a lisse sheaf and induction on the dimension of X. To help the reader, we have isolated in separate sections a number of technical points that arise in the proof. However, the reader may wish to first consult section 7 to map out the broad outlines of the proof (this is the section where the proof of 1.9 is completed).

The sections of Part I are as follows.

In section 2 we give the basic definitions of (naive) local terms and establish some basic properties. Though we are primarily interested in $\overline{\mathbb{Q}}_\ell$-coefficients and algebraic spaces, the proof of 1.9 also requires some results with X and C Deligne-Mumford stacks as well as some results with torsion coefficients. For this reason, we develop several of the foundational results also for Deligne-Mumford stacks and for various coefficient rings.

In section 3 we do some prepatory work on quasi-finite morphisms and traces which is used in section 4 where we prove 1.9 directly in the case when K is a lisse sheaf (on a Deligne-Mumford stack). The key point in this case is a result of Laumon [11].

With the case of a lisse sheaf in hand, the strategy for proving 1.9 in general is a devissage. First of all using Noether normalization we reduce to the case when X is smooth (the key point here are the results in section 5). By a standard reduction we may further assume that K is a constructible sheaf of the form $R^0j_*(K|_U)$, where $j : U \hookrightarrow X$ is the inclusion of a dense open subset (not necessarily c-invariant) such that $K|_U$ is lisse. In this case we can consider the local monodromy group of $K|_U$ around points of X (this is the topic of section 6). The main observation is that under some further assumptions on K (which we can arrange to hold), the set of points where K has finite local monodromy is a c-invariant open subset of X. This implies that there exists a dense open c-invariant subset $V \subset X$ and a Deligne-Mumford stack $\mathscr{X} \to X$ with coarse moduli space X, such that $K|_V$ lifts to a lisse sheaf on $\mathscr{X} \times_X V$. This reduces us to showing that if $j : V \subset X$ is a dense open subset which is c-invariant and K_V is a lisse sheaf on V, then the local terms of Rj_*K_V at points of $X - V$ are zero, except that we now have to allow X to also be a Deligne-Mumford stack. This vanishing of the local terms of Rj_*K_V we show by first reducing to the analogous result for $\mathbb{Z}_\ell$-coefficients and then reducing modulo ℓ^n for variable n (this is why we need to also consider torsion coefficients). The proof of 1.9 is finished in section 7.

In part II we reap the benefits of the work in part I. We introduce the notion of compatible system, following Gabber and Illusie [7]. We show that the category of compatible systems is stable under the usual functors f^*, $f_!$, f_*, and $f^!$ as well as duality. This part is organized as follows. In section 8 we introduce some basic definitions and results about twisting correspondences by powers of Frobenius. In section 9 we explain the definition of compatible system, as well as some linear algebra from [7]. We also prove in this section the key result that the Verdier dual of a compatible system is again a compatible system (this follows almost immediately from 1.9).

In the following four sections we then discuss the stability of compatible systems under various operations. The easiest is pullback f^* which we discuss in section 10. Dualizing the results for f^* we explain in section 11 the stability of compatible systems under $f^!$. In section 12 we explain how Fujiwara's theorem can be used to prove that compatible systems are stable under the functor $f_!$, and the dualizing this result we obtain stability under f_* in section 13.

Remark 1.13. Though we use Deligne-Mumford stacks in the proof of 1.9, we only prove 1.9 for algebraic spaces. We expect that the same result holds also for Deligne-Mumford stacks, but the argument presented here seems to break down in this case (in particular, the reduction using Noether normalization).

Remark 1.14. In the case of a finite group action, several more refined results on the relationship between ordinary and compactly supported cohomology have been obtained by Illusie and Zheng [9, in particular 2.2].

1.15. **Terminology and notation.** Fix a field k of characteristic p and let ℓ be a prime not equal to p. Let Λ be a Gorenstein local ring of dimension 0 and finite residue field of characteristic ℓ (the primary example we will consider is $\Lambda = \mathbb{Z}/(\ell^n)$ for some $n \geq 1$). For a separated scheme X of finite type over k we write $D^b_{ctf}(X, \Lambda)$ for the derived category of bounded complexes of Λ-modules which are of finite tor-dimension and have constructible cohomology sheaves. We will also consider $\Lambda = \overline{\mathbb{Q}}_\ell$, $\mathbb{Z}_\ell$, or $\mathbb{Q}_\ell$, in which case $D^b_{ctf}(X, \Lambda) = D^b_c(X, \Lambda)$ denotes the usual derived category of constructible Λ-modules as defined in [2, 1.1.2].

If $f : X \to Y$ is a morphism of finite type and separated k-schemes, we write

$$f_!, f_* : D^b_{ctf}(X, \Lambda) \to D^b_{ctf}(X, \Lambda), \quad f^*, f^! : D^b_{ctf}(X, \Lambda) \to D^b_{ctf}(X, \Lambda)$$

for the resulting operations on the derived category as in [6]. If we wish to consider the non-derived operations we write $R^0 f_!$, $R^0 f_*$, etc.

We write $\Omega_X \in D^b_{ctf}(X, \Lambda)$ for the dualizing complex of X, and

$$D : D^b_{ctf}(X, \Lambda) \to D^b_{ctf}(X, \Lambda), \quad K \mapsto \mathscr{R}Hom(K, \Omega_X)$$

for the Verdier duality functor.

For a scheme X of finite type over a field k, a *correspondence on* X is a morphism of schemes

$$c : C \to X \times X,$$

with C also separated and of finite type over k. We write $c_s : C \to X$ ($s = 1, 2$) for the composition of c with the projection $\mathrm{pr}_s : X \times X \to X$.

For a correspondence c on X we write

$$c^t : C \to X \times X$$

for the composite

$$C \xrightarrow{c} X \times X \xrightarrow{(x,y) \mapsto (y,x)} X \times X.$$

The correspondence c^t is called the *dual* of c. If $K \in D^b_{ctf}(X, \Lambda)$ then a *c-structure on K* is a morphism

$$u : c_1^* K \to c_2^! K$$

in $D^b_{ctf}(C, \Lambda)$. Such a c-structure induces a c^t-structure u^t on the Verdier dual DK given by

$$u^t : (c^t)_1^* DK = c_2^* DK \simeq D c_2^! K \xrightarrow{Du} D c_1^* K \simeq c_1^! DK = (c^t)_2^! DK.$$

Here we use the canonical isomorphisms

$$c_2^! \simeq D \circ c_2^* \circ D, \quad c_1^* \simeq D \circ c_1^! \circ D.$$

For a correspondence c on X, we write $\mathrm{Fix}(c)$ for the fixed points of c, which by definition is the fiber product of the diagram

$$\begin{tikzcd} & C \arrow[d, "c"] \\ X \arrow[r, "\Delta"] & X \times X. \end{tikzcd}$$

We will also need to consider correspondences with X and C Deligne-Mumford stacks. The above definitions carry over to this more general context word-for-word.

1.16. **Acknowledgements.** This work was inspired by Illusie's article [7], which provided the main idea to use naive local terms in the definition of compatible systems. I would like to thank Luc Illusie and Weizhe Zheng for helpful conversations, as well as the Mathematical Sciences Research Institute in Berkeley where much of this work was done. The author was partially supported by NSF grant DMS-0714086, NSF CAREER grant DMS-0748718, and an Alfred P. Sloan Research Fellowship.

Part I: Local terms and duality

Throughout Part I of this paper, we work over an algebraically closed field k.

Let Λ be a coefficient ring as in 1.15 (so Λ is either $\mathbb{Q}_\ell$, $\overline{\mathbb{Q}}_\ell$, $\mathbb{Z}_\ell$, or a Gorenstein local ring of dimension 0 and finite residue field of characteristic ℓ invertible in k). To ease notation for a separated finite type k-scheme X, we often write simply $D^b_{ctf}(X)$ for $D^b_{ctf}(X, \Lambda)$ if no confusion is likely to arise.

For $K \in D^b_{ctf}(\mathrm{Spec}(k))$ the natural map

$$K \otimes RHom(K, \Lambda) \to RHom(K, K)$$

is an isomorphism because K has finite tor-dimension and hence can be represented by a bounded complex of free finite type Λ-modules. In particular we get a map

$$\mathrm{tr} : RHom(K, K) \to \Lambda,$$

called the *trace map*, defined as the composite

$$RHom(K, K) \xrightarrow{\simeq} K \otimes RHom(K, \Lambda) \xrightarrow{\mathrm{ev}} \Lambda,$$

where the second map is the evaluation map. In particular, for an endomorphism $u : K \to K$ we obtain its trace $\mathrm{tr}(u) \in \Lambda$.

2. Definition of local terms and basic properties

2.1. Let

$$c : C \to X \times X$$

be a correspondence, with C and X algebraic spaces separated and of finite type over k, and assume c_2 is quasi-finite. Let $K \in D^b_{ctf}(X)$ be a complex, and let

$$u : c_1^* K \to c_2^! K$$

be a c-structure on K. For any point $y \in \mathrm{Fix}(c)(k)$ with image $x \in X(k)$ (under either c_1 or c_2), we define the *local term*, denoted

$$\mathrm{lt}_y(K, u) \in \Lambda,$$

as the trace of the composite map

$$K_x \xrightarrow{\simeq} (c_1^* K)_y \xrightarrow{u} (c_2^! K)_y \hookrightarrow \oplus_{z \in c_2^{-1}(x)} (c_2^! K)_z \xrightarrow{\simeq} (c_{2!} c_2^! K)_x \xrightarrow{c_{2!} c_2^! \to \mathrm{id}} K_x, \tag{2.1.1}$$

where the isomorphism

$$\oplus_{z \in c_2^{-1}(x)} (c_2^! K)_z \simeq (c_{2!} c_2^! K)_x$$

is given by the proper base change theorem.

Remark 2.2. The above notion is usually called the *naive local term*. Since we do not consider the *true local term*, as defined in [6], in this paper, however, we omit the adjective 'naive'.

2.3. Let $i : Z \hookrightarrow X$ be a closed immersion, and let

$$c_Z : C_Z \to Z \times Z$$

be the correspondence on Z obtained by taking the fiber product of the diagram

$$\begin{tikzcd} & Z \times Z \arrow[d, "i\times i"] \\ C \arrow[r, "c"] & X \times X. \end{tikzcd}$$

Note that $c_{Z,2} : C_Z \to Z$ is again quasi-finite. Also note that if C_i denotes the fiber product $C \times_{c_i,X,i} Z$, then we have a cartesian diagram

$$\begin{tikzcd} C_Z \arrow[r, hook, "\alpha"] \arrow[d, hook, "\beta"'] & C_2 \arrow[d, hook, "\gamma"] \\ C_1 \arrow[r, hook, "\delta"] & C, \end{tikzcd} \tag{2.3.1}$$

where all the morphisms are closed immersions.

Now let $K \in D^b_{ctf}(Z)$ be a complex, and let $u : c^*_{Z,1}K \to c^!_{Z,2}K$ be a c_Z-structure on K. We can then define a c-structure

$$i_*u : c_1^*(i_*K) \to c_2^!(i_*K)$$

as follows.

Note that if

$$\epsilon : C_Z \hookrightarrow C$$

is the inclusion (a closed immersion), then there is a natural map of functors

$$c_1^*i_* \to \epsilon_*c^*_{Z1} \tag{2.3.2}$$

obtained by adjunction from the isomorphism

$$\epsilon^*c_1^*i_* \simeq c^*_{Z1}i^*i_* \simeq c^*_{Z1}.$$

We therefore get a morphism of functors

$$c_{2!}c_1^*i_* \longrightarrow c_{2!}\epsilon_*c^*_{Z1} \xrightarrow{\epsilon_*\simeq\epsilon_!} (c_2\epsilon)_!c^*_{Z1} \xrightarrow{\simeq} i_*c_{Z2!}c^*_{Z1}. \tag{2.3.3}$$

We define

$$i_*u : c_{2!}c_1^*i_*K \to i_*K$$

to be the composite of this map $c_{2!}c_1^*i_*K \to i_*c_{Z2!}c^*_{Z1}K$ with

$$u : i_*c_{Z2!}c^*_{Z1}K \to i_*K.$$

Observe that there is a natural cartesian diagram

$$\begin{tikzcd} \mathrm{Fix}(c_Z) \arrow[r, hook] \arrow[d] & \mathrm{Fix}(c) \arrow[d] \\ Z \arrow[r, hook, "i"] & X. \end{tikzcd} \tag{2.3.4}$$

Proposition 2.4. *Let $y \in \mathrm{Fix}(c)(k)$ be a fixed point. Then*

$$\mathrm{lt}_y(i_*K, i_*u) = \mathrm{lt}_y(K, u)$$

if $y \in \mathrm{Fix}(c_Z)(k)$, and

$$\mathrm{lt}_y(i_*K, i_*u) = 0$$

otherwise.

Proof. Let $x \in X(k)$ be the image of y.

Since the square (2.3.4) is cartesian, it is immediate that $\mathrm{lt}_y(i_*K, i_*u) = 0$ for $y \notin \mathrm{Fix}(c_Z)$, for then $(i_*K)_x = 0$.

So assume $y \in \mathrm{Fix}(c_Z)$. Taking the stalk at x we obtain from (2.3.3) a map

$$\Psi : \oplus_{w \in c_2^{-1}(x)} (c_1^* i_* K)_w \to \oplus_{t \in c_{Z2}^{-1}(x)} K_t$$

defined by the commutativity of the diagram

$$\begin{tikzcd}
(c_{2!}c_1^* i_* K)_x \arrow[r] \arrow[d, "\simeq"] & (c_{2!}\epsilon_* c_{Z1}^* K)_x \arrow[r, "\epsilon_* \simeq \epsilon_!"] & ((c_2\epsilon)_! c_{Z1}^* K)_x \arrow[r, "\simeq"] & (i_* c_{Z2!} c_{Z1}^* K)_x \arrow[d, "\simeq"] \\
\oplus_{w \in c_2^{-1}(x)} (c_1^* i_* K)_w \arrow[rrr, "\Psi"] & & & \oplus_{t \in c_{Z2}^{-1}(x)} K_t.
\end{tikzcd}$$

It follows from the functoriality of the base change isomorphism, that the map Ψ is the projection induced by the natural inclusion

$$c_{Z2}^{-1}(x) \subset c_2^{-1}(x).$$

Since the local term $\mathrm{lt}_y(i_*K, i_*u)$ is by definition the trace of the composite

$$K_x \overset{y}{\hookrightarrow} \oplus_{w \in c_2^{-1}(x)} (c_1^* i_* K)_w \xrightarrow{\Psi} \oplus_{t \in c_{Z2}^{-1}(x)} K_t \xrightarrow{u} K_x$$

this implies the equality

$$\mathrm{lt}_y(i_*K, i_*u) = \mathrm{lt}_y(K, u).$$

□

2.5. Note that any c-structure

$$v : c_{2!}c_1^*(i_*K) \to i_*K$$

is of the form i_*u for some c_Z-structure u on K.

To see this, let $\chi_s : C_s \to Z$ $(s = 1, 2)$ be the map such that $i \circ \chi_1 = c_1 \circ \delta$ (resp. $i \circ \chi_2 = c_2 \circ \gamma$). By the base change theorem we have

$$c_1^* i_* K \simeq \delta_* \chi_1^* K, \quad c_2^! i_* K \simeq \gamma_* \chi_2^! K.$$

Also since (2.3.1) is cartesian we have

$$\gamma^* \delta_* \simeq \alpha_* \beta^*.$$

Therefore

$$\begin{aligned}
\mathrm{Hom}(c_1^*i_*K, c_2^!i_*K) &\simeq \mathrm{Hom}(\delta_*\chi_1^*K, \gamma_*\chi_2^!K)\\
&\simeq \mathrm{Hom}(\gamma^*\delta_*\chi_1^*K, \chi_2^!K)\\
&\simeq \mathrm{Hom}(\alpha_*\beta^*\chi_1^*K, \chi_2^!K)\\
&\simeq \mathrm{Hom}(\beta^*\chi_1^*K, \alpha^!\chi_2^!K)\\
&\simeq \mathrm{Hom}(c_{Z,1}^*K, c_{Z,2}^!K).
\end{aligned}$$

Proposition 2.6. *Under this isomorphism*

$$\mathrm{Hom}(c_{Z,1}^*K, c_{Z,2}^!K) \simeq \mathrm{Hom}(c_1^*i_*K, c_2^!i_*K) \tag{2.6.1}$$

a map $u : c_{Z1}^*K \to c_{Z2}^!K$ is sent to i_*u.

Proof. Note first of all that for any $G \in D^b_{ctf}(C)$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}(G, c_2^!i_*K) \arrow[r, "c_2^!i_*\simeq\gamma_*\chi_2^!"] \arrow[d, "\text{adj.}"] & \mathrm{Hom}(G, \gamma_*\chi_2^!K) \arrow[d, "\text{adj.}"] \\
\mathrm{Hom}(c_{2!}G, i_*K) \arrow[d, "\text{adj.}"] & \mathrm{Hom}(\gamma^*G, \chi_2^!K) \arrow[d, "\text{adj.}"] \\
\mathrm{Hom}(i^*c_{2!}G, K) \arrow[r, "i^*c_{2!}\simeq\chi_{2!}\gamma^*"] & \mathrm{Hom}(\chi_{2!}\gamma^*G, K)
\end{tikzcd}$$

commutes, as explained in [1, XVIII, 3.1.11] (in particular the commutativity of the last diagram in this paragraph). It follows that the isomorphism

$$\mathrm{Hom}(i^*c_{2!}c_1^*i_*K, K) \simeq \mathrm{Hom}(c_{Z,2!}c_{Z,1}^*K, K)$$

obtained from (2.6.1) by adjunction, is induced by the map

$$i^*c_{2!}c_1^*i_*K \to c_{Z,2!}c_{Z,1}^*K$$

obtained by going around the top and right sides of the following diagram:

$$\begin{tikzcd}
i^*c_{2!}c_1^*i_*K \arrow[rr, "i^*c_{2!}\simeq\chi_{2!}\gamma^*"] \arrow[dr, "c_1^*i_*\simeq\delta_*\chi_1^*"] \arrow[ddd, "(2.3.2)"] & & \chi_{2!}\gamma^*c_1^*i_*K \arrow[rr, "c_1^*i_*\simeq\delta_*\chi_1^*"] & & \chi_{2!}\gamma^*\delta_*\chi_1^*K \arrow[dd, "\gamma^*\delta_*\simeq\alpha_*\beta^*"] \arrow[ddll, "\mathrm{id}\to\beta_*\beta^*"'] \\
& i^*c_{2!}\delta_*\chi_1^*K \arrow[urrr, "i^*c_{2!}\simeq\chi_{2!}\gamma^*"] \arrow[ddl, "\mathrm{id}\to\beta_*\beta^*"'] & & & \\
& & \chi_{2!}\gamma^*\delta_*\beta_*\beta^*\chi_1^*K \arrow[rr, "\gamma^*\delta_*\beta_*\simeq\alpha_*"] & & \chi_{2!}\alpha_*\beta^*\chi_1^*K \arrow[d, "\simeq"] \\
i^*c_{2!}\delta_*\beta_*\beta^*\chi_1^*K \arrow[urr, "i^*c_{2!}\simeq\chi_{2!}\gamma^*"'] \arrow[rrrr, "\simeq"] & & & & c_{Z2!}c_{Z1}^*K.
\end{tikzcd} \tag{2.6.2}$$

On the other hand, by definition for $u : c_{Z2!}c_{Z1}^*K \to K$ the map i_*u is obtained by composing u with the bottom and left side of this diagram. To prove the lemma it therefore suffices to show that (2.6.2) commutes.

For this note that all the small inside diagrams clearly commute, except possibly the bottom right diagram. The commutativity of this last square amounts to the commutativity of the following diagram of functors

$$\begin{tikzcd} i^*c_{2!}\delta_*\beta_* \arrow[r, "i^*c_{2!}\simeq\chi_{2!}\gamma^*"] \arrow[d, "\simeq"] & \chi_{2!}\gamma^*\delta_*\beta_* \arrow[d, "\simeq"] \\ i^*i_*c_{Z2!} \arrow[r, "\simeq"] & \chi_{2!}\alpha_*, \end{tikzcd}$$

which follows from [1, XVII, 5.2.5] applied to the diagram

$$\begin{tikzcd} C_1 \arrow[d, "\delta"] & C_Z \arrow[l, "\beta"'] \arrow[d, "\alpha"] \\ C \arrow[d, "c_2"] & C_2 \arrow[l, "\gamma"'] \arrow[d, "\chi_2"] \\ X & Z. \arrow[l, "i"'] \end{tikzcd}$$

□

2.7. Consider a commutative diagram of algebraic spaces of finite type over k

$$\begin{tikzcd} & \widetilde{C} \arrow[ddl, "d_1"'] \arrow[d, "q"] \arrow[ddr, "d_2"] & \\ & C \arrow[dl, "c_1"] \arrow[dr, "c_2"'] & \\ X & & X, \end{tikzcd}$$

where q is proper, and c_2 and d_2 are quasi-finite. Let $K \in D^b_{ctf}(X)$ be a complex and let

$$u : d_{2!}d_1^*K \to K$$

be a d-structure. We then obtain a c-structure

$$q_*u : c_{2!}c_1^*K \to K$$

on K from the composite

$$\begin{array}{rcll} c_{2!}c_1^*K & \to & c_{2!}q_*q^*c_1^*K & (\mathrm{id} \to q_*q^*) \\ & \simeq & c_{2!}q_!q^*c_1^*K & (q_! \simeq q_*) \\ & \simeq & d_{2!}d_1^*K & \\ & \xrightarrow{u} & K. & \end{array}$$

Let

$$\gamma : \mathrm{Fix}(d) \to \mathrm{Fix}(c)$$

be the natural map. Note that since d_2 is quasi-finite, this map γ is also quasi-finite.

Proposition 2.8. *For any $y \in \mathrm{Fix}(c)(k)$ we have*

$$\mathrm{lt}_y(K, q_*u) = \sum_{z\in\gamma^{-1}(y)} \mathrm{lt}_z(K, u).$$

Proof. Let $x \in X(k)$ be the image of y. The result then follows from noting that the diagram

$$\begin{tikzcd}
K_x \arrow[r, hook, "y"] & \oplus_{z \in c_2^{-1}(x)}(c_1^*K)_z \arrow[r, "q^*"] \arrow[d, "\simeq"] & \oplus_{w \in d_2^{-1}(x)}(d_1^*K)_w \arrow[d, "\simeq"] \\
 & (c_{2!}c_1^*K)_x \arrow[r, "q^*"] \arrow[dr, "q_*u"] & (d_{2!}d_1^*K)_x \arrow[d, "u"] \\
 & & K_x
\end{tikzcd}$$

commutes. ☐

Proposition 2.9. *The two c^t-structures $q_*(u^t)$ and $(q_*u)^t$ on DK are equal.*

Proof. The map

$$q_*u : c_1^*K \to c_2^!K$$

is equal to the composite map

$$c_1^*K \xrightarrow{\mathrm{id}\to q_*q^*} q_*q^*c_1^*K \xrightarrow{u} q_*d_2^!K \xrightarrow{\simeq} q_*q^!c_2^!K \xrightarrow{q_*\simeq q_!} q_!q^!c_2^!K \xrightarrow{q_!q^!\to \mathrm{id}} c_2^!K.$$

Therefore, the map $(q_*u)^t$ is given by going around the top and right of the following diagram

$$\begin{tikzcd}
c_2^*DK \arrow[r, "\simeq"] \arrow[d, "\mathrm{id}\to q_*q^*"] & Dc_2^!K \arrow[r, "q_!q^!\to\mathrm{id}"] & Dq_!q^!c_2^!K \arrow[r, "\simeq"] & Dq_*d_2^!K \arrow[r, "u"] & Dq_*d_1^*K \arrow[d, "\simeq"] \\
q_*q^*c_2^*DK \arrow[urr, "\simeq"] \arrow[d, "\simeq"] & & & & q_!d_1^!DK \arrow[d, "\simeq"] \\
q_*d_2^*DK \arrow[d, "u^t"] & & & & q_!q^!c_1^!DK \arrow[d, "q_!q^!\to\mathrm{id}"] \\
q_*d_1^!DK \arrow[uurrrr, "q_!\simeq q_*"] \arrow[urrrr, "\simeq"] & & & & c_1^!DK.
\end{tikzcd}$$

On the other hand, the map $q_*(u^t)$ is equal to the map obtained by going around the left and bottom sides of this diagram. The lemma therefore follows from noting that this diagram commutes, as the small inside diagrams clearly commute. ☐

2.10. Though we are primarily interested in the case of algebraic spaces, we need to also consider Deligne-Mumford stacks for the proofs. We now explain how to define local terms in this setting.

