BONUS PROBLEM

MARTIN C. OLSSON

Let k be a field and let $X = \mathbb{P}^n_k$. The purpose of this exercise is to give an alternate construction of the exact sequence

$$0 \to \Omega^1_{X/k} \to \mathscr{O}_X^{n+1}(-1) \to \mathscr{O}_X \to 0. \tag{0.0.1}$$

(a) Let T_X denote $\mathscr{H}om(\Omega^1_{X/k}, \mathscr{O}_X)$. Show that giving the sequence 0.0.1 is equivalent to giving a sequence

$$0 \to \mathscr{O}_X \to \mathscr{O}_X(1)^{n+1} \to T_X \to 0. \tag{0.0.2}$$

Let $\pi : \mathscr{O}_X^{n+1} \to \mathscr{O}_X(1)$ denote the the tauotological surjection over X, and let $\mathscr{O}_X(-1) \to \mathscr{O}_X^{n+1}$ be the inclusion obtained by applying $\mathscr{H}om(-, \mathscr{O}_X)$ to π. Tensoring with $\mathscr{O}_X(1)$ we obtain an inclusion $\alpha : \mathscr{O}_X \hookrightarrow \mathscr{O}_X(1)^{n+1}$. Let Q be the cokernel. We show that $Q \simeq T_X$.

(b) Let $T_0 \hookrightarrow T$ be a square-zero closed immersion defined by an ideal $J \subset \mathscr{O}_T$, and let $x_0 : T_0 \to X$ be a morphism corresponding to a surjection $\mathscr{O}_{T_0}^{n+1} \to L_0$. Show that the set of dotted arrows x filling in the following diagram

$$\begin{array}{ccc} T_0 & \xrightarrow{x_0} & X \\ \big\downarrow & \nearrow & \\ T & & \end{array}$$

is canonically in bijection with the set of isomorphism classes of commutative diagrams of $\mathscr{O}_T$-modules

$$\begin{array}{ccc} \mathscr{O}_T^{n+1} & \twoheadrightarrow & L \\ \big\downarrow & & \big\downarrow \\ \mathscr{O}_{T_0}^{n+1} & \twoheadrightarrow & L_0, \end{array}$$

where L is an invertible sheaf of $\mathscr{O}_T$-modules, and the horizontal arrows are surjections.

(c) Suppose $U \subset X$ is an affine open subset such that $\mathscr{O}_X(1)|_U$ is trivial, and let F be a quasi-coherent sheaf on U. Show that for any commutative diagram

$$\begin{array}{ccc} \mathscr{O}_{U[F]}^{n+1} & \twoheadrightarrow & L \\ \big\downarrow & & \big\downarrow \\ \mathscr{O}_U^{n+1} & \twoheadrightarrow & L_0, \end{array}$$

as in (b), the invertible $\mathscr{O}_{U[F]}$-module L is trivial.

(d) With notation and assumptions as in (c), let $L_0[F]$ denote $L_0 \otimes_{\mathscr{O}_U} \mathscr{O}_{U[F]}$ (so as an $\mathscr{O}_U$-module we have $L_0[F] \simeq L_0 \oplus L_0 \otimes F$). Show that there is a canonical bijection between dotted arrows filling in the following diagram

$$\begin{array}{ccc} \mathscr{O}_{U[F]}^{n+1} & \dashrightarrow & L_0[F] \\ \downarrow & & \downarrow \\ \mathscr{O}_U^{n+1} & \twoheadrightarrow & L_0, \end{array}$$

and $L_0^{n+1} \otimes F = \mathscr{O}_X(1)^{n+1}|_U \otimes F$.

(e) Continuing with the notation as in (d) show that the group of automorphisms of $L_0[F]$ inducing the identity on L_0 is canonically in bijection with F. Show that if $f \in F$ is an element with corresponding automorphism a_f, and if $\gamma : \mathscr{O}_{U[F]}^{n+1} \to L_0[F]$ is a surjection corresponding to $v \in \mathscr{O}_X(1)^{n+1}|_U \otimes F$ then the composite surjection

$$\mathscr{O}_{U[F]}^{n+1} \xrightarrow{\gamma} L_0[F] \xrightarrow{a_f} L_0[F]$$

corresponds to v plus the image of f under the map

$$F \xrightarrow{\simeq} \mathscr{O}_U \otimes F \xrightarrow{\alpha \otimes 1} L_0^{n+1} \otimes F.$$

Deduce from this a canonical isomorphism $Q \otimes F \simeq T_{X/k} \otimes F$ over U. In particular, a canonical isomorphism $\epsilon_U : Q|_U \simeq T_X|_U$.

(f) Show that if $U, V \subset X$ are two open subsets as in (c), then the two isomorphisms $\epsilon_U, \epsilon_V : Q|_{U \cap V} \to T_X|_{U \cap V}$ are equal, so we obtain a global isomorphism $\epsilon : Q \simeq T_X$.