Let

$$c : \mathscr{C} \to \mathscr{X} \times \mathscr{X}$$

be a correspondence with $\mathscr{X}$ and $\mathscr{C}$ separated Deligne-Mumford stacks of finite type over k, and as in the case of spaces assume that c_2 is quasi-finite. We can then consider the *fixed* *point stack*, denoted $\mathrm{Fix}(c)$, which is defined to be the fiber product of the diagram

$$\begin{tikzcd} & \mathscr{C} \arrow[d, "c"] \\ \mathscr{X} \arrow[r, "\Delta"] & \mathscr{X} \times \mathscr{X}. \end{tikzcd}$$

Note that $\mathrm{Fix}(c)$ is again a Deligne-Mumford stack. The category $\mathrm{Fix}(c)(k)$ is the category of pairs

$$\lambda = (y \in \mathscr{C}(k), \sigma : c_1(y) \to c_2(y)),$$

where $y \in \mathscr{C}(k)$ is an object, and σ is an isomorphism in $\mathscr{X}(k)$.

As before let $K \in D^b_{ctf}(\mathscr{X})$ be a complex of Λ-modules and let

$$u : c_1^* K \to c_2^! K$$

be a c-structure. For $\lambda = (y, \sigma) \in \mathrm{Fix}(c)(k)$ we will consider two different (though closely related) notions of local terms

$$\widetilde{\mathrm{lt}}_\lambda(K, u) \quad \text{and} \quad \mathrm{lt}_\lambda(K, u)$$

defined as follows. Let $x \in \mathscr{X}(k)$ denote $c_1(y)$. As before we have a map

$$K_x \xrightarrow{\simeq} (c_1^* K)_y \xrightarrow{u} (c_2^! K)_y, \tag{2.10.1}$$

so we need to define a map

$$(c_2^! K)_y \to K_x.$$

For this note that again by the proper base change theorem we have

$$(c_{2!} c_2^! K)_x \simeq R\Gamma_c(\mathscr{C}_x^{(2)}, c_2^! K),$$

where

$$\mathscr{C}_x^{(2)} := \mathscr{C} \times_{c_2, \mathscr{X}, x} \mathrm{Spec}(k).$$

Since c_2 is quasi-finite, the stack $\mathscr{C}^{(2)}_{x,\mathrm{red}}$ is isomorphic to a finite disjoint union of stacks of the form BH, where H is a finite group. In particular, the point y defines an open and closed immersion

$$B\mathrm{Aut}(y/x) \hookrightarrow \mathscr{C}^{(2)}_{x,\mathrm{red}},$$

where

$$\mathrm{Aut}(y/x) := \mathrm{Ker}(\mathrm{Aut}(y) \overset{c_{2*}}{\to} \mathrm{Aut}(x)),$$

where we have used σ to identify $c_2(y)$ with x.

Let

$$\delta : \mathrm{Spec}(k) \to B\mathrm{Aut}(y/x)$$

be the natural projection. Then δ is finite and étale, so we have an adjunction map

$$\delta_! \delta^* \xrightarrow{\delta^* \simeq \delta^!} \delta_! \delta^! \longrightarrow \mathrm{id}.$$

This map defines a morphism

$$(c_2^! K)_y \simeq R\Gamma_c(\mathrm{Spec}(k), \delta^*(c_2^! K)|_{B\mathrm{Aut}(y/x)}) \to R\Gamma_c(\mathscr{C}_x^{(2)}, c_2^! K) \simeq (c_{2!} c_2^! K)_x,$$

and hence, by composing (2.10.1) with the composite

$$(c_2^! K)_y \to (c_{2!} c_2^! K)_x \to K_x,$$

we get the desired endomorphism

$$K_x \to K_x.$$

We define $\widetilde{\mathrm{lt}}_\lambda(K,u) \in \Lambda$ to be the trace of this map.

It is convenient to also consider the group $\mathrm{Aut}(\lambda)$, which is the set of automorphisms $\alpha : y \to y$ in $\mathscr{C}(k)$ for which the diagram in $\mathscr{X}(k)$

$$\begin{tikzcd} c_1(y) \arrow[r, "c_1(\alpha)"] \arrow[d, "\sigma"] & c_1(y) \arrow[d, "\sigma"] \\ c_2(y) \arrow[r, "c_2(\alpha)"] & c_2(y) \end{tikzcd}$$

commutes. In the case of $\Lambda = \overline{\mathbb{Q}}_\ell$, we then define

$$\mathrm{lt}_\lambda(K,u) := \frac{|\mathrm{Aut}(y)(k)|}{|\mathrm{Aut}(c_1(y))(k)| \cdot |\mathrm{Aut}(\lambda)|}\widetilde{\mathrm{lt}}_\lambda(K,u).$$

Remark 2.11. Note that the preceding paragraph shows that for any quasi-finite morphism $f : \mathscr{X} \to \mathscr{Y}$ of Deligne-Mumford stacks over k, $x \in \mathscr{X}(k)$, and $K \in D^b_{ctf}(\mathscr{Y})$, there is a canonical map

$$\tau_{K,x} : (f^!K)_x \to K_{f(x)}.$$

Example 2.12. To get a sense for the above definition of local terms for Deligne-Mumford stacks (and also for later use), consider the following special case. Let G and H be two finite groups, and let

$$\rho_1, \rho_2 : H \to G$$

be two homomorphisms. This gives rise to a correspondence

$$\begin{tikzcd} & BH \arrow[dl, "c_1"'] \arrow[dr, "c_2"] & \\ BG & & BG. \end{tikzcd}$$

Let V be a finite-dimensional representation of G over $\overline{\mathbb{Q}}_\ell$, and let ρ_i^*V denote the pullback of V to an H-representation via ρ_i (so ρ_i^*V has the same underlying vector space as V with H-action through ρ_i). A c-structure on the sheaf on BG associated to V is then equivalent to a morphism of H-representations

$$u : \rho_1^*V \to \rho_2^*V.$$

Let V_G denote the coinvariants of V. We then get an endomorphism

$$\bar{u} : V_G \to V_G$$

from the composite

$$V_G \xrightarrow{\simeq} V^G \hookrightarrow \rho_1^*V \xrightarrow{u} \rho_2^*V \twoheadrightarrow (\rho_2^*V)_H \twoheadrightarrow V_G.$$

For $g \in G$ define

$$H_g := \{h \in H | \rho_1(h) g \rho_2(h)^{-1} = g\},$$

and set

$$L_g := \frac{1}{|H_g|}\mathrm{tr}(V \xrightarrow{g} V \xrightarrow{u} V).$$

Define an equivalence relation $\sim$ on G by declaring that $g \sim g'$ if there exists $h \in H$ such that $g' = \rho_1(h)g\rho_2(h)^{-1}$. Then L_g depends only on the equivalence class of g. Indeed if $g' = \rho_1(h)g\rho_2(h)^{-1}$, then we have a commutative diagram

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d, "\rho_2(h)"] & V \arrow[r, "u"] \arrow[d, "\rho_1(h)"] & V \arrow[d, "\rho_2(h)"] \\ V \arrow[r, "g'"] & V \arrow[r, "u"] & V, \end{tikzcd}$$

where the right square commutes since u is a map of H-representations. Therefore

$$\mathrm{tr}(V \xrightarrow{g} V \xrightarrow{u} V) = \mathrm{tr}(V \xrightarrow{g'} V \xrightarrow{u} V).$$

Also we have a bijection

$$H_g \to H_{g'}, \quad \tilde{h} \mapsto h\tilde{h}h^{-1}.$$

We can therefore unambiguously write $L_{[g]}$ for a class $[g] \in G/\sim$.

Let

$$T : V \to V$$

denote the map

$$v \mapsto \frac{1}{|G|} \sum_{g \in G} g \cdot v.$$

Then T factors as

$$V \twoheadrightarrow V_G \xrightarrow{\simeq} V^G \hookrightarrow V.$$

It follows that we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}(T) \arrow[r] \arrow[d, "0"] & V \arrow[r] \arrow[d, "u \circ T"] & V_G \arrow[r] \arrow[d, "\bar{u}"] & 0 \\ 0 \arrow[r] & \mathrm{Ker}(T) \arrow[r] & V \arrow[r] & V_G \arrow[r] & 0, \end{tikzcd}$$

and therefore

$$\mathrm{tr}(\bar{u}|V_G) = \frac{1}{|G|} \sum_{g \in G} \mathrm{tr}(V \xrightarrow{g} V \xrightarrow{u} V).$$

On the other hand, by the standard orbit decomposition formula for the action of H on G given by

$$h * g := \rho_1(h)g\rho_2(h)^{-1},$$

we have

$$|H| = |H_g| \cdot |[g]|.$$

We conclude that

$$\mathrm{tr}(\bar{u}|V_G) = \frac{|H|}{|G|} \sum_{[g] \in G/\sim} L_{[g]}.$$

In terms of the correspondence c, this is equivalent to the formula

$$\mathrm{tr}(u|R\Gamma_c(BG, \mathcal{V})) = \sum_{[\lambda] \in |\mathrm{Fix}(c)(k)|} \mathrm{lt}_\lambda(\mathcal{V}, u),$$

where $\mathcal{V}$ denotes the sheaf on BG corresponding to V, and the sum on the right is over isomorphism classes of objects in $\mathrm{Fix}(c)(k)$. A more general version of this formula will be shown in 5.3.

2.13. Let $\rho : \Lambda \rightarrow \Lambda'$ be a morphism of coefficient rings (the ones we will consider are $\mathbb{Z}_\ell \hookrightarrow \overline{\mathbb{Q}}_\ell$ or the reduction maps $\mathbb{Z}_\ell \rightarrow \mathbb{Z}/(l^n)$, $n \geq 1$).

Let

$$c : \mathscr{C} \rightarrow \mathscr{X} \times \mathscr{X}$$

be a correspondence with $\mathscr{C}$ and $\mathscr{X}$ Deligne-Mumford stacks, and the maps c_1 and c_2 quasi-finite. Let $K \in D^b_{ctf}(X, \Lambda)$ be a complex with a c-structure

$$u : c_{2!}c_1^*K \rightarrow K.$$

We have a natural isomorphism [1, XVII.5.2.9]

$$c_{2!}c_1^*(K \otimes^{\mathbb{L}}_\Lambda \Lambda') \simeq (c_{2!}c_1^*K) \otimes^{\mathbb{L}}_\Lambda \Lambda',$$

so u induces a c-structure

$$u' : c_{2!}c_1^*K' \rightarrow K'$$

on $K' := K \otimes^{\mathbb{L}}_\Lambda \Lambda'$.

Proposition 2.14. *For any fixed point $\lambda \in \mathrm{Fix}(c)(k)$ we have an equality in Λ'*

$$\rho(\widetilde{\mathrm{lt}}_\lambda(K, u)) = \widetilde{\mathrm{lt}}_\lambda(K', u').$$

Proof. This is immediate from the definitions. □

2.15. In general, it is difficult to find invariant subsets for correspondences. However, one can often extend actions of correspondences as follows.

Let

$$\mathscr{C} \rightarrow \mathscr{X} \times \mathscr{X}$$

be a correspondence with $\mathscr{X}$ and $\mathscr{C}$ Deligne-Mumford stacks of finite type over k and the maps c_1 and c_2 quasi-finite. Assume that $\mathscr{X}$ is normal and irreducible. Let

$$j : \mathscr{U} \hookrightarrow \mathscr{X}$$

be a dense open immersion, and let

$$c_{\mathscr{U}} : \mathscr{C}_{\mathscr{U}} \rightarrow \mathscr{U} \times \mathscr{U}$$

be the correspondence obtained by taking fiber product of the diagram

$$\begin{tikzcd} & \mathscr{U} \times \mathscr{U} \arrow[d, "j\times j"] \\ \mathscr{C} \arrow[r, "c"] & \mathscr{X} \times \mathscr{X}. \end{tikzcd}$$

Let K be a lisse sheaf of Λ-modules on $\mathscr{U}$, and let

$$v : c_{\mathscr{U},2!}c_{\mathscr{U},1}^*K \rightarrow K$$

be a $c_{\mathscr{U}}$-structure on K.

Proposition 2.16. *The $c_{\mathscr{U}}$-structure v extends uniquely to a c-structure $\bar{v}$ on R^0j_*K.*

Proof. Note first of all that if $\rho : \mathscr{V} \hookrightarrow \mathscr{U}$ is an even smaller dense open substack, then the natural map

$$K \to R^0\rho_* K|_{\mathscr{V}}$$

is an isomorphism, since $\mathscr{X}$ is normal and K is a lisse sheaf on $\mathscr{U}$. To give a morphism

$$c_{2!}c_1^* R^0 j_* K \to R^0 j_* K,$$

is therefore equivalent to specifying a morphism

$$(c_{2!}c_1^* R^0 j_* K)|_{\mathscr{V}} \to K|_{\mathscr{V}} \tag{2.16.1}$$

for any dense open $\mathscr{V} \subset \mathscr{U}$.

Let δ denote the dimension of $\mathscr{X}$. Since c_1 is quasi-finite, for any irreducible component $\mathscr{Z} \subset \mathscr{C}$, either the map $c_1|_{\mathscr{Z}} : \mathscr{Z} \to \mathscr{X}$ is dominant, or the dimension of $\mathscr{Z}$ is strictly less than δ. Combining this with the fact that c_2 is quasi-finite we see that the dimension of

$$\overline{c_2(\mathscr{C} - c_1^{-1}(\mathscr{U}))} \subset \mathscr{X}$$

is a proper closed subset. Let $\mathscr{V}$ be the intersection of its complement with $\mathscr{U}$. Then $c_2^{-1}(\mathscr{V}) \subset c_1^{-1}(\mathscr{U})$, so

$$(c_{2!}c_1^* R^0 j_* K)|_{\mathscr{V}} = c_{\mathscr{U},2!}c_{1,\mathscr{U}}^* K|_{\mathscr{V}}.$$

We therefore get the desired map (2.16.1) from $c_{\mathscr{U}}$. □

Example 2.17. Let

$$c : \mathscr{C} \to \mathscr{X} \times \mathscr{X} \tag{2.17.1}$$

be a correspondence with $\mathscr{X}$ and $\mathscr{C}$ Deligne-Mumford stacks of finite type over k, and the maps c_1 and c_2 quasi-finite. Assume further that $\mathscr{X}$ is normal. In this case, we have a canonical c-structure on the constant sheaf Λ (where Λ is a coefficient ring as before).

To see this let us first make some general observations about quasi-finite morphisms. Let $f : \mathscr{X} \to \mathscr{Y}$ be a quasi-finite morphism between Deligne-Mumford stacks of finite type over k, and assume $\mathscr{Y}$ is normal. In this case there is a canonical map

$$f_! \Lambda \to \Lambda \tag{2.17.2}$$

defined as follows. By [13, 4.4 and 4.6], we have $f_!\Lambda \in D_c^{\leq 0}(\mathscr{Y}, \Lambda)$, so it suffices to construct a map

$$R^0 f_! \Lambda \to \Lambda. \tag{2.17.3}$$

We may further assume that $\mathscr{X}$ is reduced, in which case there exists a dense open substack $j : \mathscr{U} \hookrightarrow \mathscr{Y}$ such that $\mathscr{U}$ and $f^{-1}(\mathscr{U})$ are both smooth over k, and either $f^{-1}(\mathscr{U}) = \emptyset$ or $\mathscr{U}$ and $f^{-1}(\mathscr{U})$ are equidimensional of the same dimension. If $f^{-1}(\mathscr{U})$ is empty we define $f_!\Lambda \to \Lambda$ to be the zero map. In the second case, note that since $\mathscr{Y}$ is normal we have

$$\Lambda_{\mathscr{Y}} = R^0 j_* \Lambda_{\mathscr{U}},$$

so we have a canonical isomorphism

$$\mathrm{Hom}(R^0 f_! \Lambda, \Lambda_{\mathscr{Y}}) \simeq \mathrm{Hom}((R^0 f_! \Lambda)|_{\mathscr{U}}, \Lambda_{\mathscr{U}}).$$

It therefore suffices to construct the map (2.17.3) under the further assumption that both $\mathscr{X}$ and $\mathscr{Y}$ are smooth and of the same dimension δ. In this case we have

$$\Omega_{\mathscr{X}} \simeq \Lambda(\delta)[2\delta], \quad \Omega_{\mathscr{C}} \simeq \Lambda(\delta)[2\delta],$$

so

$$f_!\Lambda \simeq \mathscr{R}Hom(f_*\Lambda, \Lambda)$$

and

$$\Lambda \simeq \mathscr{R}Hom(\Lambda, \Lambda),$$

and we define (2.17.3) to be the map obtained by applying $\mathscr{R}Hom(-, \Lambda)$ to the canonical map

$$\Lambda \to f_*\Lambda.$$

Applying this discussion to the map $c_2 : \mathscr{C} \to \mathscr{X}$ in (2.17.1), we obtain a c-structure on the constant sheaf Λ on $\mathscr{X}$.

More generally, suppose $j : \mathscr{U} \hookrightarrow \mathscr{X}$ is an open substack such that

$$c_2^{-1}(\mathscr{U}) \subset c_1^{-1}(\mathscr{U}).$$

Let

$$c_{\mathscr{U}} : \mathscr{C}_{\mathscr{U}} \to \mathscr{U} \times \mathscr{U}$$

denote the correspondence obtained by restricting to $\mathscr{U}$ (so $\mathscr{C}_U = c_2^{-1}(\mathscr{U})$). Then the complex $Rj_*\Lambda$ has a natural c-structure given by observing that

$$\mathrm{Hom}(c_{2!}c_1^*Rj_*\Lambda, Rj_*\Lambda) \simeq \mathrm{Hom}((c_{2!}c_1^*Rj_*\Lambda)|_{\mathscr{U}}, \Lambda_{\mathscr{U}}) = \mathrm{Hom}(c_{\mathscr{U}2!}c_{\mathscr{U}1}^*\Lambda_{\mathscr{U}}, \Lambda_{\mathscr{U}}),$$

and noting that $\Lambda_{\mathscr{U}}$ has a natural $c_{\mathscr{U}}$-structure by the preceding discussion.

2.18. Let

$$c : \mathscr{C} \to \mathscr{X} \times \mathscr{X}$$

be a correspondence, with $\mathscr{X}$ and $\mathscr{C}$ Deligne-Mumford stacks of finite type over k, and the maps c_1 and c_2 quasi-finite. Let $K, L \in D^b_{ctf}(\mathscr{X}, \Lambda)$ be two complexes and suppose given c-structures

$$u : c_{2!}c_1^*K \to K, \quad v : c_{2!}c_1^*L \to L.$$

Recall that there is a natural map (see for example [1, XVII, §5.4])

$$(c_{2!}c_1^*K) \otimes^{\mathbb{L}} (c_{2!}c_1^*L) \to c_{2!}((c_1^*K) \otimes^{\mathbb{L}} (c_1^*L)). \tag{2.18.1}$$

Suppose given a c-structure

$$w : c_{2!}c_1^*(K \otimes^{\mathbb{L}} L) \to K \otimes^{\mathbb{L}} L$$

such that the diagram

$$\begin{tikzcd} c_{2!}c_1^*(K \otimes^{\mathbb{L}} L) \arrow[r, "w"] & K \otimes^{\mathbb{L}} L \\ c_{2!}((c_1^*K) \otimes^{\mathbb{L}} (c_1^*L)) \arrow[u, "\simeq"] & (c_{2!}c_1^*K) \otimes^{\mathbb{L}} (c_{2!}c_1^*L) \arrow[l, "(2.18.1)"'] \arrow[u, "u\otimes v"'] \end{tikzcd}$$

commutes.

Lemma 2.19. *For any $\lambda \in \mathrm{Fix}(c)(k)$, we have*

$$\widetilde{\mathrm{lt}}(K \otimes^{\mathbb{L}} L, w) = \widetilde{\mathrm{lt}}(K, u) \cdot \widetilde{\mathrm{lt}}(L, v).$$

Proof. This is immediate from the definitions. $\square$

2.20. For the remainder of this section, we prove some results in the case when $\Lambda = \overline{\mathbb{Q}}_\ell$, which we assume for the rest of this section.

Consider an algebraic space X/k, and a correspondence

$$d : \mathscr{C} \to X \times X$$

with $\mathscr{C}$ a Deligne-Mumford stack, such that d_1 and d_2 are quasi-finite. Let $\pi : \mathscr{C} \to C$ be the coarse moduli space, so that d factors through a correspondence

$$c : C \to X \times X,$$

with c_1 and c_2 quasi-finite. Note that since π is proper, we have for any $F \in D^b_c(C)$ a natural map

$$F \to \pi_*\pi^*F \simeq \pi_!\pi^*F,$$

which is an isomorphism by [13, 5.11]. In particular, we obtain a natural isomorphism

$$c_{2!}c_1^*K \simeq c_{2!}\pi_!\pi^*c_1^*K \simeq d_{2!}d_1^*K.$$

Therefore we have a natural bijection

$$\tau : \{d\text{-structures on } K\} \to \{c\text{-structures on } K\}. \tag{2.20.1}$$

Lemma 2.21. *Let $K \in D^b_c(X)$, and let $v : d_1^*K \to d_2^!K$ be a d-structure. Let $u : c_1^*K \to c_2^!K$ be the c-structure $\tau(v)$, and let $y \in \mathrm{Fix}(d)(k)$ be a fixed point with image $\bar{y} \in \mathrm{Fix}(c)(k)$. Then*

$$\mathrm{lt}_{\bar{y}}(K, u) = \mathrm{lt}_y(K, v).$$

Moreover,

$$\mathrm{lt}_{\bar{y}}(K, u) = \widetilde{\mathrm{lt}}_{\bar{y}}(K, u), \quad \mathrm{lt}_y(K, v) = \widetilde{\mathrm{lt}}_y(K, v).$$

Proof. Let $x \in X(k)$ be the image of y. By definition of the map τ we have a commutative diagram

$$\begin{tikzcd} d_{2!}d_1^*K \arrow[d, "v"] & c_{2!}\pi_*\pi^*c_1^*K \arrow[l, "\pi_*\simeq\pi_!"'] \\ K & c_{2!}c_1^*K. \arrow[l, "u"'] \arrow[u, "\mathrm{id}\to\pi_*\pi^*"'] \end{tikzcd} \tag{2.21.1}$$

Let $C_x^{(2)}$ denote $c_2^{-1}(x)$, and let $\mathscr{C}_x^{(2)}$ denote the fiber product

$$\mathscr{C}_x^{(2)} := \mathscr{C} \times_{d_2, X, x} \mathrm{Spec}(k).$$

Consider the diagram

$$\begin{tikzcd} (d_1^*K)_y \arrow[r] & R\Gamma_c(\mathscr{C}_x^{(2)}, d_1^*K) \arrow[dr, "v"] & \\ K_x \arrow[u, "d_1^*"] \arrow[d, "c_2^*"'] & R\Gamma_c(C_x^{(2)}, \pi_*d_1^*K) \arrow[u, "\pi_*\simeq\pi_!"] & K_x \\ (c_1^*K)_{\bar{y}} \arrow[r, hook] & R\Gamma_c(C_x^{(2)}, c_1^*K). \arrow[u, "\mathrm{id}\to\pi_*\pi^*"] \arrow[ur, "u"'] & \end{tikzcd}$$

The left pentagon commutes by the definition of the horizontal maps, and the right part of the diagram commutes by the commutativity of (2.21.1). From this and the definitions of $\mathrm{lt}_{\bar{y}}(K, u)$ and $\mathrm{lt}_y(K, v)$ the first part of lemma follows.

The second statement in the lemma is immediate from the definitions. □

2.22. Consider a correspondence

$$c : \mathscr{C} \to \mathscr{X} \times \mathscr{X},$$

with $\mathscr{X}$ and $\mathscr{C}$ separated finite type Deligne-Mumford stacks over k, and assume that c_1 and c_2 are quasi-finite. Let $i : \mathscr{Z} \hookrightarrow \mathscr{X}$ be a closed substack such that

$$c_1^{-1}(\mathscr{Z})_{\mathrm{red}} = c_2^{-1}(\mathscr{Z})_{\mathrm{red}},$$

and let $K \in D^b_c(\mathscr{X})$ be a complex with a c-structure

$$u : c_1^* K \to c_2^! K.$$

Let $\mathscr{C}_{\mathscr{Z}}$ denote $c_1^{-1}(\mathscr{Z})_{\mathrm{red}}$ so we have a correspondence

$$c_{\mathscr{Z}} : \mathscr{C}_{\mathscr{Z}} \to \mathscr{Z} \times \mathscr{Z}.$$

We then get a $c_{\mathscr{Z}}$-structure i^*u on i^*K as follows. Let

$$\tilde{i} : \mathscr{C}_{\mathscr{Z}} \hookrightarrow \mathscr{C}$$

be the inclusion, and note that for $s = 1, 2$ the map

$$\mathscr{C}_{\mathscr{Z}} \to \mathscr{C} \times_{c_s, \mathscr{X}, i} \mathscr{Z}$$

is a closed immersion defined by a nilpotent ideal. We therefore get a map

$$c^*_{\mathscr{Z}1} i^* K \simeq \tilde{i}^* c_1^* K \xrightarrow{u} \tilde{i}^* c_2^! K \xrightarrow{bc} c^!_{\mathscr{Z}2} i^* K, \tag{2.22.1}$$

where the map $\tilde{i}^* c_2^! \to c^!_{\mathscr{Z}2} i^*$ is adjoint to the composition of the base change isomorphism

$$c_{\mathscr{Z}2!} \tilde{i}^* c_2^! K \simeq i^* c_{2!} c_2^! K$$

with the adjunction map

$$i^* c_{2!} c_2^! K \to i^* K.$$

We denote the map (2.22.1) by i^*u.

Note also that there is a natural closed immersion

$$\gamma : \mathrm{Fix}(c_{\mathscr{Z}}) \hookrightarrow \mathrm{Fix}(c).$$

Lemma 2.23. *For every $\lambda \in \mathrm{Fix}(c_{\mathscr{Z}})$ we have*

$$\mathrm{lt}_\lambda(i^*K, i^*u) = \mathrm{lt}_{\gamma(\lambda)}(K, u).$$

Proof. We can without loss of generality replace $\mathscr{X}$ by an étale covering, and may therefore assume that $\mathscr{X}$ is a scheme. Furthermore, using 2.21 we can replace $\mathscr{C}$ by its coarse moduli space so we may also assume that $\mathscr{C}$ is a scheme (since an algebraic space quasi-finite over a scheme is a scheme). This reduces the proof to the case of schemes in which case the result is immediate from the definitions. □

Remark 2.24. In fact the diagram

$$\begin{tikzcd}
c_1^* K \arrow[r, "\mathrm{id}\to i_* i^*"] \arrow[d, "u"] & c_1^* i_* i^* K \arrow[d, "i_* i^* u"] \\
c_2^! K \arrow[r, "\mathrm{id}\to i_* i^*"] & c_2^! i_* i^* K
\end{tikzcd}$$

commutes. This follows from noting that the diagram

$$\begin{tikzcd}
c_1^* K \arrow[r, "\mathrm{id}\to i_* i^*"] \arrow[d, "u"] \arrow[ddr, "\mathrm{id}\to \tilde{i}_* \tilde{i}^*"] & c_1^* i_* i^* K \arrow[d, "bc"] \\
c_2^! K \arrow[d, "\mathrm{id}\to i_* i^*"] & \tilde{i}_* c^*_{\mathscr{X}1} i^* K \arrow[d, "\simeq"] \\
c_2^! i_* i^* K \arrow[d, "bc"] & \tilde{i}_* \tilde{i}^* c_1^* K \arrow[d, "u"] \\
\tilde{i}_* c^!_{\mathscr{X}2} i^* K & \tilde{i}_* \tilde{i}^* c_2^! K \arrow[l, "bc"']
\end{tikzcd}$$

commutes, which is immediate.

3. Some remarks on quasi-finite morphisms

Throughout this section we work with $\overline{\mathbb{Q}}_\ell$-coefficients.

3.1. Let $f : \mathscr{X} \to \mathscr{Y}$ be a quasi-finite morphism between separated Deligne-Mumford stacks of finite type over k.

Recall [10, 9.1.i] that for $F \in D^b_c(\mathscr{Y})$ and $K \in D^b_c(\mathscr{X})$ there is a natural isomorphism (the projection formula)

$$(f_! K) \otimes F \to f_!(K \otimes f^* F).$$

We therefore get for any $F \in D^b_c(\mathscr{Y})$ and $K \in D^b_c(\mathscr{X})$ an *evaluation map*

$$\mathrm{ev}_F : f_! \mathscr{R}Hom(f^* F, K) \otimes F \to f_! K$$

from the composite

$$f_! \mathscr{R}Hom(f^* F, K) \otimes F \xrightarrow{\mathrm{proj}} f_!(\mathscr{R}Hom(f^* F, K) \otimes f^* F) \xrightarrow{\mathrm{ev}} f_! K.$$

If $F, G \in D^b_c(\mathscr{Y})$ are two complexes we get a map

$$\tilde{\Phi}_f : f_! \mathscr{R}Hom(f^* F, f^! G) \otimes F \to G$$

by composing the evaluation map ev_F with the adjunction map $f_! f^! G \to G$. The adjoint of $\tilde{\Phi}_f$ is a map

$$\Phi_f : f_! \mathscr{R}Hom(f^* F, f^! G) \to \mathscr{R}Hom(F, G).$$

Example 3.2. Suppose $\mathscr{Y} = \mathrm{Spec}(k)$ is a point, in which case $\mathscr{X}_{\mathrm{red}}$ is isomorphic to a disjoint union of stacks of the form BH, with H a finite group.

In this case there is a canonical isomorphism

$$f_!\mathscr{R}Hom(f^*F, f^!G) \simeq \oplus_{\beta \in \pi_0(\mathscr{X})}\mathscr{R}Hom(F, G),$$

where the sum is over the set of connected components of $\mathscr{X}$. The map Φ_f is therefore in this case identified with a map

$$\oplus_{\beta \in \pi_0(\mathscr{X})}\mathscr{R}Hom(F, G) \to \mathscr{R}Hom(F, G).$$

We claim that this map is equal to the summation map

$$\Sigma : \oplus_{\beta \in \pi_0(\mathscr{X})}\mathscr{R}Hom(F, G) \to \mathscr{R}Hom(F, G).$$

To see this, it suffices to consider the case when $F = \overline{\mathbb{Q}}_\ell$, in which case the assertion is that the adjunction map

$$\oplus_\beta G \simeq f_!f^!G \xrightarrow{f_!f^! \to \mathrm{id}} G$$

is given by summation. This follows from observing that this map is dual to the adjunction map

$$DG \to f_*f^*DG \simeq \prod_\beta DG$$

which is given by the diagonal map.

3.3. By biduality the canonical map

$$G \to \mathscr{R}Hom(D(G), \Omega_{\mathscr{Y}})$$

is an isomorphism, so the map $\tilde{\Phi}_f$ corresponds by adjunction to a map

$$\Psi_f : f_!\mathscr{R}hom(f^*F, f^!G) \otimes F \otimes DG \to \Omega_{\mathscr{Y}}.$$

This map can also be described as follows. For complexes $A, B, C \in D_c^b(\mathscr{X})$ we have a natural composition map

$$\circ : \mathscr{R}Hom(A, B) \otimes \mathscr{R}Hom(B, C) \to \mathscr{R}Hom(A, C),$$

which is adjoint to the composite

$$\mathscr{R}Hom(A, B) \otimes \mathscr{R}Hom(B, C) \otimes A \xrightarrow{\text{ev on } A} B \otimes \mathscr{R}Hom(B, C) \xrightarrow{\text{ev on } B} C.$$

In particular for $K \in D_c^b(\mathscr{X})$ and $G \in D_c^b(\mathscr{Y})$ we have a natural map

$$c_G : f_!\mathscr{R}Hom(K, f^!G) \otimes DG \to f_!\mathscr{R}Hom(K, \Omega_{\mathscr{X}})$$

defined as the composite

$$\begin{tikzcd} f_!\mathscr{R}Hom(K, f^!G) \otimes DG \arrow[r, "\mathrm{proj}"] & f_!(\mathscr{R}Hom(K, f^!G) \otimes f^*DG) \arrow[r, "Df^! \simeq f^*D"] & f_!(\mathscr{R}Hom(K, f^!G) \otimes Df^!G) \arrow[d, "\circ"] \\ & & f_!\mathscr{R}hom(K, \Omega_{\mathscr{X}}). \end{tikzcd}$$

Lemma 3.4. *The map Ψ_f is equal to the composite*

$$f_!\mathscr{R}Hom(f^*F, f^!G) \otimes F \otimes DG \xrightarrow{\mathrm{ev}_F} f_!\mathscr{R}Hom(\overline{\mathbb{Q}}_\ell, f^!G) \otimes DG \xrightarrow{c_G} f_!f^!\Omega_{\mathscr{Y}} \xrightarrow{f_!f^! \to \mathrm{id}} \Omega_{\mathscr{Y}}.$$

Furthermore, the diagram

$$\begin{tikzcd}
f_!\mathscr{R}Hom(f^*F, f^!G) \otimes F \otimes DG \arrow[r, "\mathrm{ev}_F"] \arrow[d, "c_G"] & f_!\mathscr{R}Hom(\overline{\mathbb{Q}}_\ell, f^!G) \otimes DG \arrow[d, "c_G"] \\
f_!\mathscr{R}Hom(f^*F, \Omega_{\mathscr{X}}) \otimes F \arrow[r, "\mathrm{ev}_F"] & f_!f^!\Omega_{\mathscr{Y}}
\end{tikzcd} \tag{3.4.1}$$

commutes.

Proof. By definition of Ψ_f, the diagram

$$\begin{tikzcd}
(f_!\mathscr{R}Hom(f^*F, f^!G)) \otimes F \otimes DG \arrow[r, "\mathrm{proj}"] \arrow[ddrr, "\Psi_f"] & f_!(\mathscr{R}Hom(f^*F, f^!G) \otimes f^*F) \otimes DG \arrow[r, "\mathrm{ev}_F"] & f_!f^!G \otimes DG \arrow[d, "f_!f^! \to \mathrm{id}"] \\
& & G \otimes DG \arrow[d, "\mathrm{ev}_G"] \\
& & \Omega_X
\end{tikzcd}$$

commutes. It therefore suffices to show that the diagram

$$\begin{tikzcd}
f_!\mathscr{R}Hom(\overline{\mathbb{Q}}_\ell, f^!G) \otimes DG \arrow[rr, "\mathrm{proj}"] \arrow[d, "\simeq"] & & f_!(\mathscr{R}Hom(\overline{\mathbb{Q}}_\ell, f^!G) \otimes Df^!G) \arrow[d, "\circ"] \\
f_!f^!G \otimes DG \arrow[r, "\simeq"] \arrow[d, "f_!f^! \to \mathrm{id}"] & f_!(f^!G \otimes Df^!G) \arrow[r, "\mathrm{ev}"] & f_!f^!\Omega_{\mathscr{Y}} \arrow[d, "f_!f^! \to \mathrm{id}"] \\
G \otimes DG \arrow[rr, "\mathrm{ev}"] & & \Omega_{\mathscr{Y}}
\end{tikzcd}$$

commutes, which is immediate. □

By duality for any $A, B \in D_c^b(\mathscr{Y})$ we have a canonical isomorphism

$$\mathscr{R}Hom(A, B) \simeq \mathscr{R}Hom(D(B), D(A)), \tag{3.4.2}$$

and similarly for complexes on $\mathscr{X}$. A straightforward verification, which we leave to the reader, shows that for any $F \in D_c^b(\mathscr{Y})$ and $B \in D_c^b(\mathscr{X})$ the diagram

$$\begin{tikzcd}
f_!\mathscr{R}Hom(f^*F, B) \otimes F \arrow[r, "\mathrm{ev}_F"] \arrow[d, "(3.4.2)"] & f_!B \arrow[d, "\mathrm{id} \to D^2"] \\
f_!\mathscr{R}Hom(DB, Df^*F) \otimes F \arrow[r, "c_{DF}"] & f_!D^2B
\end{tikzcd} \tag{3.4.3}$$

commutes, and similarly for any $A \in D_c^b(\mathscr{X})$ and $G \in D_c^b(\mathscr{Y})$ the diagram

(3.4.4)
$$\begin{tikzcd}
f_!\mathscr{R}Hom(A, f^!G) \otimes DG \arrow[rr, "c_G"] \arrow[d, "(3.4.2)"] & & f_!DA \\
f_!\mathscr{R}Hom(Df^!G, DA) \otimes DG \arrow[r, "Df^! \simeq f^*D"] & f_!\mathscr{R}Hom(f^*DG, DA) \otimes DG \arrow[ur, "\mathrm{ev}_{DG}"'] &
\end{tikzcd}$$

commutes.

Lemma 3.5. *For any $F, G \in D_c^b(\mathscr{Y})$ the diagram*

$$\begin{tikzcd}
f_!\mathscr{R}Hom(f^*F, f^!G) \arrow[r, "\Phi_f"] \arrow[d, "(3.4.2)"] & \mathscr{R}Hom(F, G) \arrow[dd, "(3.4.2)"] \\
f_!\mathscr{R}Hom(Df^!G, Df^*F) \arrow[d, "\alpha"] & \\
f_!\mathscr{R}Hom(f^*DG, f^!DF) \arrow[r, "\Phi_f"] & \mathscr{R}Hom(DG, DF)
\end{tikzcd}$$

commutes, where α is induced by the biduality isomorphisms $Df^! \simeq f^*D$ and $Df^* \simeq f^!D$.

Proof. By adjunction it suffices to show that the following diagram

$$\begin{tikzcd}
f_!\mathscr{R}Hom(f^*F, f^!G) \otimes F \otimes DG \arrow[rr, "(\mathrm{ev}_F, c_G)"] \arrow[d, "(3.4.2)"] & & f_!f^!\Omega_{\mathscr{Y}} \\
f_!\mathscr{R}Hom(Df^!G, Df^*F) \otimes F \otimes DG \arrow[r, "\simeq"] & f_!\mathscr{R}Hom(f^*DG, f^!DF) \otimes F \otimes DG \arrow[ur, "(\mathrm{ev}_{DG}, c_{DF})"'] &
\end{tikzcd}$$

commutes. This follows from the commutativity of (3.4.1), (3.4.3), and (3.4.4). $\square$

Corollary 3.6. *The diagram*

$$\begin{tikzcd}
f_!f^!\overline{\mathbb{Q}}_\ell \arrow[r, "\simeq"] \arrow[d, "f_!f^! \to \mathrm{id}"] & f_!\mathscr{R}Hom(f^*\Omega_{\mathscr{Y}}, f^!\Omega_{\mathscr{Y}}) \arrow[d, "\Phi_f"] \\
\overline{\mathbb{Q}}_\ell \arrow[r, "\mathrm{id} \to D^2"] & \mathscr{R}Hom(\Omega_{\mathscr{Y}}, \Omega_{\mathscr{Y}})
\end{tikzcd}$$

commutes.

Proof. Take $F = G = \overline{\mathbb{Q}}_\ell$ in 3.5. $\square$

Lemma 3.7. *Let $i : \mathscr{Y}' \to \mathscr{Y}$ be a separated finite type morphism of Deligne-Mumford stacks, and let $\mathscr{X}'$ denote the fiber product $\mathscr{X} \times_{\mathscr{Y}} \mathscr{Y}'$ so we have a cartesian square*

$$\begin{tikzcd}
\mathscr{X}' \arrow[r, "i'"] \arrow[d, "f'"] & \mathscr{X} \arrow[d, "f"] \\
\mathscr{Y}' \arrow[r, "i"] & \mathscr{Y}.
\end{tikzcd}$$

Then for any $F, G \in D^b_c(\mathscr{Y})$ the diagram

$$\begin{tikzcd}
i^* f_! \mathscr{R}Hom(f^*F, f^!G) \arrow[r, "\Phi_f"] \arrow[d, "bc"] & i^* \mathscr{R}Hom(F, G) \arrow[d] \\
f'_! i'^* \mathscr{R}Hom(f^*F, f^!G) \arrow[d] & \mathscr{R}Hom(i^*F, i^*G) \\
f'_! \mathscr{R}Hom(i'^* f^* F, i'^* f^! G) \arrow[r, "i'^* f^! \to f'^! i^*"] & f'_! \mathscr{R}Hom(f'^* i^* F, f'^! i^* G) \arrow[u, "\Phi_{f'}"']
\end{tikzcd}$$

commutes, where 'bc' denotes 'base change' and the map $i'^* f^! \to f'^! i^*$ is obtained by adjunction from the map

$$f'_! i'^* f^! \xrightarrow{f'_! i'^* \simeq i^* f_!} i^* f_! f^! \xrightarrow{f_! f^! \to \mathrm{id}} i^*.$$

Proof. By adjunction it suffices to show that the big outside diagram in the following diagram commutes

$$\begin{tikzcd}
i^* f_! \mathscr{R}Hom(f^*F, f^!G) \otimes i^*F \arrow[r, "\simeq"] \arrow[d, "bc"] & i^*(f_! \mathscr{R}Hom(f^*F, f^!G) \otimes F) \arrow[d, "\mathrm{ev}_F"] & \\
f'_! i'^* \mathscr{R}Hom(f^*F, f^!G) \otimes i^*F \arrow[d] & i^* f_! f^! G \arrow[r, "f_! f^! \to \mathrm{id}"] \arrow[d, "bc"] & i^*G \\
f'_! \mathscr{R}Hom(f'^* i^* F, i'^* f^! G) \otimes i^*F \arrow[r, "\mathrm{ev}_{i^*F}"] \arrow[d, "i'^* f^! \to f'^! i^*"] & f'_! i'^* f^! G \arrow[r, "i'^* f^! \to f'^! i^*"] & f'_! f'^! i^* G. \arrow[u, "f'_! f'^! \to \mathrm{id}"'] \\
f'_! \mathscr{R}Hom(f'^* i^* F, f'^! i^* G) \otimes i^*F \arrow[urr, "\mathrm{ev}_{i^*F}"'] & &
\end{tikzcd}$$

This follows by noting that each of the small inside diagrams clearly commute. □

3.8. Consider now the case when $\mathscr{Y}' = \mathrm{Spec}(k)$ is a point so i is given by a point $y \in \mathscr{Y}(k)$. In this case the stack $\mathscr{X}'_{\mathrm{red}}$ is isomorphic to a disjoint union of stacks of the form BG, for G a finite group. Consider a point

$$z : \mathrm{Spec}(k) \to \mathscr{X}'.$$

The map

$$i'^* f^! G \to f'^! i^* G$$

then induces my pulling back to z a map

$$(f^!G)_z \to G_y. \tag{3.8.1}$$

Lemma 3.9. *The map (3.8.1) is equal to $\tau_{z,G}$ defined in 2.11.*

Proof. This is immediate from the definitions. □

3.10. Suppose now that $\mathscr{X}$ and $\mathscr{Y}$ are algebraic spaces, which we denote by X and Y to avoid confusion, and again that $\mathscr{Y}' = \mathrm{Spec}(k)$ is a point. Then combining the above discussion with 3.2 we get that the following diagram commutes:

$$\begin{tikzcd}
(f_!\mathscr{R}Hom(f^*F, f^!G))_y \arrow[r, "\simeq"] \arrow[d, "\Phi_f"] & \oplus_{z\in f^{-1}(y)}\mathscr{R}Hom(f^*F, f^!G)_z \arrow[d] \\
\mathscr{R}Hom(F,G)_y \arrow[d] & \oplus_{z\in f^{-1}(y)}\mathrm{RHom}(F_y, (f^!G)_z) \arrow[d, "\oplus\tau_{G,z}"] \\
\mathrm{RHom}(F_y, G_y) & \oplus_{z\in f^{-1}(y)}\mathrm{RHom}(F_y, G_y). \arrow[l, "\Sigma"']
\end{tikzcd}$$

Lemma 3.11. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a quasi-finite morphism between separated finite type Deligne-Mumford stacks over k. Then for any point $x : \mathrm{Spec}(k) \to \mathscr{X}$ the diagram*

$$\begin{tikzcd}
(f^!\overline{\mathbb{Q}}_\ell)_x \arrow[r] \arrow[d, "\tau_{\overline{\mathbb{Q}}_\ell, x}"] & \mathrm{RHom}(\Omega_{\mathscr{Y},f(x)}, \Omega_{\mathscr{X},x}) \arrow[d, "\tau_{\Omega_{\mathscr{Y}},x}"] \\
\overline{\mathbb{Q}}_\ell \arrow[r] & \mathrm{RHom}(\Omega_{\mathscr{Y},f(x)}, \Omega_{\mathscr{Y},f(x)})
\end{tikzcd}$$

commutes.

Proof. The assertion is étale local on $\mathscr{Y}$ and $\mathscr{X}$, so we may assume that they are both schemes. In this case the result follows from 3.10 and 3.6. □

3.12. In general, for $K \in D^b_c(\mathrm{Spec}(k), \overline{\mathbb{Q}}_\ell)$ the composite map

$$\overline{\mathbb{Q}}_\ell \longrightarrow \mathrm{RHom}(K, K) \xrightarrow{\mathrm{tr}} \overline{\mathbb{Q}}_\ell$$

is equal to multiplication by

$$\chi(K) := \sum_i (-1)^i \dim_{\overline{\mathbb{Q}}_\ell} H^i(K).$$

Proposition 3.13. *Let $\mathscr{X}$ be a separated Deligne-Mumford stack of finite type over k. Then for any $x \in \mathscr{X}(k)$ we have*

$$\chi(\Omega_{\mathscr{X},x}) = 1.$$

Proof. The assertion is étale local on $\mathscr{X}$, so it suffices to consider the case when $\mathscr{X}$ is a scheme, which for consistency of notation we denote by X. Let

$$i : \mathrm{Spec}(k) \hookrightarrow X$$

be the inclusion. Then $\Omega_{X,x}$ is dual to $i^!\overline{\mathbb{Q}}_\ell$, so it suffices to show that

$$\chi(i^!\overline{\mathbb{Q}}_\ell) = 1.$$

For this let $j : U \hookrightarrow X$ be the complement of x, and consider the distinguished triangle

$$i_*i^!\overline{\mathbb{Q}}_\ell \to \overline{\mathbb{Q}}_\ell \to j_*j^*\overline{\mathbb{Q}}_\ell \to i^!\overline{\mathbb{Q}}_\ell[1].$$

To prove that $\chi(i^!\overline{\mathbb{Q}}_\ell) = 1$, it suffices to show that

$$\chi(i^*j_*j^*\overline{\mathbb{Q}}_\ell) = 0.$$

This follows from a more general statement of Laumon, which implies that in fact $i^*j_*j^*\overline{\mathbb{Q}}_\ell = 0$ in the Grothendieck group of $\overline{\mathbb{Q}}_\ell$-vector spaces [11, 1.1]. $\square$

Combining 3.11 and 3.13, we obtain the following:

Corollary 3.14. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a quasi-finite morphism of separated Deligne-Mumford stacks of finite type over k, and let $x \in \mathscr{X}(k)$ be a point. Then the map*

$$\tau_{\overline{\mathbb{Q}}_\ell,x} : (f^!\overline{\mathbb{Q}}_\ell)_x \to \overline{\mathbb{Q}}_\ell$$

is equal to the composite map

$$(f^!\overline{\mathbb{Q}}_\ell)_x \longrightarrow \mathrm{RHom}(\Omega_{\mathscr{Y},f(x)}, \Omega_{\mathscr{X},x}) \xrightarrow{\tau_{\Omega_{\mathscr{Y}},x}} \mathrm{RHom}(\Omega_{\mathscr{Y},f(x)}, \Omega_{\mathscr{Y},f(x)}) \xrightarrow{\mathrm{tr}} \overline{\mathbb{Q}}_\ell.$$

4. A special case

Throughout this section we work with $\overline{\mathbb{Q}}_\ell$-coefficients.

4.1. Let

$$c : \mathscr{C} \to \mathscr{X} \times \mathscr{X}$$

be a correspondence, with $\mathscr{X}$ and $\mathscr{C}$ separated finite type Deligne-Mumford stacks over k, and c_1 and c_2 quasi-finite. Let L be a lisse sheaf on $\mathscr{X}$ and let

$$u : c_1^*L \to c_2^!L$$

be a c-structure.

The main result of this section is the following theorem, which in the case of algebraic spaces yields 1.9 when K is a lisse sheaf. The proof of 1.9 in general will be a rather elaborate devisage reducing 1.9 to the following result for Deligne-Mumford stacks.

Theorem 4.2. *For any $\lambda \in \mathrm{Fix}(c)(k)$ we have*

$$\widetilde{\mathrm{lt}}_\lambda(L, u) = \widetilde{\mathrm{lt}}_\lambda(DL, u^t).$$

Proof. Write $\lambda = (y \in \mathscr{C}(k), \sigma : c_1(y) \to c_2(y))$, and let x denote $c_1(y)$. Recall from 2.11 that for any $K \in D_c^b(\mathscr{X})$ we have a canonical map

$$\tau_{K,y} : (c_2^!K)_y \to K_x,$$

where we identify $K_{c_2(y)}$ with K_x using σ. In particular, we have a map

$$\tau_{\overline{\mathbb{Q}}_\ell,y} : (c_2^!\overline{\mathbb{Q}}_\ell)_y \to \overline{\mathbb{Q}}_\ell.$$

In general, if $K \in D_c^b(\mathscr{X})$ then there is a canonical map

$$\rho : c_2^*K \otimes c_2^!\overline{\mathbb{Q}}_\ell \to c_2^!K$$

defined as the adjoint of the composite

$$c_{2!}(c_2^*K \otimes c_{2!}\overline{\mathbb{Q}}_\ell) \xrightarrow{\mathrm{proj}} K \otimes c_{2!}c_2^!\overline{\mathbb{Q}}_\ell \xrightarrow{c_{2!}c_2^! \to \mathrm{id}} K.$$

Note that the definition of this map also makes sense for torsion coefficients.

In the case of the lisse sheaf L, the map

$$\rho : c_2^*L \otimes c_2^!\overline{\mathbb{Q}}_\ell \to c_2^!L$$

is an isomorphism. This follows from a standard reduction to the case of torsion coefficients, where the result is immediate.

Therefore the map u can equivalently be viewed as an element of

$$u \in \Gamma(\mathscr{C}, (c_1^*L^\wedge \otimes c_2^*L) \otimes c_2^!\overline{\mathbb{Q}}_\ell),$$

where $L^\wedge$ denotes the lisse sheaf $\mathscr{H}om(L, \overline{\mathbb{Q}}_\ell)$. In particular, looking at the stalk at y we obtain an element

$$u_y \in (L_x^\wedge \otimes L_x) \otimes (c_2^!\overline{\mathbb{Q}}_\ell)_y.$$

Lemma 4.3. *The number*

$$\widetilde{\mathrm{lt}}_\lambda(L, u) \in \overline{\mathbb{Q}}_\ell$$

is equal to the image of u_y under the map

$$(L_x^\wedge \otimes L_x) \otimes (c_2^!\overline{\mathbb{Q}}_\ell)_y \xrightarrow{\langle\cdot,\cdot\rangle \otimes \tau_{\overline{\mathbb{Q}}_\ell, y}} \overline{\mathbb{Q}}_\ell \otimes \overline{\mathbb{Q}}_\ell \simeq \overline{\mathbb{Q}}_\ell,$$

where

$$\langle\cdot,\cdot\rangle : L_x^\wedge \otimes L_x \to \overline{\mathbb{Q}}_\ell$$

is the natural pairing.

Proof. This follows from noting that the diagram

$$\begin{tikzcd}
L_x \arrow[r, "\simeq"] & (c_1^*L)_y \arrow[r] & (c_2^!L)_y \arrow[r, "\tau_L"] & L_x \\
 & & (c_2^*L)_y \otimes (c_2^!\overline{\mathbb{Q}}_\ell)_y \arrow[u, "\rho"] \arrow[r, "\tau_{\overline{\mathbb{Q}}_\ell}"] & L_x \otimes \overline{\mathbb{Q}}_\ell \arrow[u, "\simeq"']
\end{tikzcd}$$

commutes, which in turn follows from observing that the diagram

$$\begin{tikzcd}
c_{2!}(c_2^*L \otimes c_2^!\overline{\mathbb{Q}}_\ell) \arrow[r, "\mathrm{proj}"] \arrow[d, "\rho"] & (c_{2!}c_2^!\overline{\mathbb{Q}}_\ell) \otimes L \arrow[d, "c_{2!}c_2^! \to \mathrm{id}"] \\
c_{2!}c_2^!L \arrow[r, "c_{2!}c_2^! \to \mathrm{id}"] & L
\end{tikzcd}$$

commutes, which follows from the definition of ρ. $\square$

To calculate the local term $\widetilde{\mathrm{lt}}_\lambda(DL, u^t)$, note first that we have a natural isomorphism

$$DL \simeq L^\wedge \otimes \Omega_{\mathscr{X}},$$

so

$$c_2^*DL \simeq c_2^*L^\wedge \otimes c_2^*\Omega_{\mathscr{X}}$$

and

$$c_1^!DL \simeq c_1^!(L^\wedge \otimes \Omega_{\mathscr{X}}) \simeq c_1^*L^\wedge \otimes c_1^!\Omega_{\mathscr{X}} \simeq c_1^*L^\wedge \otimes \Omega_{\mathscr{C}}.$$

Therefore u^t can be viewed as a map

$$c_2^*L^\wedge \otimes c_2^*\Omega_{\mathscr{X}} \to c_1^*L^\wedge \otimes \Omega_{\mathscr{C}},$$

or equivalently as a map

$$u^t : c_2^*\Omega_{\mathscr{X}} \to (c_1^*L^\wedge \otimes c_2^*L) \otimes \Omega_{\mathscr{C}}.$$

In particular, looking at the stalk at y we get a map

$$(u^t)_y : \Omega_{\mathscr{X},x} \to (L_x^\wedge \otimes L_x) \otimes \Omega_{\mathscr{C},y}.$$

Lemma 4.4. *The number* $\widetilde{\mathrm{lt}}_\lambda(DL, u^t)$ *is equal to the trace of the composite map*

$$\Omega_{\mathscr{X},x} \xrightarrow{(u^t)_y} (L_x^\wedge \otimes L_x) \otimes \Omega_{\mathscr{C},y} \simeq (L_x^\wedge \otimes L_x) \otimes (c_2^!\Omega_{\mathscr{X}})_y \xrightarrow{\langle\cdot,\cdot\rangle\otimes\tau_{\Omega_{\mathscr{X}},y}} \Omega_{\mathscr{X},x}. \tag{4.4.1}$$

Proof. The local term $\widetilde{\mathrm{lt}}_\lambda(DL, u^t)$ is defined to be the trace of the composite of the map

$$(DL)_x \xrightarrow{\simeq} (c_2^*DL)_y \xrightarrow{u^t} (c_1^!DK)_y$$

with the map

$$(c_1^!DL)_y \xrightarrow{\tau_{DL}} (DL)_x.$$

On the other hand, the trace of (4.4.1) is by definition equal to the trace of the map

$$L_x^\wedge \otimes \Omega_{\mathscr{X},x} \to L_x^\wedge \otimes \Omega_{\mathscr{X},x}$$

obtained from the bottom of the following diagram

$$\begin{tikzcd}
(DL)_x \arrow[r, "\simeq"] \arrow[d, "\simeq"] & (c_2^*DL)_y \arrow[r, "u^t"] \arrow[d, "\simeq"] & (c_1^!DK)_y \arrow[r, "\tau_{DL}"] \arrow[d, "\simeq"] & (DL)_x \arrow[dr, "\simeq"] & \\
L_x^\wedge \otimes \Omega_{\mathscr{X},x} \arrow[r, "\simeq"] & (c_2^*L^\wedge \otimes c_2^*\Omega_{\mathscr{X}})_y \arrow[r] & (c_1^*L^\wedge \otimes c_1^!\Omega_{\mathscr{X}})_y \arrow[r, "\simeq"] & L_x^\wedge \otimes \Omega_{\mathscr{C},y} \arrow[r, "\tau_{\Omega_{\mathscr{X}}}"] & L_x^\wedge \otimes \Omega_{\mathscr{X},x},
\end{tikzcd}$$

where the two left squares commute by definition of the vertical and bottom horizontal arrows. Therefore it suffices to show that the right pentagon in this diagram commutes. This follows from noting that the diagram

$$\begin{tikzcd}
c_{1!}c_1^!DL \arrow[r, "c_{1!}c_1^!\to\mathrm{id}"] \arrow[d, "\mathrm{proj}"] & DL \arrow[dd, "\simeq"] \\
c_{1!}(c_1^*L^\wedge \otimes c_1^!\Omega_{\mathscr{X}}) \arrow[d, "\simeq"] & \\
(c_{1!}c_1^!\Omega_{\mathscr{X}}) \otimes L^\wedge \arrow[r, "c_{1!}c_1^!\to\mathrm{id}"] & L^\wedge \otimes \Omega_{\mathscr{X}}
\end{tikzcd}$$

commutes. □

We have

$$c_2^!\overline{\mathbb{Q}}_\ell \simeq \mathscr{R}Hom(c_2^*\Omega_{\mathscr{X}}, \Omega_{\mathscr{C}}),$$

so there is a natural map

$$\varphi_y : (c_2^!\overline{\mathbb{Q}}_\ell)_y \to RHom(\Omega_{\mathscr{X},x}, \Omega_{\mathscr{C},y}).$$

By 3.14, the diagram

$$\begin{tikzcd}(c_2^!\overline{\mathbb{Q}}_\ell)_y \arrow[r, "\varphi_y"] \arrow[d, "\tau_{\overline{\mathbb{Q}}_\ell, y}"] & RHom(\Omega_{\mathscr{X},x}, \Omega_{\mathscr{C},y}) \arrow[d, "\tau_{\Omega_{\mathscr{X}},y}"] \\ \overline{\mathbb{Q}}_\ell & RHom(\Omega_{\mathscr{X},x}, \Omega_{\mathscr{X},x}) \arrow[l, "\mathrm{tr}"']\end{tikzcd} \tag{4.4.2}$$

commutes. Combining 4.3, 4.4, and the commutativity of (4.4.2), we see that to complete the proof of 4.2, it suffices to prove the following.

Lemma 4.5. *The image of u_y under the map*

$$H^0((L_x^\wedge \otimes L_x) \otimes (c_2^!\overline{\mathbb{Q}}_\ell)_y) \xrightarrow{\varphi_y} (L_x^\wedge \otimes L_x) \otimes \mathrm{Hom}(\Omega_{\mathscr{X},x}, \Omega_{\mathscr{C},y}) \simeq \mathrm{Hom}(\Omega_{\mathscr{X},x}, (L_x^\wedge \otimes L_x) \otimes \Omega_{\mathscr{C},y})$$

is equal to $(u^t)_y$.

Proof. This follows from noting that the following diagram of isomorphisms commutes

$$\begin{tikzcd}\mathscr{R}Hom(c_1^*L, c_2^!L) \arrow[r, "D"] \arrow[d] & \mathscr{R}Hom(c_2^*DL, c_1^!DL) \arrow[d] \\ c_1^*L^\wedge \otimes c_2^!L \arrow[d] & \mathscr{R}Hom(c_2^*L^\wedge \otimes c_2^*\Omega_{\mathscr{X}}, c_1^!(L^\wedge \otimes \Omega_{\mathscr{X}})) \arrow[d] \\ c_1^*L^\wedge \otimes c_2^*L \otimes c_2^!\overline{\mathbb{Q}}_\ell \arrow[r] & c_1^*L^\wedge \otimes c_2^*L \otimes \mathscr{R}Hom(c_2^*\Omega_{\mathscr{X}}, \Omega_{\mathscr{C}}).\end{tikzcd}$$

□

This completes the proof of 4.2. □

5. Local terms and finite pushforwards

5.1. Let

$$c : \mathscr{C} \to \mathscr{X} \times \mathscr{X}$$

be a correspondence with $\mathscr{C}$ and $\mathscr{X}$ Deligne-Mumford stack of finite type over k, and with c_2 quasi-finite. Let $K \in D_c^b(\mathscr{X}, \overline{\mathbb{Q}}_\ell)$ be a complex on $\mathscr{X}$ with c-structure u.

Let $\pi : \mathscr{X} \to \mathscr{Y}$ be a proper quasi-finite morphism of stacks, and let

$$d : \mathscr{C} \to \mathscr{Y} \times \mathscr{Y}$$

be the correspondence obtained by composing c with

$$\pi \times \pi : \mathscr{X} \times \mathscr{X} \to \mathscr{Y} \times \mathscr{Y}.$$

Let $\pi_!u$ be the d-structure on $\pi_!K$ defined by the composite

$$\begin{array}{rcll} d_1^*\pi_!K & \simeq & c_1^*\pi^*\pi_*K & (\pi_! \simeq \pi_* \text{ since } \pi \text{ is proper}) \\ & \to & c_1^*K & (\pi^*\pi_* \to \mathrm{id}) \\ & \xrightarrow{u} & c_2^!K & \\ & \to & c_2^!\pi^!\pi_!K & (\mathrm{id} \to \pi^!\pi_!) \\ & \simeq & d_2^!\pi_!K. & \end{array}$$

5.2. Now let $\lambda \in \mathrm{Fix}(d)(k)$ be a fixed point of d, say λ is given by a pair

$$\lambda = (y \in \mathscr{C}(k), \bar{\sigma} : d_1(y) \to d_2(y)).$$

Let $I_{\bar{\sigma}}$ denote the scheme classifying isomorphisms $\sigma : c_1(y) \to c_2(y)$ in $\mathscr{X}$ such that $\pi(\sigma) = \bar{\sigma}$. The scheme $I_{\bar{\sigma}}$ is the fiber product of the diagram

$$\begin{tikzcd} & \mathrm{Spec}(k) \arrow[d, "{(c_1(y),c_2(y),\bar{\sigma})}"] \\ \mathscr{X} \arrow[r, "\Delta"] & \mathscr{X} \times_{\mathscr{Y}} \mathscr{X}. \end{tikzcd}$$

Let G_λ be the group scheme of automorphisms α of y in $\mathscr{C}(k)$ which induce the identity automorphism after applying both d_1 and d_2. Then G_λ acts on $I_{\bar{\sigma}}$ by the formula

$$\alpha * \sigma = c_2(\alpha)^{-1} \circ \sigma \circ c_1(\alpha).$$

Note also that since $\mathscr{C}$ is a Deligne-Mumford stack and k is separably closed, the group scheme G_λ is a constant group scheme.

Let

$$\gamma : \mathrm{Fix}(c) \to \mathrm{Fix}(d) \tag{5.2.1}$$

be the projection. Then the fiber product of the diagram

$$\begin{tikzcd} & \mathrm{Fix}(c) \arrow[d, "\gamma"] \\ \mathrm{Spec}(k) \arrow[r, "\lambda"] & \mathrm{Fix}(d) \end{tikzcd}$$

is isomorphic to $I_{\bar{\sigma}}$.

This isomorphism is obtained by associating to each point $\sigma : c_1(y) \to c_2(y)$ of $I_{\bar{\sigma}}(k)$ the fixed point.

$$\lambda_\sigma := (y, \sigma) \in \mathrm{Fix}(c)(k).$$

Define an equivalence relation on $I_{\bar{\sigma}}(k)$ by declaring that $\sigma \sim \sigma'$, if σ and σ' are in the same orbit of the G_λ-action on $I_{\bar{\sigma}}$. Note that if $\sigma \sim \sigma'$ then $\lambda_\sigma \simeq \lambda_{\sigma'}$ in $\mathrm{Fix}(c)(k)$ so

$$\widetilde{\mathrm{lt}}_{\lambda_\sigma}(K, u) = \widetilde{\mathrm{lt}}_{\lambda_{\sigma'}}(K, u).$$

It therefore makes sense to write $\widetilde{\mathrm{lt}}_{\lambda_{[\sigma]}}(K, u)$ for an equivalence class $[\sigma] \in I_{\bar{\sigma}}/\sim$.

Proposition 5.3. *We have*

$$\widetilde{\mathrm{lt}}_\lambda(\pi_! K, \pi_! u) = \sum_{[\sigma] \in I_{\bar{\sigma}}/\sim} C_{[\sigma]} \cdot \widetilde{\mathrm{lt}}_{\lambda_{[\sigma]}}(K, u),$$

where $C_{[\sigma]}$ is the constant

$$C_{[\sigma]} := \frac{|\mathrm{Ker}(\mathrm{Aut}(y) \to \mathrm{Aut}(d_1(y)))|}{|\mathrm{Ker}(\mathrm{Aut}(c_1(y)) \to \mathrm{Aut}(d_1(y)))| \cdot |\mathrm{Ker}(\mathrm{Aut}(\lambda_\sigma) \to \mathrm{Aut}(d_1(y)))|}.$$

Proof. The assertion is étale local on $\mathscr{Y}$, so we may assume that $\mathscr{Y}$ is a scheme (which we denote by Y to avoid confusion). Let $x \in Y(k)$ be the image of λ. Replacing $\mathscr{C}$ by the complement of the closed substack

$$(c_1^{-1}(x) \cup c_2^{-1}(x)) - \{\text{image of } y : \operatorname{Spec}(k) \to \mathscr{C}\},$$

we may assume that $c_1^{-1}(x)_{\mathrm{red}} = c_2^{-1}(x)_{\mathrm{red}}$, and that this stack is isomorphic to BH for some group H. By 2.23 we may therefore base change to x, and hence may assume that $\mathscr{Y}$ is a point. In this case, $\mathscr{X}_{\mathrm{red}}$ is isomorphic to a disjoint union of stacks of the form BG, with G a finite group, and $\mathscr{C}_{\mathrm{red}} \simeq BH$ for some group H. If the images of BH under c_1 and c_2 are disjoint, then clearly the trace of $\pi_! u$ on $\pi_! K$ is zero and the formula holds. So assume this is not the case. Then replacing $\mathscr{X}$ by the image of $\mathscr{C}$ we may assume that $\mathscr{X} \simeq BG$ for some group G. So now we are considering a correspondence

$$c : BH \to BG \times BG.$$

Now any morphism $BH \to BG$ is induced by a homomorphism $H \to G$, so in this case the proposition follows from 2.12. $\square$

Remark 5.4. In the case when $\mathscr{C}$ and $\mathscr{X}$ are algebraic spaces the map (5.2.1) is a monomorphism and we find that

$$\mathrm{lt}_\lambda(\pi_! K, \pi_! u) = \begin{cases} \mathrm{lt}_\lambda(K, u) & \text{if } \lambda \in \mathrm{Fix}(c) \\ 0 & \text{otherwise.} \end{cases}$$

5.5. Since π is proper, we have an isomorphism $\pi_! \simeq \pi_*$. We therefore have a natural isomorphism

$$D(\pi_! K) \simeq \pi_! D(K).$$

Lemma 5.6. *The two d^t-structures on $\pi_!(DK)$ given by*

$$\pi_!(u^t), \quad \textit{and} \quad (\pi_! u)^t$$

are equal.

Proof. This follows from noting that the diagram

$$\begin{tikzcd}
Dc_1^*\pi^*\pi_! K \arrow[d, "\simeq"] & Dc_1^*\pi^*\pi_* K \arrow[l, "D(\pi_!\simeq\pi_*)"'] \arrow[d, "\simeq"] & Dc_1^* K \arrow[l, "D(\pi^*\pi_*\to\mathrm{id})"'] \arrow[d, "\simeq"] & Dc_2^! K \arrow[l, "Du"'] \arrow[d, "\simeq"] & Dc_2^!\pi^!\pi_! K \arrow[l, "D(\mathrm{id}\to\pi^!\pi_!)"'] \arrow[d, "\simeq"] \\
c_1^!\pi^!\pi_* DK \arrow[d, "\pi_*\simeq\pi_!"] & c_1^!\pi^!\pi_! DK \arrow[l, "\pi_!\simeq\pi_*"'] & c_1^* DK \arrow[l, "\mathrm{id}\to\pi^!\pi_!"'] & c_2^* DK \arrow[l, "u^t"'] & c_2^*\pi^*\pi_* DK \arrow[l, "\pi^*\pi_*\to\mathrm{id}"'] \arrow[d, "\simeq"] \\
c_1^!\pi^!\pi_! DK & & & & c_2^*\pi^*\pi_! DK \arrow[llll, "\pi_!(u^t)"']
\end{tikzcd}$$

commutes. $\square$

5.7. One useful application of the above discussion is the following. Let

$$c : \mathscr{C} \to \mathscr{X} \times \mathscr{X}$$

be a correspondence with $\mathscr{C}$ and $\mathscr{X}$ Deligne-Mumford stacks of finite type over k, and the maps c_1 and c_2 quasi-finite. Let

$$d : C \to X \times X$$

be the correspondence obtained by passing to the coarse moduli spaces, so we have a commutative diagram

$$\begin{tikzcd} & \mathscr{C} \arrow[dl, "c_1"'] \arrow[dr, "c_2"] \arrow[dd, "\tilde{\pi}"] & \\ \mathscr{X} \arrow[dd, "\pi"'] & & \mathscr{X} \arrow[dd, "\pi"] \\ & C \arrow[dl, "d_1"'] \arrow[dr, "d_2"] & \\ X & & X. \end{tikzcd}$$

Let $\mathscr{C}_i$ $(i = 1, 2)$ denote the fiber product

$$\mathscr{C}_i := \mathscr{X} \times_{X, c_i} C.$$

Assume that the diagram

(5.7.1)
$$\begin{tikzcd} \mathscr{C} \arrow[r] \arrow[d] & \mathscr{C}_2 \arrow[d] \\ \mathscr{C}_1 \arrow[r] & C \end{tikzcd}$$

is cartesian, and let

$$\gamma : \mathrm{Fix}(c) \to \mathrm{Fix}(d)$$

be the projection. Let $K \in D^b_c(\mathscr{X}, \overline{\mathbb{Q}}_\ell)$ be a complex with a c-structure

$$u : c_1^* K \to c_2^! K.$$

Let

$$c' : \mathscr{C} \to X \times X$$

be the correspondence obtained by composing c with

$$\pi \times \pi : \mathscr{X} \times \mathscr{X} \to X \times X.$$

As in 5.1 we then have a c'-structure

$$u' : c'_{2!} c'^*_1 \pi_! K \to \pi_! K.$$

Through the bijection (2.20.1) we then also get a d-structure

$$v : d_{2!} d_1^* \pi_! K \to \pi_! K.$$

Proposition 5.8. *For any $\lambda \in \mathrm{Fix}(c)(k)$ we have*

$$\widetilde{\mathrm{lt}}_\lambda(K, u) = \widetilde{\mathrm{lt}}_{\gamma(\lambda)}(\pi_! K, v).$$

Proof. Let $\lambda' \in \mathrm{Fix}(c')(k)$ be the image of λ.

By 2.21 we have

$$\widetilde{\mathrm{lt}}_{\gamma(\lambda)}(\pi_! K, v) = \widetilde{\mathrm{lt}}_{\lambda'}(\pi_! K, u'),$$

so it suffices to show that

$$\widetilde{\mathrm{lt}}_{\lambda'}(\pi_! K, u') = \widetilde{\mathrm{lt}}_\lambda(K, u).$$

Write $\lambda = (y, \sigma)$, and let $x \in \mathscr{X}(k)$ be the image of y. Let H denote the automorphism group of x. Then since the square (5.7.1) is cartesian we have a natural isomorphism

$$\mathrm{Aut}(y) \simeq H \times H.$$

From this it follows that the set $I_{\bar{\sigma}}/\sim$ occuring in 5.3 consists of a single element, and that for this element the constant $C_{[\sigma]}$ is equal to 1. $\square$

6. Remarks on local fundamental groups

6.1. Let $\mathscr{X}$ be a normal Deligne-Mumford stack of finite type over k. Let $j : \mathscr{U} \hookrightarrow \mathscr{X}$ be a dense open immersion, and let F be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathscr{U}$, where ℓ is a prime invertible in k. For a point $x \in \mathscr{X}(k)$, let $X_{(x)}$ denote the scheme

$$X_{(x)} := \mathscr{U} \times_{\mathscr{X}} \mathrm{Spec}(\mathscr{O}_{\mathscr{X},x}),$$

and let $F_{(x)}$ denote the pullback of F to $X_{(x)}$ (here $\mathscr{O}_{\mathscr{X},x}$ denotes the local ring for the étale topology). For any geometric point $\bar{y} \to X_{(x)}$ we then obtain a representation

$$\rho_{\bar{y}} : \pi_1(X_{(x)}, \bar{y}) \to GL(F_{(x),\bar{y}}).$$

We say that F has *finite local monodromy at* x if the image of this homomorphism $\rho_{\bar{y}}$ is finite. Note that this is independent of the choice of $\bar{y}$.

Proposition 6.2. *There exists a dense open substack $\mathscr{V} \subset \mathscr{X}$ containing $\mathscr{U}$ such that a point $x \in \mathscr{X}(k)$ is in $\mathscr{V}(k)$ if and only if F has finite local monodromy at x.*

Proof. The sheaf F is by definition obtained from a lisse K-sheaf, where K is a finite extension of $\mathbb{Q}_\ell$. Forgetting the K-structure we see that it suffices to consider the case when F is obtained from a lisse $\mathbb{Z}_\ell$-sheaf $\mathcal{F}$ by tensoring with $\overline{\mathbb{Q}}_\ell$. For every $n \geq 1$ let $\mathcal{F}_n$ denote the locally constant sheaf $\mathcal{F}/\ell^n$. Let r be the rank of F (a locally constant function on $\mathscr{U}$), and let $I_n \to \mathscr{X}$ be the normalization of the finite étale $\mathscr{U}$-scheme

$$\underline{\mathrm{Isom}}((\mathbb{Z}/\ell^n)^r, \mathcal{F}_n).$$

Then there is a natural action of $GL_r(\mathbb{Z}/\ell^n)$ on I_n. Let $\mathscr{P}_n \to \mathscr{X}$ be the stack theoretic quotient

$$\mathscr{P}_n := [I_n/GL_r(\mathbb{Z}/\ell^n)].$$

The map $\mathscr{P}_n \to \mathscr{X}$ is an isomorphism over $\mathscr{U}$, and identifies $\mathscr{X}$ with the relative coarse moduli space of $\mathscr{P}_n \to \mathscr{X}$ (this follows for example from [12, 2.9 (ii)]). Let $W_n \subset \mathscr{P}_n$ be the maximal open subset over which F extends to a lisse sheaf (if $j : \mathscr{U} \hookrightarrow \mathscr{P}_n$ is the inclusion, then W_n is the open subset where $R^0 j_* F$ has maximal rank), and let $\mathscr{U}_n \subset \mathscr{X}$ be the image of W_n. The set $\mathscr{U}_n$ is an open subset (since pullback identifies open subsets of $\mathscr{X}$ with open subsets of $\mathscr{P}_n$ since $\mathscr{X}$ is the relative coarse moduli space of $\mathscr{P}_n$), containing $\mathscr{U}_n$. We claim that

$$\mathscr{V} := \cup_{n \geq 1} \mathscr{U}_n$$

is the desired open subset of $\mathscr{X}$.

To see this, let $x \in \mathscr{X}(k)$ be a point, and let $P_{n,x}$ denote the fiber product

$$P_{n,x} := \mathscr{P}_n \times_{\mathscr{X}} \mathrm{Spec}(\mathscr{O}_{\mathscr{X},x}).$$

Fix a geometric point $\bar{y} \to X_{(x)}$. Using the natural dense open immersion $X_{(x)} \hookrightarrow P_{n,x}$, we get a surjection

$$\pi_1(X_{(x)}, \bar{y}) \twoheadrightarrow \pi_1(P_{n,x}, \bar{y}).$$

On the other hand, the base change

$$I_n \times_{\mathscr{X}} \mathrm{Spec}(\mathscr{O}_{\mathscr{X},x}) \to P_{n,x}$$

is a finite étale Galois cover with group $GL_r(\mathbb{Z}/\ell^n)$. Fix an isomorphism $\iota_n : \mathcal{F}_{n,\bar{y}} \simeq (\mathbb{Z}/\ell^n)^r$, defining a lifting

$$\tilde{y} : \mathrm{Spec}(k) \to I_n$$

of $\bar{y}$. Since $I_n \times_{\mathscr{X}} \mathrm{Spec}(\mathscr{O}_{\mathscr{X},x})$ is equal to a finite disjoint union of spectra of strictly henselian local rings, we get from the choice of ι_n an identification of $\pi_1(P_{n,x}, \bar{y})$ with a subgroup of $GL_r(\mathbb{Z}/\ell^n)$ (namely the subgroup of elements preserving ι_n).

On the other hand, we also get a homomorphism

$$\rho_{\bar{y}} : \pi_1(X_{(x)}, \bar{y}) \to GL_r(\mathbb{Z}/(\ell^n)),$$

defined as the composite

$$\pi_1(X_{(x)}, \bar{y}) \longrightarrow GL(\mathcal{F}_{n,\bar{y}}) \xrightarrow{\iota_n} GL_r(\mathbb{Z}/\ell^n).$$

It follows from the construction that the image is precisely $\pi_1(\mathscr{P}_n, \bar{y})$. From this we conclude that F extends to $P_{n,x}$ if and only if the image $\mathcal{G}$ of $\pi_1(X_{(x)}, \bar{y})$ in $GL(F_{(x),\bar{y}})$ maps injectively to $GL(\mathcal{F}_{n,\bar{y}})$.

To deduce the proposition from this, simply note that F has finite local monodromy at x if and only if there exists n for which the map

$$\mathcal{G} \to GL(F_{n,\bar{y}})$$

is injective. □

Proposition 6.3. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a quasi-finite dominant morphism of normal Deligne-Mumford stacks of finite type over k. Let $\mathscr{U} \subset \mathscr{Y}$ be a dense open substack, let $\mathscr{W} \subset \mathscr{X}$ be its preimage in $\mathscr{X}$, and let F be a lisse sheaf on $\mathscr{U}$. Let $\mathscr{V} \subset \mathscr{Y}$ (resp. $\mathscr{R} \subset \mathscr{X}$) be the open substack such that a point $y \in \mathscr{Y}(k)$ (resp. $x \in \mathscr{X}(k)$) is in $\mathscr{V}(k)$ (resp. $\mathscr{R}(k)$) if and only if F (resp. f^*F) has finite local monodromy at y (resp. x). Then*

$$\mathscr{R} = f^{-1}(\mathscr{V}).$$

Proof. The assertion is étale local on both $\mathscr{X}$ and $\mathscr{Y}$, so we may assume that both are schemes (and for consistency of notation we write X, Y etc. for $\mathscr{X}$, $\mathscr{Y}$, etc.). Let $x \in X(k)$ be a point mapping to $y \in Y(k)$, and consider the induced map

$$X_{(x)} \to Y_{(y)}.$$

Let $\bar{z} \to X_{(y)}$ be a geometric point. It then suffices to show that the image of the induced homomorphism

$$\pi_1(X_{(x)}, \bar{z}) \to \pi_1(Y_{(y)}, f(\bar{z}))$$

has finite index in $\pi_1(Y_{(y)}, f(\bar{z}))$. Let K (resp. L) denote the fraction field of $\mathscr{O}_{Y,y}$ (resp. $\mathscr{O}_{X,x}$) so we have an inclusion of fields $K \subset L$. To prove that the image of the map on fundamental groups has finite index, we may without loss of generality assume that $\bar{z}$ is given by a separable closure $L \hookrightarrow L^s$. Let $K^s \subset L^s$ be the separable closure of K in L^s. Then we have a commutative diagram

$$\begin{tikzcd} \mathrm{Gal}(L^s/L) \arrow[r] \arrow[d] & \mathrm{Gal}(K^s/K) \arrow[d] \\ \pi_1(X_{(x)}, \bar{z}) \arrow[r] & \pi_1(Y_{(y)}, f(\bar{z})), \end{tikzcd}$$

where the vertical maps are surjections. It therefore suffices to show that the image of the map

$$\mathrm{Gal}(L^s/L) \to \mathrm{Gal}(K^s/K)$$

has finite index. This follows from standard field theory. □

6.4. Let $f : X \to Y$ be a quasi-finite dominant morphism of finite type, separated, and normal connected k-schemes. Let $U \subset X$ be a dense open subset, F a lisse sheaf on U with finite local monodromy at every point of X. Let $V \subset Y$ be a dense open subset such that $f_!F$ is smooth over V.

Proposition 6.5. *The sheaf $(f_!F)|_V$ has finite local monodromy at every point in the image $f(X)$ of X in Y.*

Proof. Let $x \in X(k)$ be a point mapping to a point $y \in Y(k)$. Let K (resp. L) denote the function field of Y (resp. X), and let $K_{(y)}$ (resp. $L_{(x)}$) denote the field of fractions of $\mathscr{O}_{Y,y}$ (resp. $\mathscr{O}_{X,x}$). Fix also an algebraic closure $L_{(x)} \subset \Omega$. So we have a commutative diagram of fields

$$\begin{tikzcd} & & \Omega \\ L \arrow[r, hook] & L_{(x)} \arrow[ur, hook] & \\ K \arrow[r, hook] \arrow[u, hook] & K_{(y)} \arrow[u, hook] & \end{tikzcd}$$

Let $\bar{\eta} \to Y_{(y)}$ denote the geometric generic point defined by the inclusion $\tau : K_{(y)} \subset \Omega$. We then have an isomorphism

$$(L \otimes_K \Omega)_{\mathrm{red}} \simeq \prod_{\sigma} \Omega_{(\sigma)},$$

where the product is taken over embeddings $\sigma : L \hookrightarrow \Omega$ over τ and $\Omega_{(\sigma)}$ is the field Ω (with the subscript as a place holder). The given embedding $\sigma_0 : L \hookrightarrow \Omega$ defines a geometric point of X. Let V denote the representation of $\mathrm{Aut}(\Omega/\sigma_0(L))$ defined by F, and let $\tilde{\sigma}_0 : L_{(x)} \hookrightarrow \Omega$ be the given extension of σ_0. Then by assumption the action of the subgroup $\mathrm{Aut}(\Omega/\tilde{\sigma}_0(L_{(x)})) \subset \mathrm{Aut}(\Omega/\sigma_0(L))$ on V factors through a finite quotient.

Any embedding

$$\sigma : L \hookrightarrow \Omega$$

over τ defines a lifting $\tilde{\eta} : \mathrm{Spec}(\Omega) \to X$ of $\bar{\eta}$. Let V_σ denote the representation of $\mathrm{Aut}(\Omega/\sigma(L))$ defined by F. Then as a vector space we have

$$(f_!F)_{\bar{\eta}} \simeq \prod_{\tilde{\eta}} V_\sigma.$$

Let S denote the set of embeddings $L \hookrightarrow \Omega$ over τ. An element $\alpha \in \mathrm{Aut}(\Omega/K)$ defines an automorphism $S \to S$ by sending σ to $\alpha \circ \sigma$. This defines a homomorphism

$$\mathrm{Aut}(\Omega/K) \to \mathrm{Aut}(S), \quad \alpha \mapsto \bar{\alpha}$$

whose kernel H is the subgroup of elements which fix the compositum of the fields $\sigma(L)$ ($\sigma \in S$). In particular H is contained in $\mathrm{Aut}(\Omega/\sigma_0(L))$. An element $\alpha \in \mathrm{Aut}(\Omega/K)$ sends $V_\sigma \subset (f_!F)_{\bar{\eta}}$ to $V_{\bar{\alpha}(\sigma)}$.

The representation V_σ can be described as follows. Choose an automorphism $g \in \mathrm{Aut}(\Omega/K)$ such that $g \circ \sigma_0 = \sigma$. Then $g(\sigma_0(L)) = \sigma(L)$, and hence we obtain an isomorphism

$$\mathrm{Aut}(\Omega/\sigma(L)) \to \mathrm{Aut}(\Omega/\sigma_0(L)), \quad \alpha \mapsto g^{-1} \circ \alpha \circ g.$$

The representation V_σ is isomorphic to the representation obtained from V through this isomorphism. Let $H_y \subset H$ denote the intersection of H with $\mathrm{Aut}(\Omega/K_{(y)})$. The group H_y is a subgroup of

$$\mathrm{Aut}(\Omega/\sigma_0(L) \cdot K_{(y)}) = \mathrm{Aut}(\Omega/L_{(x)}).$$

In particular, the group H_y acts through a finite quotient on each V_σ. Since H_y has finite index in $\mathrm{Aut}(\Omega/K_{(y)})$, this implies that the action of $\mathrm{Aut}(\Omega/K_{(y)})$ on $(f_!F)_{\bar{\eta}}$ factors through a finite quotient. □

6.6. Consider now a correspondence

$$c : \mathscr{C} \to \mathscr{X} \times \mathscr{X}$$

with $\mathscr{C}$ and $\mathscr{X}$ Deligne-Mumford stacks, and the maps c_1 and c_2 quasi-finite. Assume that $\mathscr{X}$ is normal, let $j : \mathscr{U} \subset \mathscr{X}$ be a dense open subset, and let F be a lisse sheaf on $\mathscr{U}$. Let G denote $R^0 j_* F$, and suppose given a c-structure

$$u : c_{2!}c_1^* G \to G.$$

Proposition 6.7. *Assume that the map u is generically surjective, that $\mathscr{C}$ is irreducible, and that the maps c_i are dominant. Let $\mathscr{V} \subset \mathscr{X}$ denote the open subset characterized by the property that a point $x \in \mathscr{X}(k)$ lies in $\mathscr{V}(k)$ if and only if F has finite local monodromy at x. Then*

$$c_1^{-1}(\mathscr{V}) \subset c_2^{-1}(\mathscr{V}). \tag{6.7.1}$$

Proof. Let $\widetilde{\mathscr{C}} \to \mathscr{C}$ be the normalization of $\mathscr{C}$, and let $\tilde{c}$ denote the composite

$$\widetilde{\mathscr{C}} \longrightarrow \mathscr{C} \xrightarrow{c} \mathscr{X} \times \mathscr{X}.$$

Then G also has a natural $\tilde{c}$-structure (this follows from 2.16), and since $\widetilde{\mathscr{C}} \to \mathscr{C}$ is surjective it suffices to consider $\widetilde{\mathscr{C}}$. We may therefore assume that $\mathscr{C}$ is normal.

By 2.16 we can freely replace $\mathscr{C}$ by an open substack containing $c_1^{-1}(\mathscr{V})$, so we may assume that in fact $\mathscr{C} = c_1^{-1}(\mathscr{V})$. So the sheaf $c_1^* G$ has finite local monodromy everywhere on $\mathscr{C}$. Let $\mathscr{W} \subset \mathscr{Y}$ be an open substack such that the restriction of $c_{2!}c_1^* G$ to $\mathscr{W}$ is lisse. Then $(c_{2!}c_1^* G)|_{\mathscr{W}}$ has finite local monodromy at every point of $c_2(\mathscr{C})$ by 6.5. Since this sheaf also surjects onto $F|_{\mathscr{W}}$ we conclude that F also has finite local monodromy at every point of $c_2(\mathscr{C})$. This implies the inclusion (6.7.1). □

7. Proof of theorem 1.9

7.1. For a complex $K \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$ on an algebraic space X, let K^{ss}, called the *semi-simplification of* K, denote the sheaf

$$K^{ss} := \oplus_{i\in\mathbb{Z}}\mathscr{H}^i(K).$$

If

$$c : C \to X \times X$$

is a correspondence with c_2 quasi-finite, then since c_1^* and $c_{2!}$ are exact functors, any c-structure

$$u : c_{2!}c_1^*K \to K$$

induces for every $i \in \mathbb{Z}$ a c-structure

$$u_i : c_{2!}c_1^*\mathscr{H}^i(K) \to \mathscr{H}^i(K).$$

Define

$$u^{ss} : c_{2!}c_1^*K^{ss} \to K^{ss}$$

to be the c-structure given by

$$u^{ss} := \oplus_i(-1)^iu_i.$$

Lemma 7.2. *For any $y \in \mathrm{Fix}(c)(k)$ we have*

$$\mathrm{lt}_y(K, u) = \mathrm{lt}_y(K^{ss}, u^{ss}), \quad \mathrm{lt}_y(DK, u^t) = \mathrm{lt}_y(D(K^{ss}), (u^{ss})^t).$$

Proof. The equality

$$\mathrm{lt}_y(K, u) = \mathrm{lt}_y(K^{ss}, u^{ss})$$

is immediate. For the second equality, we proceed by induction on the integer n such that there exists $a \in \mathbb{Z}$ such that $K \in D_c^{[a,a+n]}(X)$.

If $n = 0$ we have $K = K^{ss}$ so the result holds in this case. For the inductive step, suppose $K \in D_c^{[a,a+n]}(X)$ and consider the distinguished triangle

$$\tau_{<a+n}K \to K \to \mathscr{H}^{a+n}(K)[-a-n] \to (\tau_{<a+n}K)[1].$$

Note that since the functors c_1^* and $c_{2!}$ are exact, the map

$$u : c_{2!}c_1^*K \to K$$

induces maps

$$u' : c_{2!}c_1^*\tau_{<a+n}K \to \tau_{<a+n}K, \quad u_{a+n} : c_{2!}c_1^*\mathscr{H}^{a+n}(K)[-a-n] \to \mathscr{H}^{a+n}(K)[-a-n]$$

such that the diagram

$$\begin{tikzcd}
c_{2!}c_1^*\tau_{<a+n}K \arrow[r] \arrow[d, "u'"] & c_{2!}c_1^*K \arrow[r] \arrow[d, "u"] & c_{2!}c_1^*\mathscr{H}^{a+n}(K)[-a-n] \arrow[r] \arrow[d, "u_{a+n}"] & c_{2!}c_1^*(\tau_{<a+n}K)[1] \arrow[d, "u'[1]"] \\
\tau_{<a+n}K \arrow[r] & K \arrow[r] & \mathscr{H}^{a+n}(K)[-a-n] \arrow[r] & (\tau_{<a+n}K)[1]
\end{tikzcd}$$

commutes. Dualizing we obtain a morphism of distinguished triangles

$$\begin{tikzcd}
(c^t)_{2!}(c^t)_1^* D(\tau_{<a+n}K) \arrow[d, "u'^t"] & (c^t)_{2!}(c^t)_1^* DK \arrow[l] \arrow[d, "u^t"] & (c^t)_{2!}(c^t)_1^* D\mathscr{H}^{a+n}(K)[-a-n] \arrow[l] \arrow[d, "u^t_{a+n}"] \\
D(\tau_{<a+n}K) & DK \arrow[l] & D(\mathscr{H}^{a+n}(K)[-a-n]). \arrow[l]
\end{tikzcd}$$

By additivity of traces and induction on n we therefore get that

$$\mathrm{lt}_y(DK, u^t) = \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{lt}_y(D(\mathscr{H}^i(K)), u_i^t) = \mathrm{lt}_y(D(K^{ss}), (u^{ss})^t).$$

□

We now begin the proof of 1.9. The proof is by induction on the dimension δ of X.

7.3. **Base case $\delta = 0$.** If X is zero-dimensional, then 1.9 holds in the case when K is a sheaf, as any constructible sheaf on X is lisse so we can apply 4.2. Using 7.2 we therefore also get 1.9 for K a complex in this case.

So now assume that 1.9 holds whenever X has dimension $< \delta$, and suppose given a correspondence

$$c : C \to X \times X$$

with c_1 and c_2 quasi-finite and X of dimension δ. Suppose further that $K \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$ is a complex with a c-structure $u : c_{2!}c_1^*K \to K$. We need to show that

$$\mathrm{lt}_y(K, u) = \mathrm{lt}_y(DK, u^t)$$

for every $y \in \mathrm{Fix}(c)(k)$. We do this through a sequence of reductions.

7.4. **Reduction to the case when K is a sheaf.**

This is immediate from 7.2.

7.5. **Reduction to the case when K is a sheaf and X is smooth and geometrically irreducible.** To prove 1.9 we may work étale locally on X, and may therefore assume that X is affine. By the invariance of the étale site under infinitesimal thickenings, we may further assume that X is reduced. In this case, by Noether normalization [3, 13.3], there exists a finite morphism

$$\pi : X \to Y$$

with Y smooth, geometrically irreducible, and of the same dimension δ as X (in fact we can arrange for Y to be $\mathbb{A}^\delta$).

Let

$$d : C \to Y \times Y$$

be the correspondence with $d_i = \pi \circ c_i$ $(i = 1, 2)$, and let

$$\gamma : \mathrm{Fix}(c) \hookrightarrow \mathrm{Fix}(d)$$

be the natural inclusion of fixed points. By 5.1 the complex $\pi_! K$ has a d-structure $\pi_! u$. Moreover, by 5.3 we have for every $\lambda \in \mathrm{Fix}(d)$ an equality

$$\mathrm{lt}_\lambda(\pi_! K, \pi_! u) = \begin{cases} \mathrm{lt}_\lambda(K, u) & \text{if } \lambda \in \mathrm{Fix}(c), \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, by 5.3 and 5.6 we have

$$\mathrm{lt}_\lambda(D(\pi_! K), (\pi_! u)^t) = \begin{cases} \mathrm{lt}_\lambda(DK, u^t) & \text{if } \lambda \in \mathrm{Fix}(c), \\ 0 & \text{otherwise.} \end{cases}$$

To prove 1.9, we may therefore replace X by Y, and (K, u) by $(\pi_! K, \pi_! u)$.

7.6. **Reduction to the case when K is a torsion free sheaf and X is smooth and geometrically irreducible.** Let Z be any normal Deligne-Mumford stack over k, and let F be a constructible sheaf on Z. Then there is a canonical morphism of constructible sheaves on Z

$$\alpha : F \to \overline{F}$$

defined as follows. Let $j : U \hookrightarrow Z$ be a dense open immersion such that $F|_U$ is a lisse sheaf. Then define

$$\overline{F} := R^0 j_*(F|_U),$$

and let α be the adjunction map. This sheaf $\overline{F}$ and the map α are independent of the choice of $U \subset Z$. Indeed if $s : V \hookrightarrow U$ is the inclusion of an even smaller dense open subset, then the adjunction map

$$F|_U \to R^0 s_*(F|_V)$$

is an isomorphism, since $F|_U$ is a lisse sheaf and Z is normal.

Definition 7.7. We say that a constructible sheaf F on Z is *torsion free* if the map $\alpha : F \to \overline{F}$ is an isomorphism.

Lemma 7.8. *Let F be a torsion free sheaf on Z, and let $j : U \hookrightarrow Z$ be a dense open immersion with $F|_U$ lisse. Then for any constructible sheaf G on Z the natural map*

$$\mathrm{Hom}_Z(G, F) \to \mathrm{Hom}_U(G|_U, F|_U)$$

is an isomorphism

Proof. This is immediate as $F = R^0 j_*(F|_U)$. □

Now consider again our correspondence

$$c : C \to X \times X$$

with c_1 and c_2 quasi-finite and X smooth and irreducible, and the constructible sheaf K with c-structure u. Let $\alpha : K \to \overline{K}$ be the canonical morphism to a torsion free sheaf.

Lemma 7.9. *There exists a dense open subset $j : U \hookrightarrow X$ such that the map induced by α*

$$c_{2!} c_1^* K \to c_{2!} c_1^* \overline{K}$$

is an isomorphism over U.

Proof. Since c_1 is quasi-finite, and X is smooth and irreducible of dimension δ, for any irreducible component $Z \subset C$ either $c_1|_Z : Z \to X$ is dominant, or the dimension of Z is strictly less than δ. It follows that there exists an open subset $V \subset C$ such that the map

$$c_1^* K \to c_1^* \overline{K}$$

is an isomorphism over V, and such that the dimension of every irreducible component of $C - V$ is strictly less than δ. Combining this with the fact that c_2 is quasi-finite, we conclude that the closed subset

$$\overline{c_2(C - V)} \subset X$$

is a proper closed subset. Let $U \subset X$ be its complement. Then $c_2^{-1}(U) \subset V$, which gives the lemma. □

We get a c-structure on $\overline{K}$ by choosing $j : U \hookrightarrow X$ as in 7.9, and then noting that by 7.8 we have

$$\mathrm{Hom}(c_{2!}c_1^*K, \overline{K}) \simeq \mathrm{Hom}((c_{2!}c_1^*K)|_U, \overline{K}|_U) \simeq \mathrm{Hom}((c_{2!}c_1^*\overline{K})|_U, \overline{K}|_U) \simeq \mathrm{Hom}(c_{2!}c_1^*\overline{K}, \overline{K}).$$

We define

$$\bar{u} : c_{2!}c_1^*\overline{K} \to \overline{K}$$

to be the map obtained from these isomorphisms from the composite

$$c_{2!}c_1^*K \xrightarrow{u} K \xrightarrow{\alpha} \overline{K}.$$

Note that the diagram

$$\begin{tikzcd}
c_{2!}c_1^*K \arrow[r, "\alpha"] \arrow[d, "u"] & c_{2!}c_1^*\overline{K} \arrow[d, "\bar{u}"] \\
K \arrow[r, "\alpha"] & \overline{K}
\end{tikzcd}$$

commutes. In particular if T (resp. T') denotes the kernel (resp. cokernel) of α, then T (resp. T') obtains an induced c-structure u_T (resp. $u_{T'}$).

Let

$$\mathcal{T} := T \oplus T'[-1]$$

and let

$$u_{\mathcal{T}} : c_{2!}c_1^*\mathcal{T} \to \mathcal{T}$$

be the c-structure given by $u_T \oplus u_{T'}$. We then have a distinguished triangle

$$\mathcal{T} \to K \to \overline{K} \to \mathcal{T}[1]$$

such that the diagram

$$\begin{tikzcd}
c_{2!}c_1^*\mathcal{T} \arrow[r] \arrow[d, "u_{\mathcal{T}}"] & c_{2!}c_1^*K \arrow[r] \arrow[d, "u"] & c_{2!}c_1^*\overline{K} \arrow[r] \arrow[d, "\bar{u}"] & c_{2!}c_1^*\mathcal{T}[1] \arrow[d, "u_{\mathcal{T}}[1]"] \\
\mathcal{T} \arrow[r] & K \arrow[r] & \overline{K} \arrow[r] & \mathcal{T}[1]
\end{tikzcd}$$

commutes. It follows that for any $\lambda \in \mathrm{Fix}(c)(k)$ we have

$$\mathrm{lt}_\lambda(K, u) = \mathrm{lt}_\lambda(\overline{K}, \bar{u}) + \mathrm{lt}_\lambda(\mathcal{T}, u_{\mathcal{T}}).$$

Similarly we have

$$\mathrm{lt}_\lambda(DK, u^t) = \mathrm{lt}_\lambda(D\overline{K}, \bar{u}^t) + \mathrm{lt}_\lambda(D\mathcal{T}, u^t_{\mathcal{T}}).$$

On the other hand, the complex $\mathcal{T}$ is supported on a proper closed subspace $i : Z \hookrightarrow X$. Let $\mathcal{T}_Z$ denote the restriction of $\mathcal{T}$ to Z so $\mathcal{T} = i_*\mathcal{T}_Z$. Furthermore, by 2.5 the c-structure $u_{\mathcal{T}}$ is obtained by applying i_* from a unique c_Z-structure

$$v : c^*_{Z1}\mathcal{T}_Z \to c^!_{Z2}\mathcal{T}_Z,$$

where $c_Z : C_Z \to Z \times Z$ is the pullback of c. By induction we therefore have

$$\mathrm{lt}_\lambda(D\mathcal{T}, u^t_{\mathcal{T}}) = \mathrm{lt}_\lambda(\mathcal{T}, u_{\mathcal{T}}).$$

To prove the equality

$$\mathrm{lt}_\lambda(K, u) = \mathrm{lt}_\lambda(DK, u^t),$$

it therefore suffices to show that

$$\mathrm{lt}_\lambda(\overline{K}, \bar{u}) = \mathrm{lt}_\lambda(D\overline{K}, \bar{u}^t).$$

In addition to the reductions made so far, we may therefore further assume that K is torsion free.

7.10. **Proof in the case when X is smooth and irreducible, K is torsion free, and K has finite local monodromy everywhere.** Let $j : U \subset X$ be an open subset such that $K|_U$ is lisse. As in 6.2, choose a stack $\pi : \mathscr{X} \to X$ with the following properties:

(1) π identifies X with the coarse moduli space of $\mathscr{X}$, and π is an isomorphism over U.
(2) If $\tilde{j} : U \hookrightarrow \mathscr{X}$ is the inclusion, then the sheaf $\mathcal{F} := R^0\tilde{j}_*(K|_U)$ is a lisse sheaf on $\mathscr{X}$.

Let $\mathscr{C}$ denote the fiber product

$$\mathscr{C} := C \times_{X\times X} (\mathscr{X} \times \mathscr{X}),$$

so we obtain a correspondence

$$d : \mathscr{C} \to \mathscr{X} \times \mathscr{X}.$$

There is a d-structure

$$v : d_{2!}d_1^*\mathcal{F} \to \mathcal{F}$$

defined as follows. Note that as usual if $V \subset \mathscr{X}$ is a dense open substack, then the natural map

$$\mathrm{Hom}(d_{2!}d_1^*\mathcal{F}, \mathcal{F}) \to \mathrm{Hom}((d_{2!}d_1^*\mathcal{F})|_V, \mathcal{F}|_V)$$

is an isomorphism. It therefore suffices to specify the d-structure on a dense open substack of $\mathscr{X}$. Now observe that if $\mathscr{Z} \subset \mathscr{X}$ is the complement (with the reduced structure) of the maximal open substack $V \subset \mathscr{X}$ which is an algebraic space, then the dimension of $\mathscr{Z}$ is strictly less than δ. As in the proof of 7.9, we therefore can find a dense open subset $U \subset \mathscr{X}$ which is an algebraic space (and hence maps isomorphically to a dense open subspace of X), and such that $c_2^{-1}(U)$ is contained in $c_1^{-1}(V)$. It follows that over U we have

$$(d_{2!}d_1^*\mathcal{F})|_U \simeq (c_{2!}c_1^*K)|_U, \quad \mathcal{F}|_U \simeq K|_U.$$

The c-structure u, therefore defines a d-structure v on $\mathcal{F}$.

7.11. By 4.2 (the case of a lisse sheaf on a stack!), for any $\lambda \in \mathrm{Fix}(d)(k)$, we have

$$\mathrm{lt}_\lambda(\mathcal{F}, v) = \mathrm{lt}_\lambda(D\mathcal{F}, v^t).$$

We use this equality to get the corresponding statement for our original sheaf with c-structure (K, u).

Let C' denote the coarse moduli space of $\mathscr{C}$, so we have a factorization

$$\mathscr{C} \xrightarrow{a} C' \xrightarrow{b} C.$$

Since the diagram

$$\begin{tikzcd} \mathscr{C} \arrow[r] \arrow[d] & \mathscr{X} \times \mathscr{X} \arrow[d, "\pi\times\pi"] \\ C \arrow[r, "c"] & X \times X \end{tikzcd}$$

is cartesian and $\pi \times \pi$ identifies $X \times X$ with the coarse moduli space of $\mathscr{X} \times \mathscr{X}$, the map b is a universal homeomorphism (indeed since π is a proper quasi-finite morphism which is a bijection on geometric points, the map b is a proper morphism which is a bijection on geometric points, whence b is a radicial surjective morphism). It follows that pullback along b identifies the étale sites of C and C'. In particular, if

$$c' : C' \to X \times X$$

denotes $c \circ b$, then K has a c'-structure

$$u' : c_1'^* K \to c_2'^! K$$

given by

$$c_1'^* K \xrightarrow{\simeq} b^* c_1^* K \xrightarrow{u} b^* c_2^! K \xrightarrow{b^* \simeq b^!} c_2'^! K.$$

Lemma 7.12. *(i) The map*

$$\gamma : \mathrm{Fix}(c')(k) \to \mathrm{Fix}(c)(k)$$

is a bijection.

(ii) For any $y \in \mathrm{Fix}(c')(k)$ we have

$$\mathrm{lt}_y(K, u') = \mathrm{lt}_{\gamma(y)}(K, u), \quad \mathrm{lt}_y(DK, (u')^t) = \mathrm{lt}_{\gamma(y)}(DK, u^t).$$

Proof. This is immediate. □

Replacing C by C', we may therefore assume that C is the coarse moduli space of $\mathscr{C}$.

Note that since $\pi_! \simeq \pi_*$, there is a natural map

$$\beta : K \to \pi_! \mathcal{F}.$$

Lemma 7.13. *The map β is an isomorphism.*

Proof. This is clear, for if $j : U \hookrightarrow X$ is the inclusion of an open subset over which π is an isomorphism, then $R^0 j_* = \pi_* \circ R^0 \tilde{j}_*$. □

7.14. Let
$$\tilde{c} : \mathscr{C} \to X \times X$$
denote the composite
$$\mathscr{C} \xrightarrow{d} \mathscr{X} \times \mathscr{X} \xrightarrow{\pi \times \pi} X \times X.$$

Then by 5.6 and 5.8, for every $\lambda \in \mathrm{Fix}(\tilde{c})(k)$ we have
$$\mathrm{lt}_\lambda(K, \pi_! v) = \mathrm{lt}_\lambda(DK, (\pi_! v)^t).$$
By 2.21, this in turn implies that for every $y \in \mathrm{Fix}(c)(k)$, we have
$$\mathrm{lt}_y(K, s) = \mathrm{lt}_y(DK, s^t),$$
where s is the c-structure on K obtained from $\pi_! v$ under the bijection (2.20.1). To complete the proof of 1.9 in the case when K has finite local monodromy everywhere, it therefore suffices to note that $s = u$, which is immediate as these two maps $c_{2!}c_2^*K \to K$ become equal on a dense open subspace of X, and K is torsion free.

This completes the proof of 1.9 in the case when K is a generically pure torsion free sheaf, X is smooth and geometrically irreducible, and K has finite local monodromy at every point of X.

7.15. Finally we prove 1.9 under the assumption that X is smooth and irreducible, and that K is torsion free. Let $j : U \hookrightarrow X$ be a dense open subset over which K is lisse, so that $K = R^0 j_*(K|_U)$.

7.16. **Reduction to the case when X is smooth and irreducible, K is torsion free, and u and u^t are generically surjective.** Suppose the map
$$u : c_{2!}c_1^*K \to K$$
is not generically surjective. Let $I \subset K$ be the image, and let $I \to \overline{I}$ be the universal map to a torsion free sheaf. Since K is torsion free, the inclusion $I \subset K$ extends uniquely to an inclusion $\overline{I} \hookrightarrow K$, and since u is not generically surjective the sheaf $\overline{I}$ has generic rank strictly smaller than the generic rank of K. Let
$$u' : c_{2!}c_1^*\overline{I} \to \overline{I}$$
be the map obtained from the composite
$$c_{2!}c_1^*\overline{I} \hookrightarrow c_{2!}c_1^*K \xrightarrow{u} I \subset \overline{I}.$$

Then the local terms of $(\overline{I}, u')$ and (K, u) are equal, as are those of their duals, so we can replace (K, u) by $(\overline{I}, u')$. Proceeding by induction on the generic rank of K, we may therefore assume that u is generically surjective.

Let G denote the dual of the lisse sheaf $K|_U$. Then G has a c_U^t-structure given by a map
$$u^t : c^t_{U2,!}c^{t*}_{U1}G \to G.$$
We say that u^t *is generically surjective* if this map is generically surjective.

Suppose this is not the case. As above, let $J \subset G$ be the torsion free subsheaf characterized by the property that its restriction to some dense open of U is equal to the image of u^t. Then J is a lisse sheaf, and we have a factorization of u^t as

$$c^t_{U2,!}c^{t*}_{U1}G \to J \hookrightarrow G.$$

Let L be the dual of J so we have a surjection

$$K|_U \twoheadrightarrow L.$$

Then by construction the map

$$c_U : c_{U2!}c^*_{U1}K \to K$$

factors as

$$c_{U2!}c^*_{U1}K|_U \twoheadrightarrow c_{U2!}c^*_{U1}L \xrightarrow{\chi} K|_U.$$

Let $\mathscr{L}$ denote the sheaf R^0j_*L, and let

$$v : c_{2!}c^*_1\mathscr{L} \to \mathscr{L}$$

be the c-structure obtained from the c_U-structure on L given by

$$c_{U2!}c^*_{U1}L \xrightarrow{\chi} K|_U \twoheadrightarrow L.$$

We then have a map (which need no longer be a surjection)

$$\rho : K \to \mathscr{L}$$

such that the diagram

$$\begin{tikzcd} c_{2!}c^*_1K \arrow[r, "\rho"] \arrow[d, "u"] & c_{2!}c^*_1\mathscr{L} \arrow[d, "v"] \arrow[dl, "\chi"'] \\ K \arrow[r, "\rho"] & \mathscr{L} \end{tikzcd}$$

commutes. From this it follows that the local terms of (K, u) and $(\mathscr{L}, v)$ agree, and the same holds for their duals. We can therefore replace (K, u) by $(\mathscr{L}, v)$, and $\mathscr{L}$ has strictly smaller generic rank than K (since we assumed that u^t was not surjective). Proceeding by induction on the generic rank of K we may therefore further assume that u^t is generically surjective.

7.17. **Reduction to the case when X is smooth and irreducible, K is torsion free, u and u^t are generically surjective, C is normal and irreducible, and the maps c_i are dominant.** We may without loss of generality assume that C is reduced. Let $\widetilde{C} \to C$ be its normalization, and let

$$\tilde{c} : \widetilde{C} \to X \times X$$

be the induced correspondence. Then by 2.16 the c-structure on K extends to a $\tilde{c}$-structure on K, and using 2.8 and 2.9 it suffices to prove the result for $\widetilde{C}$. We may therefore assume that C is irreducible.

If the projections $C \to X$ are not dominant, then the c-structure u must be zero, so we may further assume that the maps c_i are dominant.

7.18. **Completion of proof of 1.9.** We now finish the proof of 1.9, under the assumptions of 7.17.

By 6.2, there exists a maximal dense open subset $\alpha : U \subset X$ such that $K|_U$ has finite local monodromy everywhere. Let $Z \subset X$ be the complement with the reduced structure. By 6.7 (applied to both c and c^t) we have

$$c_1^{-1}(Z) = c_2^{-1}(Z), \quad c_1^{-1}(U) = c_2^{-1}(U).$$

We need to show that for a point $y \in \mathrm{Fix}(c)$ with image in Z we have

$$\mathrm{lt}_y(K, u) = \mathrm{lt}_y(DK, u^t).$$

Let $j : V \hookrightarrow X$ be a dense open subset such that $K|_V$ is lisse. By definition of a lisse $\overline{\mathbb{Q}}_\ell$-sheaf, there exists a finite subextension $\mathbb{Q}_\ell \subset M \subset \overline{\mathbb{Q}}_\ell$ with ring of integers $\Lambda \subset M$, and a lisse, torsion free sheaf of Λ-modules on V such that $F \otimes_\Lambda \overline{\mathbb{Q}}_\ell \simeq K|_V$. For $n \geq 1$ let Λ_n denote $\Lambda/\ell^n\Lambda$. After possibly replacing u by $\ell^m u$ for some $m \geq 0$ and M by a bigger subextension we may assume that u is induced by a c-structure (which we denote by the same letter u)

$$u : c_{2!}c_1^* R^0 j_* F \to R^0 j_* F.$$

In this case the local terms

$$\mathrm{lt}_y(K, u), \quad \mathrm{lt}_y(DK, u^t)$$

are in Λ. We show that for all $n \geq 1$ we have

$$\mathrm{lt}_y(K, u) \equiv \mathrm{lt}_y(DK, u^t) \pmod{\ell^n}. \tag{7.18.1}$$

For $n \geq 1$ let $T_n \to X$ denote the normalization of X in the finite étale V-scheme

$$\underline{\mathrm{Isom}}(\Lambda_n^r, F \otimes \mathbb{Z}/(\ell^n)),$$

where r is the rank of F. There is a natural action of $GL_r(\Lambda_n)$ on T_n, and we define

$$\pi : \mathscr{X}_n \to X$$

to be the quotient

$$\mathscr{X}_n := [T_n/GL_r(\mathbb{Z}/\ell^n)].$$

Let

$$c^{(n)} : \mathscr{C}^{(n)} \to \mathscr{X}_n \times \mathscr{X}_n$$

be the fiber product of the diagram

$$\begin{array}{ccc} & & \mathscr{X}_n \times \mathscr{X}_n \\ & & \downarrow{\scriptstyle \pi\times\pi} \\ C & \xrightarrow{c} & X \times X. \end{array}$$

Also define

$$\alpha_n : \mathscr{U}_n \hookrightarrow \mathscr{X}_n, \quad (\text{resp. } \beta_n : \mathscr{Z}_n \hookrightarrow \mathscr{X}_n)$$

to be the preimage of U (resp. Z), where $\mathscr{Z}_n$ is given the reduced structure. Since K has finite local monodromy over U, there exists an integer n_0 such that for every $n \geq n_0$ the sheaf F over V extends uniquely to a lisse sheaf of Λ-modules $\mathscr{F}^{(n)}$ on $\mathscr{U}_n$ (see the proof of 6.2).

We can compute the local terms of K and DK using $K^{(n)} := R^0\alpha_{n*}\mathscr{F}^{(n)}$ as follows. Let $u^{(n)}$ denote the unique $c^{(n)}$-structure on $K^{(n)}$ extending $u|_V$.

Note first that the natural map

$$\mathrm{Fix}(c^{(n)})(k) \to \mathrm{Fix}(c)(k)$$

induces a bijection on isomorphism classes (this is immediate from the definition of $c^{(n)}$ and the fact that X is the coarse moduli space of $\mathscr{X}$). Let $\lambda \in \mathrm{Fix}(c^{(n)})(k)$ be a lifting of $y \in \mathrm{Fix}(c)(k)$. Then by 5.6 and 5.8 we have

$$\widetilde{\mathrm{lt}}_\lambda(K^{(n)}, u^{(n)}) = \mathrm{lt}_y(K, u),$$

and

$$\widetilde{\mathrm{lt}}_\lambda(DK^{(n)}, u^{(n),t}) = \mathrm{lt}_y(DK, u^t).$$

To show the congruence (7.18.1) it therefore suffices to show that

$$\widetilde{\mathrm{lt}}_\lambda(K^{(n)}, u^{(n)}) \equiv \widetilde{\mathrm{lt}}_\lambda(DK^{(n)}, u^{(n),t}) \pmod{\ell^n}.$$

Set

$$Q^{(n)} := \tau_{\geq 1} R\alpha_{n*}\mathscr{F}^{(n)},$$

so we have a distinguished triangle

$$K^{(n)} \to R\alpha_{n*}\mathscr{F}^{(n)} \to Q^{(n)} \to K^{(n)}[1].$$

Since $\mathscr{U}_n$ is $c^{(n)}$-invariant we also have a $c^{(n)}$-structure v (resp. w) on $R\alpha_{n*}\mathscr{F}^{(n)}$ (resp. $Q^{(n)}$) such that the diagram

$$\begin{tikzcd}
c^{(n)}_{2!}c^{(n)*}_1K^{(n)} \arrow[r] \arrow[d, "u^{(n)}"] & c^{(n)}_{2!}c^{(n)*}_1R\alpha_{n*}\mathscr{F}^{(n)} \arrow[r] \arrow[d, "v"] & c^{(n)}_{2!}c^{(n)*}_1Q^{(n)} \arrow[r] \arrow[d, "w"] & c^{(n)}_{2!}c^{(n)*}_1K^{(n)}[1] \arrow[d, "u^{(n)}[1]"] \\
K^{(n)} \arrow[r] & R\alpha_{n*}\mathscr{F}^{(n)} \arrow[r] & Q^{(n)} \arrow[r] & K^{(n)}[1]
\end{tikzcd}$$

commutes. It follows that

$$\widetilde{\mathrm{lt}}_\lambda(K^{(n)}, u^{(n)}) = \widetilde{\mathrm{lt}}_\lambda(R\alpha_{n*}\mathscr{F}^{(n)}, v) - \widetilde{\mathrm{lt}}_\lambda(Q^{(n)}, w),$$

and by dualizing

$$\widetilde{\mathrm{lt}}_\lambda(DK^{(n)}, u^{(n),t}) = \widetilde{\mathrm{lt}}_\lambda(DR\alpha_{n*}\mathscr{F}^{(n)}, v^t) - \widetilde{\mathrm{lt}}_\lambda(DQ^{(n)}, w^t).$$

Now we have

$$DR\alpha_{n*}\mathscr{F}^{(n)} \simeq R\alpha_{n!}D\mathscr{F}^{(n)},$$

which is supported on $\mathscr{U}_n$. Therefore

$$\widetilde{\mathrm{lt}}_\lambda(DR\alpha_{n*}\mathscr{F}^{(n)}, v^t) = 0$$

and we have

$$\widetilde{\mathrm{lt}}_\lambda(DK^{(n)}, u^{(n),t}) = -\widetilde{\mathrm{lt}}_\lambda(DQ^{(n)}, w^t).$$

On the other hand, set

$$Q := \tau_{\geq 1} R\alpha_*(K|_U),$$

and let $\bar{w}$ be the natural c-structure on Q.

By 5.6 and 5.8 we have

$$\widetilde{\mathrm{lt}}_\lambda(Q^{(n)}, w) = \mathrm{lt}_y(\pi_!Q^{(n)}, \pi_!w) = \mathrm{lt}_y(Q, \bar{w}),$$

and

$$\widetilde{\mathrm{lt}}_\lambda(DQ^{(n)}, w^t) = \mathrm{lt}_y(\pi_! DQ^{(n)}, \pi_! w^t) = \mathrm{lt}_y(DQ, \bar{w}^t),$$

and these two numbers are equal by induction. We conclude that to prove that

$$\widetilde{\mathrm{lt}}_\lambda(K^{(n)}, u^{(n)}) \equiv \widetilde{\mathrm{lt}}_\lambda(DK^{(n)}, u^{(n),t}) \pmod{\ell^n},$$

it suffices to show that

$$\widetilde{\mathrm{lt}}_\lambda(R\alpha_{n*}\mathscr{F}^{(n)}, v) \equiv 0 \pmod{\ell^n}.$$

Equivalently, by 2.14 we need to show that

$$\widetilde{\mathrm{lt}}_\lambda(R\alpha_{n*}(\mathscr{F}^{(n)} \otimes^{\mathbb{L}} (\mathbb{Z}/\ell^n), v) \in \Lambda_n$$

is zero. On the other hand, by construction the sheaf $\mathscr{F}^{(n)} \otimes^{\mathbb{L}} (\mathbb{Z}/\ell^n)$ extends to a lisse sheaf $\mathscr{G}^{(n)}$ of $\mathbb{Z}/(\ell^n)$-modules on all of $\mathscr{X}_n$. We therefore have

$$R\alpha_{n*}(\mathscr{F}^{(n)} \otimes^{\mathbb{L}} (\mathbb{Z}/\ell^n)) \simeq \mathscr{G}^{(n)} \otimes^{\mathbb{L}} R\alpha_{n*}(\mathbb{Z}/\ell^n).$$

Let

$$\tilde{v} : c_{2!}^{(n)} c_1^{(n)*}(\mathscr{G}^{(n)} \otimes^{\mathbb{L}} R\alpha_{n*}(\mathbb{Z}/\ell^n)) \to \mathscr{G}^{(n)} \otimes^{\mathbb{L}} R\alpha_{n*}(\mathbb{Z}/\ell^n)$$

denote the $c^{(n)}$-structure induced by pushing forward v, let

$$a : c_{2!}^{(n)} c_1^{(n)*}\mathscr{G}^{(n)} \to \mathscr{G}^{(n)}$$

be the $c^{(n)}$-structure on $\mathscr{G}^{(n)}$ obtained by applying $\mathscr{H}^0$ to $\tilde{v}$, and let

$$b : c_{2!}^{(n)} c_1^{(n)*} R\alpha_{n*}\mathbb{Z}/(\ell^n) \to R\alpha_{n*}\mathbb{Z}/(\ell^n)$$

be the canonical $c^{(n)}$-structure (see 2.17). Then the diagram

$$\begin{tikzcd}
c_{2!}^{(n)}(c_1^{(n)*}\mathscr{G}^{(n)}) \otimes^{\mathbb{L}} c_{2!}^{(n)} c_1^{(n)*} R\alpha_{n*}\mathbb{Z}/(\ell^n) \arrow[d] \arrow[dr, "a\otimes b"] & \\
c_{2!}^{(n)} c_1^{(n)*}(\mathscr{G}^{(n)} \otimes^{\mathbb{L}} R\alpha_{n*}(\mathbb{Z}/\ell^n)) \arrow[r, "\tilde{v}"] & \mathscr{G}^{(n)} \otimes^{\mathbb{L}} R\alpha_{n*}(\mathbb{Z}/\ell^n)
\end{tikzcd}$$

commutes, where the vertical arrow is the natural morphism (see (2.18.1)). Indeed this can be verified after restricting to $\mathscr{U}_n$ where it is immediate.

By 2.19, it therefore suffices to show that

$$\widetilde{\mathrm{lt}}_\lambda(R\alpha_{n*}\mathbb{Z}/(\ell^n), b) = 0.$$

For this in turn it suffices (using 2.14) to show that

$$\widetilde{\mathrm{lt}}_\lambda(R\alpha_{n*}\mathbb{Z}_\ell, b) = 0,$$

and for this it suffices to show that

$$\widetilde{\mathrm{lt}}_\lambda(R\alpha_{n*}\overline{\mathbb{Q}}_\ell, b) = 0.$$

By the discussion above with $\mathscr{F}^{(n)}$ replaced by $\overline{\mathbb{Q}}_\ell$ this vanishing is equivalent to the equality

$$\widetilde{\mathrm{lt}}_\lambda(\overline{\mathbb{Q}}_\ell, b) = \widetilde{\mathrm{lt}}_\lambda(\Omega_{\mathscr{X}_n}, b^t),$$

which follows from the case of a lisse sheaf 4.2.

This completes the proof of 1.9. $\square$

Part II: Compatible systems and the standard operations

Throughout part II of this paper, we work over a finite field $\mathbb{F}_q$ with q elements and characteristic p (so $q = p^a$ for some a). We fix an algebraic closure $\mathbb{F}_q \hookrightarrow k$. We usually denote a scheme (or stack) over $\mathbb{F}_q$ with a subscript "0", and its base change to k by the corresponding unadorned symbol (so X_0 denotes a scheme over $\mathbb{F}_q$ and X denotes the base change $X_0 \otimes_{\mathbb{F}_q} k$).

8. Twisting by Frobenius

8.1. Let $\mathscr{X}_0$ be a separated Deligne-Mumford stack of finite type over $\mathbb{F}_q$, and let

$$F_{\mathscr{X}} : \mathscr{X} \to \mathscr{X}$$

denote the relative Frobenius of $\mathscr{X}/k$.

Definition 8.2. An *ℓ-adic Weil complex* on $\mathscr{X}$ (or just *Weil complex* if the reference to ℓ is clear) is a pair (K, ϵ), where $K \in D^b_c(\mathscr{X}, \overline{\mathbb{Q}}_\ell)$ and

$$\epsilon : F^*_{\mathscr{X}} K \to K$$

is an isomorphism.

Let

$$c : \mathscr{C}_0 \to \mathscr{X}_0 \times \mathscr{X}_0$$

be a correspondence over $\mathbb{F}_q$, with $\mathscr{C}_0$ and $\mathscr{X}_0$ Deligne-Mumford stacks.

Definition 8.3. A *c-structure* on a Weil complex (K, ϵ) is a morphism

$$u : c_1^* K \to c_2^! K$$

such that the diagram

(8.3.1)
$$\begin{tikzcd}
c_1^* F^*_{\mathscr{X}} K \arrow[r, "\epsilon"] \arrow[d, "\simeq"] & c_1^* K \arrow[d, "u"] \\
F^*_{\mathscr{C}} c_1^* K \arrow[d, "u"] & c_2^! K \\
F^*_{\mathscr{C}} c_2^! K \arrow[r] & c_2^! F^*_{\mathscr{X}} K \arrow[u, "\epsilon"]
\end{tikzcd}$$

commutes, where the bottom horizontal arrow is obtained from the natural isomorphisms $F^*_{\mathscr{X}} \simeq F^!_{\mathscr{X}}$ and $F^*_{\mathscr{C}} \simeq F^!_{\mathscr{C}}$.

8.4. For natural numbers $n, m \in \mathbb{N}$ we write $c^{(n,m)}$ for the correspondence

$$\begin{tikzcd}
 & & \mathscr{C}_0 \arrow[dl, "c_1"'] \arrow[dr, "c_2"] & & \\
 & \mathscr{X}_0 \arrow[dl, "F^n_{\mathscr{X}_0}"'] & & \mathscr{X}_0 \arrow[dr, "F^m_{\mathscr{X}_0}"] & \\
\mathscr{X}_0 & & & & \mathscr{X}_0.
\end{tikzcd}$$

If (K, ϵ, u) is an ℓ-adic Weil sheaf with c-structure, then for every $n, m \in \mathbb{N}$ the complex K has a natural $c^{(n,m)}$-structure $u^{(n,m)}$ defined as the composite

$$c_1^{(n,m)*}K \xrightarrow{\simeq} c_1^* F_{\mathscr{X}}^{n*}K \xrightarrow{\epsilon^n} c_1^*K \xrightarrow{u} c_2^!K \xrightarrow{\epsilon^{-m}} c_2^! F_{\mathscr{X}}^{m*}K \xrightarrow{\simeq} c_2^{(n,m)!}K,$$

where the last isomorphism is obtained from the canonical isomorphism $F_{\mathscr{X}}^! \simeq F_{\mathscr{X}}^*$ (since $F_{\mathscr{X}}$ is a universal homeomorphism).

8.5. Let

$$f : \mathscr{X}_0 \to \mathscr{Y}_0$$

be a proper morphism between separated Deligne-Mumford stacks over $\mathbb{F}_q$. Let

$$c : \mathscr{C}_0 \to \mathscr{X}_0 \times \mathscr{X}_0$$

be a correspondence, and let

$$d : \mathscr{C}_0 \to \mathscr{Y}_0 \times \mathscr{Y}_0$$

be the composite

$$\mathscr{C}_0 \xrightarrow{c} \mathscr{X}_0 \times \mathscr{X}_0 \xrightarrow{f \times f} \mathscr{Y}_0 \times \mathscr{Y}_0.$$

Let (K, ϵ, u) be an ℓ-adic Weil sheaf with c-structure on $\mathscr{X}$. We give $f_!K \in D_c^b(\mathscr{Y}, \overline{\mathbb{Q}}_\ell)$ the structure of a Weil sheaf with d-structure as follows.

Consider the commutative diagram

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "g"] \arrow[rr, bend left, "F_{\mathscr{X}}"] \arrow[dr, "f"'] & \mathscr{X}' \arrow[r, "\pi"] \arrow[d, "f'"] & \mathscr{X} \arrow[d, "f"] \\
 & \mathscr{Y} \arrow[r, "F_{\mathscr{Y}}"] & \mathscr{Y},
\end{tikzcd}$$

where the square is cartesian, and g is the relative Frobenius of $\mathscr{X}/\mathscr{Y}$. We then get an isomorphism

$$F_{\mathscr{Y}}^* f_!K \xrightarrow{bc} f'_!\pi^*K \xrightarrow{\mathrm{id} \to g_*g^*} f'_!g_*g^*\pi^*K \xrightarrow{g_* \simeq g_!} f_!F_{\mathscr{X}}^*K \xrightarrow{\epsilon} f_!K,$$

which we denote by $f_!\epsilon$ (here the adjunction map $\mathrm{id} \to g_*g^*$ is an isomorphism since g is a universal homeomorphism).

As in 5.1, define

$$f_!u : d_1^*f_!K \to d_2^!f_!K$$

to be the composite map

$$\begin{array}{rclr}
d_1^*f_!K & \simeq & c_1^*f^*f_*K & (f_! \simeq f_* \text{ since } f \text{ is proper}) \\
 & \to & c_1^*K & (f^*f_* \to \mathrm{id}) \\
 & \xrightarrow{u} & c_2^!K & \\
 & \to & c_2^!f^!f_!K & (\mathrm{id} \to f^!f_!) \\
 & \simeq & d_2^!f_!K. &
\end{array}$$

Lemma 8.6. *The data $(f_!K, f_!\epsilon, f_!u)$ is a Weil complex with d-structure.*

Proof. We need to prove that the diagram

$$\begin{tikzcd}
d_1^*F_{\mathscr{Y}}^*f_!K \arrow[r, "f_!\epsilon"] \arrow[d, "\simeq"] & d_1^*f_!K \arrow[d, "f_!u"] \\
F_{\mathscr{C}}^*d_1^*f_!K \arrow[d, "f_!u"] & d_2^!f_!K \\
F_{\mathscr{C}}^*d_2^!f_!K \arrow[r] & d_2^!F_{\mathscr{Y}}^*f_!K \arrow[u, "f_!\epsilon"']
\end{tikzcd}$$

commutes. Expanding out the definitions of $f_!\epsilon$ and $f_!u$, this amounts to showing that the big outside diagram in the following diagram commutes:

(8.6.1)
$$\begin{tikzcd}
c_1^*f^*F_{\mathscr{Y}}^*f_!K \arrow[r, "bc"] \arrow[d, "\simeq"] \arrow[dr] & c_1^*f^*f_!'\pi^*K \arrow[r, "\mathrm{id}\to g_*g^*"] & c_1^*f^*f_!'g_*g^*\pi^*K \arrow[r, "\simeq"] & c_1^*f^*f_!F_{\mathscr{X}}^*K \arrow[d, "\epsilon"] \\
F_{\mathscr{C}}^*c_1^*f^*f_!K \arrow[d, "f_!\simeq f_*"] & c_1^*F_{\mathscr{X}}^*f^*f_!K \arrow[d, "f_!\simeq f_*"] & & c_1^*f^*f_!K \arrow[d, "f_!\simeq f_*"] \\
F_{\mathscr{C}}^*c_1^*f^*f_*K \arrow[d, "f^*f_*\to\mathrm{id}"] & c_1^*F_{\mathscr{X}}^*f^*f_*K \arrow[d, "f^*f_*\to\mathrm{id}"] & & c_1^*f^*f_*K \arrow[d, "f^*f_*\to\mathrm{id}"] \\
F_{\mathscr{C}}^*c_1^*K \arrow[r, "\simeq"] \arrow[d, "u"] & c_1^*F_{\mathscr{X}}^*K \arrow[rr, "\epsilon"] & & c_1^*K \arrow[d, "u"] \\
F_{\mathscr{C}}^*c_2^!K \arrow[d, "\simeq"] & & & c_2^!K \arrow[ddd, "\mathrm{id}\to f^!f_!"] \\
c_2^!F_{\mathscr{X}}^*K \arrow[urrr, "\epsilon"] \arrow[d, "\mathrm{id}\to f^!f_!"] & & & \\
c_2^!F_{\mathscr{X}}^*f^!f_!K \arrow[d, "\simeq"] & & & \\
c_2^!f^!F_{\mathscr{Y}}^*f_!K \arrow[rrr, "f_!\epsilon"] & & & c_2^!f^!f_!K.
\end{tikzcd}$$

The top left inner diagram clearly commutes, and the center inner diagram commutes since (K, ϵ, u) is a Weil complex with c-structure. The commutativity of the top right diagram follows from noting that the diagram

$$\begin{tikzcd}
f^*F^*_{\mathscr{Y}}f_!K \arrow[r, "bc"] \arrow[d, "\simeq"] & f^*f'_!\pi^*K \arrow[r, "\mathrm{id}\to g_*g^*"] \arrow[d, "f'_!\simeq f'_*"] & f^*f'_!g_*g^*\pi^*K \arrow[r, "\simeq"] \arrow[d, "f'_!\simeq f'_*"] & f^*f_!F^*_{\mathscr{X}}K \arrow[d, "f_*\simeq f_!"] \\
F^*_{\mathscr{X}}f^*f_!K \arrow[ur, "bc"] \arrow[d, "f_!\simeq f_*"] & f^*f'_*\pi^*K \arrow[r, "\mathrm{id}\to g_*g^*"] & f^*f'_*g_*g^*\pi^*K \arrow[r, "\simeq"] & f^*f_*F^*_{\mathscr{X}}K \arrow[dll, "f^*f_*\to\mathrm{id}"'] \arrow[d, "\epsilon"] \\
F^*_{\mathscr{X}}f^*f_*K \arrow[ur, "bc"] \arrow[r, "f^*f_*\to\mathrm{id}"] & F^*_{\mathscr{X}}K \arrow[r, "\epsilon"] & K & f^*f_*K \arrow[l, "f^*f_*\to\mathrm{id}"']
\end{tikzcd}$$

commutes, and the commutativity of the bottom inner diagram follows from noting that

$$\begin{tikzcd}
F^*_{\mathscr{X}}K \arrow[r, "\epsilon"] \arrow[d, "\mathrm{id}\to f^!f_!"'] \arrow[ddrrr, "\mathrm{id}\to f^!f_!"] & K \arrow[rr, "\mathrm{id}\to f^!f_!"] & & f^!f_!K \\
F^*_{\mathscr{X}}f^!f_!K \arrow[d, "\simeq"] & & & \\
f^!F^*_{\mathscr{Y}}f_!K \arrow[r, "bc"] & f^!f'_!\pi^*K \arrow[r, "\mathrm{id}\to g_*g^*"] & f^!f'_!g_*g^*\pi^*K \arrow[r, "\simeq"] & f^!f_!F^*_{\mathscr{X}}K \arrow[uu, "\epsilon"]
\end{tikzcd}$$

commutes. □

8.7. Assume now that $\mathscr{C}_0$ and $\mathscr{X}_0$ are algebraic spaces, which we henceforth denote by C_0 and X_0 to avoid confusion. Let $y \in \mathrm{Fix}(c)(k)$ be a fixed point with image $x \in X(k)$ defined over $\mathbb{F}_q$ (so x is obtained from a point $x_0 \in X(\mathbb{F}_q)$). Let

$$f_x : K_x \to K_x$$

denote the automorphism defined by the Weil complex structure, and let

$$u_y : K_x \to K_x$$

denote the endomorphism (2.1.1).

Observe that since x is defined over $\mathbb{F}_q$, for any $n, m \in \mathbb{N}$ we also have $y \in \mathrm{Fix}(c^{(n,m)})$.

Lemma 8.8. *For any $n, m \in \mathbb{N}$ we have*

$$\mathrm{lt}_y(K, u^{(n,m)}) = \mathrm{tr}(f_x^{-m}u_yf_x^n|K_x).$$

Proof. The local term $\mathrm{lt}_y(K, u^{(n,m)})$ is by definition the trace of the composite

$$K_x \xrightarrow{c_1^{(n,m)*}} (c_1^{(n,m)*}K)_y \hookrightarrow \oplus_{z\in c_2^{(n,m)-1}(x)}(c_1^{(n,m)*}K)_z \xrightarrow{\simeq} (c_{2!}^{(n,m)}c_1^{(n,m)*}K)_x \xrightarrow{u^{(n,m)}} K_x.$$

This sequence of morphisms fits into the following larger commutative diagram

$$\begin{tikzcd}
K_x \arrow[r, "c_1^{(n,m)*}"] \arrow[dd, "f_x^n"] & (c_1^{(n,m)*}K)_y \arrow[r, hook] \arrow[d, "\simeq"] & \oplus_{z\in c_2^{(n,m)-1}(x)}(c_1^{(n,m)*}K)_z \arrow[r, "\simeq"] \arrow[d, "\simeq"] & (c_{2!}^{(n,m)}c_1^{(n,m)*}K)_x \arrow[r, "u^{(n,m)}"] \arrow[d, "\simeq"] & K_x \\
 & (c_1^*F_X^{n*}K)_y \arrow[r, hook] \arrow[d, "\epsilon^n"] & \oplus_{z\in(F_X^m c_2)^{-1}(x)}(c_1^*F_X^{n*}K)_z \arrow[r, "\simeq"] \arrow[d, "\epsilon^n"] & (F_{X!}^m c_{2!}c_1^*F_X^{n*}K)_x \arrow[d, "\epsilon^n"] & K_x \arrow[u, "(f_x^m)^{-1}"'] \\
K_x \arrow[r, "\simeq"] & (c_1^*K)_y \arrow[r, hook] & \oplus_{z\in(F_X^m c_2)^{-1}(x)}(c_1^*K)_z \arrow[r, "\simeq"] & (F_{X!}^m c_{2!}c_1^*K)_x \arrow[r, "u"] & (F_{X*}^m K)_x \arrow[u, "\simeq"']
\end{tikzcd}$$

which implies the lemma, as the composite map $K_x \to K_x$ obtained by going around the left, bottom, and right side of the diagram is $f_x^{-m}u_y f_x^n$. $\square$

Lemma 8.9. *The endomorphisms u_y and f_x of K_x commute. In particular, for $n \geq m$ we have*

$$\mathrm{lt}_y(K, u^{(n,m)}) = \mathrm{lt}_y(K, u^{(n-m,0)}).$$

Proof. Taking the stalk at y of the diagram (8.3.1) we get that the diagram

$$\begin{tikzcd}
K_x \arrow[r, "f_x"] \arrow[d, "\simeq"] & K_x \arrow[d, "\simeq"] \\
(c_1^*K)_{F_C(y)} \arrow[d, "u"] & (c_1^*K)_y \arrow[d, "u"] \\
(c_2^!K)_{F(y)} \arrow[r, "\epsilon"] \arrow[d, "\tau_{K,F_C(y)}"] & (c_2^!K)_y \arrow[d, "\tau_{K,y}"] \\
K_x \arrow[r, "f_x"] & K_x
\end{tikzcd}$$

commutes, which implies the lemma. $\square$

9. Basic definitions and duality

9.1. Let E be a field of characteristic 0, and let I be a set. Assume given for every $\alpha \in I$ a pair $(\ell_\alpha, \iota_\alpha)$, where ℓ_α is a prime not equal to p, and $\iota_\alpha : E \hookrightarrow \overline{\mathbb{Q}}_{\ell_\alpha}$ is an embedding.

Let

$$c : C_0 \to X_0 \times X_0$$

be a correspondence over $\mathbb{F}_q$ with c_1 and c_2 quasi-finite.

Definition 9.2. An I-*system* consists of a ℓ_α-adic Weil complex with c-structure $(K_\alpha, \epsilon_\alpha, u_\alpha)$ for every $\alpha \in I$. An I-*compatible system* (or just *compatible system* if the reference to I is clear) is an I-system $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}$ such that for every $n \geq 0$ and $y \in \mathrm{Fix}(c^{(n,0)})$ we have

$$\mathrm{lt}_y(K_\alpha, u_\alpha^{(n,0)}) \in \iota_\alpha(E) \subset \overline{\mathbb{Q}}_{\ell_\alpha},$$

and

$$\mathrm{lt}_y(K_\alpha, u_\alpha^{(n,0)}) = \mathrm{lt}_y(K_\beta, u_\beta^{(n,0)}),$$

for all $\alpha, \beta \in I$, where these local terms are viewed as elements of E via ι_α and ι_β respectively.

9.3. Recall the following facts of linear algebra (see [7, §8]). Let E be a field of characteristic 0, and let $V = V_0 \oplus V_1$ be a finite-dimensional $\mathbb{Z}/(2)$-graded E-vector space. Let

$$u, f : V \to V$$

be two commuting, graded, endomorphisms of V with f bijective. Define a function

$$S(t) := \sum_{n \geq 1} \mathrm{tr}(uf^n)t^n \in t \cdot E[[t]].$$

Lemma 9.4. *(i) $S(t) \in E(t)$.*

(ii) Let $s = 1/t$. Then $S(t)$ does not have a pole at $s = 0$ and $-\mathrm{tr}(u)$ is equal to the value of $S(t)$ at $s = 0$.

(iii) Suppose $K \subset E$ is a subfield, and that $S(t) \in t \cdot K[[t]] \subset t \cdot E[[t]]$. Then in fact $S(t) \in K(t) \subset E(t)$ and in particular $\mathrm{tr}(u) \in K$.

Proof. See [7, §8]. □

The main application for us of this linear algebra is the following:

Lemma 9.5. *Let $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}_{\alpha \in I}$ be an I-system.*

(i) Suppose that for every $\alpha, \beta \in I$ there exists an integer n_0 such that for every $n \geq n_0$ we have

$$\mathrm{lt}_y(K_\alpha, u_\alpha^{(n,0)}) \in \iota_\alpha(E), \quad \mathrm{lt}_y(K_\beta, u_\beta^{(n,0)}) \in \iota_\beta(E),$$

and

$$\mathrm{lt}_y(K_\alpha, u_\alpha^{(n,0)}) = \mathrm{lt}_y(K_\beta, u_\beta^{(n,0)})$$

for all $y \in \mathrm{Fix}(c^{(n,0)})$ (where again we view these local terms as elements of E). Then $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}$ is a compatible system.

(ii) If $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}$ is a compatible system, then for every $\alpha, \beta \in I$, $n, m \in \mathbb{N}$, and $y \in \mathrm{Fix}(c^{(n,m)})$ we have

$$\mathrm{lt}_y(K_\alpha, u_\alpha^{(n,m)}) \in \iota_\alpha(E), \quad \mathrm{lt}_y(K_\beta, u_\beta^{(n,m)}) \in \iota_\beta(E)$$

and

$$\mathrm{lt}_y(K_\alpha, u_\alpha^{(n,m)}) = \mathrm{lt}_y(K_\beta, u_\beta^{(n,m)}).$$

Proof. For (i), let $n \geq 0$ be an integer and let $y \in \mathrm{Fix}(c^{(n,0)})$ be a fixed point. Also choose $\alpha, \beta \in I$. We need to show that

$$\mathrm{lt}_y(K_\alpha, u_\alpha^{(n,0)}) \in \iota_\alpha(E), \quad \mathrm{lt}_y(K_\beta, u_\beta^{(n,0)}) \in \iota_\beta(E), \tag{9.5.1}$$

and that

$$\mathrm{lt}_y(K_\alpha, u_\alpha^{(n,0)}) = \mathrm{lt}_y(K_\beta, u_\beta^{(n,0)}).$$

Replacing c by $c^{(n,0)}$ and u by $u^{(n,0)}$, we may assume that $n = 0$. Furthermore, after making a field extension $\mathbb{F}_q \to \mathbb{F}_{q^r}$, we may assume that $y \in C(\mathbb{F}_q)$, and that $n_0 = 1$. Let $x \in X(k)$ be the image of y, and set

$$V_\alpha = \oplus_{i \in \mathbb{Z}} H^i(K_{\alpha,x}),$$

a $\mathbb{Z}/(2)$-graded $\overline{\mathbb{Q}}_{\ell_\alpha}$-vector space. Let

$$u_\alpha : V_\alpha \to V_\alpha$$

be the endomorphism defined by the action of the correpondence on $K_{\alpha,x}$, and let

$$f_\alpha : V_\alpha \to V_\alpha$$

be the automorphism defined by the Weil complex structure. Then u_α and f_α are commuting endomorphisms of V_α, by 8.9. Set

$$S_\alpha(t) := \sum_{n\geq 1} \mathrm{tr}(u_\alpha f_\alpha^n | V_\alpha),$$

which by (9.4 (i)) is an element of $\overline{\mathbb{Q}}_{\ell_\alpha}(t)$. Similarly define $(V_\beta, u_\beta, f_\beta)$ and $S_\beta(t) \in \overline{\mathbb{Q}}_{\ell_\beta}(t)$. By (9.4 (iii)) we have

$$S_\alpha(t), S_\beta(t) \in E(t),$$

and

$$S_\alpha(t) = S_\beta(t),$$

so from (9.4 (ii)) we have (9.5.1) and

$$\mathrm{tr}(u_\alpha | V_\alpha) = \mathrm{tr}(u_\beta | V_\beta),$$

which is equivalent to the equality

$$\mathrm{lt}_y(K_\alpha, u_\alpha) = \mathrm{lt}_y(K_\beta, u_\beta).$$

For (ii), note that by 8.9 it suffices to show that for $m \geq 1$ we have

$$\mathrm{lt}_y(K_\alpha, u_\alpha^{(0,m)}) \in \iota_\alpha(E), \quad \mathrm{lt}_y(K_\beta, u_\beta^{(0,m)}) \in \iota_\beta(E)$$

and

$$\mathrm{lt}_y(K_\alpha, u_\alpha^{(0,m)}) = \mathrm{lt}_y(K_\beta, u_\beta^{(0,m)}).$$

For this note that by 8.9 again, we have for $n \geq m$

$$\mathrm{lt}_y(K_\alpha, u_\alpha^{(n,m)}) \in \iota_\alpha(E), \quad \mathrm{lt}_y(K_\beta, u_\beta^{(n,m)}) \in \iota_\beta(E),$$

and

$$\mathrm{lt}_y(K_\alpha, u_\alpha^{(n,m)}) = \mathrm{lt}_y(K_\beta, u_\beta^{(n,m)}).$$

Statement (ii) therefore follows from (i) applied to the I-system with $c^{(0,m)}$-structure

$$\{(K_\alpha, \epsilon_\alpha^{(0,m)}, u_\alpha^{(0,m)})\}.$$

□

Theorem 9.6. *Let $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}$ be a compatible system of Weil complexes with c-structure. Then $\{(DK_\alpha, \epsilon_\alpha^t, u_\alpha^t)\}$ is a compatible system of Weil complexes with c^t-structure.*

Proof. Note that

$$(c^t)^{(n,0)} = (c^{(0,n)})^t$$

and that for $\alpha \in I$ we have

$$(u_\alpha^t)^{(n,0)} = (u_\alpha^{(0,n)})^t.$$

Therefore for any $\alpha \in I$, $n \geq 0$, and

$$y \in \mathrm{Fix}((c^t)^{(n,0)}) = \mathrm{Fix}((c^{(0,n)})^t),$$

we have by 1.9

$$\mathrm{lt}_y(DK_\alpha, (u_\alpha^t)^{(n,0)}) = \mathrm{lt}_y(K_\alpha, u_\alpha^{(0,n)}).$$

From this and (9.5 (ii)) the result follows. □

10. Pullback f^*

10.1. Consider a commutative diagram

$$\begin{tikzcd} & C_0 \arrow[dl, "c_1"'] \arrow[d, "g"] \arrow[dr, "c_2"] & \\ X_0 \arrow[dd, "f"] & D_0 \arrow[dl, "d_1"'] \arrow[dr, "d_2"] & X_0 \arrow[dd, "f"] \\ & & \\ Y_0 & & Y_0, \end{tikzcd}$$

with c_1, c_2, d_1, and d_2 quasi-finite, and let $\ell \neq p$ be a prime.

Assume that the map

$$\chi : C \to D \times_{d_2, Y, f} X$$

is a universal homeomorphism. In this case we have a base change isomorphism

$$bc : c_{2!} g^* \simeq f^* d_{2!}.$$

Remark 10.2. A key case is when f is a closed immersion, and C_0 is the fiber product of the diagram

$$\begin{tikzcd} & X_0 \times X_0 \arrow[d] \\ D_0 \arrow[r, "d"] & Y_0 \times Y_0, \end{tikzcd}$$

which is the scheme-theoretic intersection $d_1^{-1}(X) \cap d_2^{-1}(X)$. In this case, the condition that χ is a universal homeomorphism is equivalent to the condition that $d_2^{-1}(X)_{\mathrm{red}} \subset d_1^{-1}(X)_{\mathrm{red}}$.

10.3. Let (K, ϵ, u) be an ℓ-adic Weil complex with d-structure on Y. Define

$$f^*\epsilon : F_X^*(f^*K) \to f^*K$$

to be the isomorphism

$$F_X^* f^* K \xrightarrow{\simeq} f^* F_Y^* K \xrightarrow{\epsilon} f^* K,$$

and let

$$f^* u : c_{2!} c_1^* f^* K \to f^* K$$

be the morphism

$$c_{2!} c_1^* f^* K \xrightarrow{\simeq} c_{2!} g^* d_1^* K \xrightarrow{bc} f^* d_{2!} d_1^* K \xrightarrow{u} f^* K.$$

Lemma 10.4. *The data $(f^*K, f^*\epsilon, f^*u)$ is a Weil complex with c-structure on X.*

Proof. The condition that the diagram

$$\begin{tikzcd}
c_{2!}c_1^*F_X^*f^*K \arrow[r, "f^*\epsilon"] \arrow[d, "\simeq"] & c_{2!}c_1^*f^*K \arrow[dd, "f^*u"] \\
F_X^*c_{2!}c_1^*f^*K \arrow[d, "f^*u"] & \\
F_X^*f^*K \arrow[r, "f^*\epsilon"] & f^*K
\end{tikzcd}$$

commutes, is equivalent to the commutativity of the big outside diagram in the following:

(10.4.1)
$$\begin{tikzcd}
c_{2!}c_1^*F_X^*f^*K \arrow[r, "F_X^*f^*\simeq f^*F_Y^*"] \arrow[d, "\simeq"] & c_{2!}c_1^*f^*F_Y^*K \arrow[rr, "\epsilon"] \arrow[d, "c_1^*f^*\simeq g^*d_1^*"] & & c_{2!}c_1^*f^*K \arrow[d, "c_1^*f^*\simeq g^*d_1^*"] \\
F_X^*c_{2!}c_1^*f^*K \arrow[d, "c_1^*f^*\simeq g^*d_1^*"] & c_{2!}g^*d_1^*F_Y^*K \arrow[rr, "\epsilon"] \arrow[d, "bc"] & & c_{2!}g^*d_1^*K \arrow[d, "u"] \\
F_X^*c_{2!}g^*d_1^*K \arrow[d, "bc"] & f^*d_{2!}d_1^*F_Y^*K \arrow[r, "\epsilon"] \arrow[d, "\simeq"] & f^*d_{2!}d_1^*K \arrow[dr, "u"] & c_{2!}g^*d_2^!K \arrow[d, "bc"] \\
F_X^*f^*d_{2!}d_1^*K \arrow[r, "F_X^*f^*\simeq f^*F_Y^*"] & f^*F_Y^*d_{2!}d_1^*K \arrow[d, "u"] & & f^*d_{2!}d_2^!K \arrow[d, "d_{2!}d_2^!\to\mathrm{id}"] \\
& f^*F_Y^*K \arrow[rr, "\epsilon"] & & f^*K.
\end{tikzcd}$$

This implies the lemma as all the small inside diagram in (10.4.1) clearly commute. ∎

Let

$$\gamma : \mathrm{Fix}(c) \to \mathrm{Fix}(d)$$

be the natural map.

Lemma 10.5. *For every $y \in \mathrm{Fix}(c)(k)$, we have*

$$\mathrm{lt}_y(f^*K, f^*u) = \mathrm{lt}_{\gamma(y)}(K, u).$$

Proof. Let x denote $c_1(y) = c_2(y)$. The lemma follows by noting that the diagram

$$\begin{tikzcd}
K_x \arrow[r, hook, "y"] & \oplus_{z\in c_2^{-1}(x)}(c_1^*f^*K)_z \arrow[r, "\simeq"] & (c_{2!}c_1^*f^*K)_x \arrow[d, "bc"] \arrow[dr, "f^*u"] & \\
& & (f^*d_{2!}d_1^*K)_x \arrow[r, "u"] \arrow[d, "\simeq"] & K_x \\
K_{f(x)} \arrow[uu, "f^*"] \arrow[r, hook, "\gamma(y)"] & \oplus_{w\in d_2^{-1}(f(x))}(d_1^*K)_w \arrow[r, "\simeq"] & (d_{2!}d_1^*K)_{f(x)} \arrow[r, "u"] & K_{f(x)} \arrow[u, "f^*"]
\end{tikzcd}$$

commutes. ∎

Proposition 10.6. *Let $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}$ be an I-compatible system of Weil complexes with d-structure on Y. Then*

$$\{(f^*K_\alpha, f^*\epsilon_\alpha, f^*u_\alpha)\}$$

is a compatible system of Weil complexes with c-structure on X.

Proof. This follows from 10.5 and the observation that

$$f^*(u^{(n,0)}) = (f^*u)^{(n,0)}.$$

□

11. Extraordinary inverse image $f^!$

11.1. We can dualize the results of the previous section. Consider a commutative diagram

$$\begin{tikzcd}
 & C_0 \arrow[dl, "c_1"'] \arrow[dr, "c_2"] \arrow[dd, "g"] & \\
X_0 \arrow[dd, "f"] & & X_0 \arrow[dd, "f"] \\
 & D_0 \arrow[dl, "d_1"'] \arrow[dr, "d_2"] & \\
Y_0 & & Y_0,
\end{tikzcd}$$

with c_1, c_2, d_1, and d_2 quasi-finite, and let $\ell \neq p$ be a prime. Assume that the map

$$\chi : C \to X \times_{f,Y,d_1} D$$

is a universal homeomorphism. This implies that there is a base change morphism

$$bc : c_1^* f^! \to g^! d_1^*$$

adjoint to the composite

$$g_! c_1^* f^! \xrightarrow{g_! c_1^* \simeq d_1^* f_!} d_1^* f_! f^! \xrightarrow{f_! f^! \to \mathrm{id}} d_1^*.$$

For an ℓ-adic Weil complex with d-structure (K, ϵ, u), define

$$f^!\epsilon : F_X^* f^! K \to f^! K$$

to be the composite isomorphism

$$F_X^* f^! K \xrightarrow{\simeq} f^! F_Y^* K \xrightarrow{\epsilon} f^! K,$$

and let

$$f^! u : c_1^* f^! K \to c_2^! f^! K$$

be the composite morphism

$$c_1^* f^! K \xrightarrow{bc} g^! d_1^* K \xrightarrow{u} g^! d_2^! K \xrightarrow{\simeq} c_2^! f^! K.$$

Proposition 11.2. *Let $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}$ be a compatible system of Weil complexes with d-structure on Y. Then $\{(f^! K_\alpha, f^! \epsilon_\alpha, f^! u_\alpha)\}$ is a compatible system of Weil complexes with c-structure on X.*

Proof. This follows from 9.6 and 10.6, together with the observation that under the canonical isomorphism

$$f^! K_\alpha \simeq D f^* D(K_\alpha)$$

the morphism $f^! \epsilon_\alpha$ (resp. $f^! u_\alpha$) corresponds to $(f^*(\epsilon_\alpha^t))^t$ (resp. $(f^*(u_\alpha^t))^t$), which is immediate from the definitions. $\square$

12. Compactly supported cohomology $f_!$

12.1. Consider a commutative diagram

$$\begin{tikzcd}
& C_0 \arrow[dl, "c_1"'] \arrow[dd, "g"] \arrow[dr, "c_2"] & \\
X_0 \arrow[dd, "f"] & & X_0 \arrow[dd, "f"'] \\
& D_0 \arrow[dl, "d_1"'] \arrow[dr, "d_2"] & \\
Y_0 & & Y_0,
\end{tikzcd}$$

with c_1, c_2, d_1, and d_2 quasi-finite. Let ℓ be a prime number not equal to p, and let $K \in D_c^b(X, \overline{\mathbb{Q}}_\ell)$ be a complex with a c-structure

$$u : c_1^* K \to c_2^! K.$$

Let P_1 denote the fiber product

$$P_1 := D \times_{d_1, Y, f} X,$$

let $\epsilon_1 : P_1 \to X$ be the projection, and let

$$C \xrightarrow{\beta} P_1 \xrightarrow{h} D \tag{12.1.1}$$

be the natural factorization of g.

Assume that the map β in 12.1.1 is proper. Then we get a d-structure

$$f_! u : d_1^* f_! K \to d_2^! f_! K$$

on $f_! K$ from the composite

$$d_{2!} d_1^* f_! K \xrightarrow{d_1^* f_! \simeq h_! \epsilon_1^*} d_{2!} h_! \epsilon_1^* K \xrightarrow{\mathrm{id} \to \beta_* \beta^*} d_{2!} h_! \beta_* \beta^* \epsilon_1^* K \xrightarrow{\beta_* \simeq \beta_!} d_{2!} h_! \beta_! c_1^* K \xrightarrow{\simeq} f_! c_{2!} c_1^* K \xrightarrow{u} f_! K.$$

12.2. Suppose now that (K, ϵ, u) is a Weil complex with c-structure on X, and define

$$f_! \epsilon : F_Y^* f_! K \to f_! K$$

to be the composite isomorphism

$$F^*f_!K \xrightarrow{\simeq} f_!F_X^*K \xrightarrow{\epsilon} f_!K.$$

Lemma 12.3. *The data $(f_!K, f_!\epsilon, f_!u)$ is a Weil complex with d-structure on Y.*

Proof. To see that the diagram

$$\begin{tikzcd}
d_{2!}d_1^*F_Y^*f_!K \arrow[r, "f_!\epsilon"] \arrow[d, "\simeq"] & d_{2!}d_1^*f_!K \arrow[dd, "f_!u"] \\
F_Y^*d_{2!}d_1^*f_!K \arrow[d, "f_!u"] & \\
F_Y^*f_!K \arrow[r, "f_!\epsilon"] & f_!K
\end{tikzcd}$$

commutes, note that this is equivalent to showing that the big outside diagram in the following diagram commutes:

$$\begin{tikzcd}
d_{2!}d_1^*F_Y^*f_!K \arrow[r, "\simeq"] \arrow[d, "\simeq"] & d_{2!}d_1^*f_!F_X^*K \arrow[r, "\epsilon"] \arrow[d, "bc"] & d_{2!}d_1^*f_!K \arrow[d, "bc"] \\
F_Y^*d_{2!}d_1^*f_!K \arrow[d, "bc"] & d_{2!}g_{1!}\epsilon_1^*F_X^*K \arrow[r, "\epsilon"] \arrow[d, "\mathrm{id}\to\beta_*\beta^*"] & d_{2!}g_{1!}\epsilon_1^*K \arrow[d, "\mathrm{id}\to\beta_*\beta^*"] \\
F_Y^*d_{2!}g_{1!}\epsilon_1^*K \arrow[d, "\mathrm{id}\to\beta_*\beta^*"] & d_{2!}g_{1!}\beta_*\beta^*\epsilon_1^*F_X^*K \arrow[r, "\epsilon"] \arrow[d, "\beta_*\simeq\beta_!"] & d_{2!}g_{1!}\beta_*\beta^*\epsilon_1^*K \arrow[d, "\beta_*\simeq\beta_!"] \\
F_Y^*d_{2!}g_{1!}\beta_*\beta^*\epsilon_1^*K \arrow[d, "\beta_*\simeq\beta_!"] & d_{2!}g_{1!}\beta_!c_1^*F_X^*K \arrow[r, "\epsilon"] \arrow[d, "\simeq"] & d_{2!}g_{1!}\beta_!c_1^*K \arrow[d, "\simeq"] \\
F_Y^*d_{2!}g_{1!}\beta_!c_1^*K \arrow[d, "\simeq"] & f_!c_{2!}c_1^*F_X^*K \arrow[r, "\epsilon"] \arrow[d, "\simeq"] & f_!c_{2!}c_1^*K \arrow[dd, "u"] \\
F_Y^*f_!c_{2!}c_1^*K \arrow[r, "\simeq"] \arrow[d, "u"] & f_!F_X^*c_{2!}c_1^*K \arrow[d, "u"] & \\
F_Y^*f_!K \arrow[r, "\simeq"] & f_!F_X^*K \arrow[r, "\epsilon"] & f_!K.
\end{tikzcd}$$

This follows from observing that all the small inside diagrams in this diagram commute. □

12.4. Let $i : Z \hookrightarrow Y$ be a closed subscheme such that

$$d_1^{-1}(Z)_{\mathrm{red}} = d_2^{-1}(Z)_{\mathrm{red}}.$$

Let

$$a : X_Z \hookrightarrow X$$

be the inverse image of Z, and let

$$e : D_Z \to Z \times Z$$

and

$$b : C_Z \to X_Z \times X_Z$$

be the pullbacks of D and C respectively (so $D_Z = d_1^{-1}(Z) \cap d_2^{-1}(Z)$, and $C_Z = c_1^{-1}(X_Z) \cap c_2^{-1}(X_Z)$). Let

$$h : X_Z \to Z$$

be the base change of f.

Lemma 12.5. *Let (K, ϵ, u) be an ℓ-adic Weil complex with c-structure on X. Then the diagrams*

(12.5.1)
$$\begin{tikzcd} F_Z^* i^* f_! K \arrow[r, "i^* f_! \epsilon"] \arrow[d, "i^* f_! \simeq h_! a^*"] & i^* f_! K \arrow[d, "i^* f_! \simeq h_! a^*"] \\ F_Z^* h_! a^* K \arrow[r, "h_! a^* \epsilon"] & h_! a^* K \end{tikzcd}$$

and

(12.5.2)
$$\begin{tikzcd} e_{2!} e_1^* i^* f_! K \arrow[r, "i^* f_! u"] \arrow[d, "i^* f_! \simeq h_! a^*"] & i^* f_! K \arrow[d, "i^* f_! \simeq h_! a^*"] \\ e_{2!} e_1^* h_! a^* K \arrow[r, "h_! a^* u"] & h_! a^* K \end{tikzcd}$$

commute.

Proof. For the commutativity of (12.5.1) note that the diagram

$$\begin{tikzcd} F_Z^* i^* f_! K \arrow[r, "\simeq"] \arrow[d, "i^* f_! \simeq h_! a^*"] & i^* F_Y^* f_! K \arrow[r, "\simeq"] & i^* f_! F_X^* K \arrow[r, "\epsilon"] \arrow[d, "i^* f_! \simeq h_! a^*"] & i^* f_! K \arrow[d, "i^* f_! \simeq h_! a^*"] \\ F_Z^* h_! a^* K \arrow[r, "\simeq"] & h_! F_{X_Z}^* a^* K \arrow[r, "\simeq"] & h_! a^* F_X^* K \arrow[r, "\epsilon"] & h_! a^* K \end{tikzcd}$$

commutes.

For the commutativity of (12.5.2) it suffices, by expanding out the definitions of $i^*f_!u$ and $h_!a^*u$, to show that the following diagram commutes:

(12.5.3)
$$\begin{tikzcd}
e_{2!}e_1^*i^*f_!K \arrow[r, "\simeq"] \arrow[d, "i^*f_!\simeq h_!a^*"] & e_{2!}\sigma^*d_1^*f_!K \arrow[r, "e_{2!}\sigma^*\simeq i^*d_{2!}"] \arrow[dr, "d_1^*f_!\simeq g_{1!}\epsilon_1^*"] & i^*d_{2!}d_1^*f_!K \arrow[r, "d_1^*f_!\simeq g_{1!}\epsilon_1^*"] & i^*d_{2!}g_{1!}\epsilon_1^*K \arrow[d, "\mathrm{id}\to\beta_*\beta^*"] \\
e_{2!}e_1^*h_!a^*K \arrow[d, "e_1^*h_!\simeq\nu_{1!}\tau^*"] & e_{2!}\nu_{1!}t^*\epsilon_1^*K \arrow[dl, "\simeq"'] & e_{2!}\sigma^*g_{1!}\epsilon_1^*K \arrow[l, "\nu_{1!}t^*\simeq\sigma^*g_{1!}"'] \arrow[d, "\mathrm{id}\to\beta_*\beta^*"] & i^*d_{2!}g_{1!}\beta_*\beta^*\epsilon_1^*K \arrow[d, "\beta_*\simeq\beta_!"] \\
e_{2!}\nu_{1!}\tau^*a^*K \arrow[d, "\mathrm{id}\to\gamma_*\gamma^*"] & e_{2!}\nu_{1!}t^*\beta_*\beta^*\epsilon_1^*K \arrow[dl, "bc"'] & e_{2!}\sigma^*g_{1!}\beta_*\beta^*\epsilon_1^*K \arrow[l, "bc"] \arrow[d, "\simeq"] \arrow[ur, "bc"] & i^*d_{2!}g_{1!}\beta_!c_1^*K \arrow[d, "\simeq"] \arrow[ddl, "\simeq"'] \\
e_{2!}\nu_{1!}\gamma_*\gamma^*\tau^*a^*K \arrow[d, "\gamma_*\simeq\gamma_!"] & e_{2!}\nu_{1!}t^*\beta_!c_1^*K \arrow[dl, "bc"'] \arrow[d, "\simeq"] & e_{2!}\sigma^*g_{1!}\beta_!c_1^*K \arrow[l, "bc"'] \arrow[ur, "bc"] & i^*f_!c_{2!}c_1^*K \arrow[d, "u"] \arrow[ddl, "bc"] \\
e_{2!}\nu_{1!}\gamma_!b_1^*a^*K \arrow[d, "\simeq"] & (e_2\nu_1)_!t^*\beta_!c_1^*K \arrow[r, "bc"] \arrow[d, "bc"] & (d_2g_1)_!\beta_!c_1^*K & i^*f_!K \arrow[d, "bc"] \\
h_!b_{2!}b_1^*a^*K \arrow[r, "\simeq"] & h_!b_{2!}\tilde{a}^*c_1^*K \arrow[r, "bc"] & h_!a^*c_{2!}c_1^*K \arrow[r, "u"] & h_!a^*K.
\end{tikzcd}$$

Here the notation is as follows. Let Q_1 denote the fiber product $X_Z \times_{h,Z,e_1} D$ and let

$$\gamma : C_Z \to Q_1$$

be the natural map. Let $\tilde{a} : C_Z \hookrightarrow C$ be the inclusion, so we have cartesian squares

$$\begin{tikzcd}
Q_1 \arrow[r, "\nu_1"] \arrow[d, "\tau"] & D_Z \arrow[d, "e_1"] \\
X_Z \arrow[r, "h"] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
D_Z \arrow[r, "\sigma"] \arrow[d, "e_2"] & D \arrow[d, "d_2"] \\
Z \arrow[r, "i"] & Y,
\end{tikzcd}$$

and

$$\begin{tikzcd}
Q_1 \arrow[r, "t"] \arrow[d, "\nu_1"] & P_1 \arrow[d, "g_1"] \\
D_Z \arrow[r, "\sigma"] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
C_Z \arrow[r, "\tilde{a}"] \arrow[d, "\gamma"] & C \arrow[d, "\beta"] \\
Q_1 \arrow[r, "t"] & P_1.
\end{tikzcd}$$

The lemma then follows by noting that (12.5.3) commutes, as each of the small inside squares commutes by standard properties of the base change isomorphisms. □

Theorem 12.6. *Let $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}_{\alpha\in I}$ be a compatible system of Weil complexes with c-structure on X. Then $\{(f_!K_\alpha, f_!\epsilon_\alpha, f_!u_\alpha)\}$ is a compatible system of Weil complexes with d-structure on Y.*

Proof. Let $z \in \mathrm{Fix}(d^{(n,0)})(k)$ be a fixed point for some $n \geq 0$. We have to show that

$$\mathrm{lt}_z(f_!K, f_!u_\alpha^{(n,0)}) \in \iota_\alpha(E)$$

for all $\alpha \in I$, and that the resulting elements of E agree. Replacing d by $d^{(n,0)}$, c by $c^{(n,0)}$, and possibly making a field extension $\mathbb{F}_q \to \mathbb{F}_{q^r}$, we may assume that $n = 0$ and that z is defined over $\mathbb{F}_q$. Let $y_0 \in Y_0(\mathbb{F}_q)$ be the image of z. After removing

$$(d_1^{-1}(y) \cup d_2^{-1}(y)) - \{z\}$$

from D and replacing C by the corresponding inverse image, we may further assume that $d_1^{-1}(y)_{\mathrm{red}} = d_2^{-1}(y)_{\mathrm{red}} = \{z\}$.

Let

$$i : \mathrm{Spec}(\mathbb{F}_q) \hookrightarrow Y_0$$

be the inclusion defined by y_0, let X_y denote $f^{-1}(y)$, and let C_z denote $g^{-1}(z)$. We then have a correspondence

$$e : C_z \to X_y \times X_y,$$

with e_1 proper (even finite). Let $\{(L_\alpha, \eta_\alpha, v_\alpha)\}$ be the compatible system of Weil complexes with e-structure on X_y obtained from $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}$ by pullback 10.6. By 12.5 we then have

$$\mathrm{lt}_z(f_! K_\alpha, f_! u_\alpha) = \mathrm{tr}(R\Gamma_c(v_\alpha)|R\Gamma_c(X_y, L_\alpha)).$$

Now by Fujiwara's theorem 1.3, there exists an integer n_0 such that for all $n \geq n_0$ we have

$$\mathrm{lt}_z(f_! K_\alpha, f_! u_\alpha) \in \iota_\alpha(E),$$

and such that these numbers all agree (in fact they are given by a sum of local terms of L_α). By 9.5 we therefore get the theorem. $\square$

13. Ordinary cohomology f_*

13.1. Consider a commutative diagram

$$\begin{tikzcd}
 & C_0 \arrow[dl, "c_1"'] \arrow[dd, "g"] \arrow[dr, "c_2"] & \\
X_0 \arrow[dd, "f"] & & X_0 \arrow[dd, "f"'] \\
 & D_0 \arrow[dl, "d_1"'] \arrow[dr, "d_2"] & \\
Y_0 & & Y_0,
\end{tikzcd}$$

with c_1, c_2, d_1, and d_2 quasi-finite. Let ℓ be a prime number not equal to p, and let $K \in D_c^b(X, \overline{\mathbb{Q}}_\ell)$ be a complex with a c-structure

$$u : c_1^* K \to c_2^! K.$$

Let P_2 denote the fiber product

$$P_2 := D \times_{d_2, Y, f} X,$$

let $\epsilon_2 : P_2 \to X$ be the projection, and let

$$C \xrightarrow{\beta} P_2 \xrightarrow{h} D \qquad (13.1.1)$$

be the natural factorization of g.

Assume that the map β in 13.1.1 is proper.

We then get a d-structure f_*u on f_*K defined as the composite morphism

$$d_1^*f_*K \longrightarrow g_*c_1^*K \xrightarrow{u} g_*c_2^!K \xrightarrow{\simeq} h_*\beta_*\beta^!\epsilon_2^!K \xrightarrow{\beta_*\simeq\beta_!} h_*\beta_!\beta^!\epsilon_2^!K \xrightarrow{\beta_!\beta^!\to \mathrm{id}} h_*\epsilon_2^!K \xrightarrow{bc} d_2^!f_*K.$$

If

$$\epsilon : F_X^*K \to K$$

is a Weil complex structure on K, we also get a Weil complex structure $f_*\epsilon$ on f_*K from the composite

$$F_Y^*f_*K \xrightarrow{\simeq} f_*F_X^*K \xrightarrow{\epsilon} f_*K.$$

Proposition 13.2. *Let $\{(K_\alpha, \epsilon_\alpha, u_\alpha)\}$ be a compatible system of Weil complexes with c-structure on X. Then $\{(f_*K_\alpha, f_*\epsilon_\alpha, f_*u_\alpha)\}$ is a compatible system of Weil complexes with d-structure on Y.*

Proof. This follows from 12.6 and 9.6, and the observation that under the canonical isomorphism

$$f_*K_\alpha \simeq Df_!DK_\alpha$$

the map $f_*\epsilon_\alpha$ (resp. f_*u_α) corresponds to $(f_!(\epsilon_\alpha^t))^t$ (resp. $(f_!(u_\alpha^t))^t$), which is immediate from the definitions. □

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