MATH 115: NOTES ON CURVE THEORY 2

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1. Lines

As usual let k be a field, and consider 2-dimensional projective space $\mathbb{P}^n(k)$ with coordinates $X_0, \ldots, X_2$. A *line* in $\mathbb{P}^2(k)$ is a subset of the form

$$V(L) \subset \mathbb{P}^2(k),$$

where S is a nonzero homogeneous linear polynomial

$$S = \alpha_0 X_0 + \alpha_1 X_1 + \alpha_2 X_2.$$

Let us verify some basic properties of lines.

Lemma 1.1. *Let $A, B \in \mathbb{P}^2(k)$ be two distinct points. Then there exists a unique line L through A and B.*

Proof. Write

$$A = [a_0 : a_1 : a_2], \quad B = [b_0 : b_1 : b_2].$$

If S defines a line through A and B then we must have

$$\alpha_0 a_0 + \alpha_1 a_1 + \alpha_2 a_2 = 0, \quad \alpha_0 b_0 + \alpha_1 b_1 + \alpha_2 b_2 = 0. \tag{1.1.1}$$

We claim that these two equations have a unique solution $(\alpha_0, \alpha_1, \alpha_2)$ up to multiplication by an element of $k - \{0\}$. If your linear algebra course developed linear algebra over a general field this is immediate, as vectors $(\alpha_0, \alpha_1, \alpha_2)$ satisfying the two equations correspond to elements of the nullspace of the matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{pmatrix}$$

and since the two points are distinct the two rows are linearly independent which implies that the null space has dimension 1.

But we can also proceed directly as follows. To see that the two equations 1.1.1 have a unique solution up to scalar, note first that after relabeling the coordinates we may assume that $a_0 \neq 0$, in which case we may even assume that $a_0 = 1$. Then from the first equation we get

$$\alpha_0 = -\alpha_1 a_1 - \alpha_2 a_2.$$

Plugging this into the second equation we get

$$(-\alpha_1 a_1 - \alpha_2 a_2) b_0 + \alpha_1 b_1 + \alpha_2 b_2 = 0,$$

which is equivalent to the equation

$$(b_1 - a_1 b_0)\alpha_1 + (b_2 - a_2 b_0)\alpha_2 = 0.$$

Since the two points are distinct, the left side of this equation is not identically zero, and therefore there exists a unique solution (α_1, α_2) up to scalar. $\square$

2. Tangent lines

Now consider a homogeneous polynomial F in variables X_0, X_1, X_2 defining

$$V := V(F) \subset \mathbb{P}^2(k).$$

For a point $P = [a_0 : a_1 : a_2]$, define the *tangent line to V at P* to be the line defined by the equation

$$\frac{\partial F}{\partial X_0}(a_0, a_1, a_2)X_0 + \frac{\partial F}{\partial X_1}(a_0, a_1, a_2)X_1 + \frac{\partial F}{\partial X_1}(a_0, a_1, a_2)X_2 = 0.$$

Note that this is not always well-defined as all the partial derivatives might be zero. Let T_PV denote this tangent line. Note that it is independent of the choice of the representative $[a_0 : a_1 : a_2]$ for P.

Lemma 2.1. *Let F be a homogeneous polynomial of degree r in variables $X_0, \ldots, X_n$. Then*

$$\sum_{i=0}^{n} \frac{\partial F}{\partial X_i} \cdot X_i = rF.$$

Proof. Both sides of the stated equality commute with taking linear combinations of polynomials, so it suffices to consider the case when

$$F = X_0^{i_0} \cdots X_n^{i_n},$$

with $i_0 + \cdots + i_n = r$. In this case we have

$$\frac{\partial F}{\partial X_s} \cdot X_s = i_s X_0^{i_0} \cdots X_n^{i_n}.$$

Summing over all indices s we therefore get

$$\sum_{s=0}^{n} \frac{\partial F}{\partial X_s} \cdot X_s = (\sum_{s=0}^{n} i_s)F = rF.$$

$\square$

Corollary 2.2. *The tangent line T_PV passes through P.*

Proof. We have

$$\sum_{i=0}^{2} \frac{\partial F}{\partial X_i}(a_0, a_1, a_2)a_i = (\text{deg}F)F(a_0, a_1, a_2) = 0.$$

$\square$

Definition 2.3. A point $P = [a_0 : a_1 : a_2] \in V$ is called *nonsingular* if at least one of the numbers

$$\frac{\partial F}{\partial X_0}(a_0, a_1, a_2),\ \frac{\partial F}{\partial X_1}(a_0, a_1, a_2),\ \frac{\partial F}{\partial X_2}(a_0, a_1, a_2)$$

is nonzero.

Example 2.4. Consider $V \subset \mathbb{P}^2(k)$ defined by the equation

$$F = X^2 - dY^2 - Z^2.$$

The the partial derivatives are

$$\frac{\partial F}{\partial X} = 2X, \quad \frac{\partial F}{\partial Y} = -2dY, \ \frac{\partial F}{\partial Z} = -2Z.$$

If $2d$ is not zero in the field k, then the only simultaneous solutions of these equations are given by the point $(0, 0, 0)$ which does not define a point of projective space, so in this case V is nonsingular at every point. If $2 = 0$ in the field, however, then all these partial derivatives are zero, and so there are no nonsingular points. Finally if $2 \neq 0$ in k, but $d = 0$ in k, then the only simultaneous solution of the equations of the partial derivatives is given by $[0 : 1 : 0]$, which is also a solution of F, so in this case all points except $[0 : 1 : 0]$ are nonsingular.

Example 2.5. Consider $V \subset \mathbb{P}^2(k)$ defined by the equation

$$F = Y^2Z - X^3 + AXZ^2 + BZ^3,$$

and assume that $6 \neq 0$ in k. The condition that V is nonsingular at every point can then be ensured by the condition that the *disciminant*

$$\Delta := 16(4A^3 - 27B^2)$$

is nonzero. This can be seen as follows. First we compute the partial derivatives

$$\frac{\partial F}{\partial X} = -3X^2 + AZ^2, \frac{\partial F}{\partial Y} = 2YZ, \quad \frac{\partial F}{\partial Z} = Y^2 + 2AXZ + 3BZ^2.$$

Let $P = [x, y, z]$ be a point of V which is singular. Then from the vanishing of $\frac{\partial F}{\partial Y}$ we get that either y or z is zero. We consider each of these cases separately.

For the case $z = 0$, note that then we have (since P is a point of V)

$$-x^3 = 0,$$

which implies that $x = 0$. We are therefore looking at the point $[0 : 1 : 0]$. But for this point we have $\frac{\partial F}{\partial Z}(0, 1, 0) \neq 0$ so this case provides no singular points.

In the case when $y = 0$, we may assume that $P = [x : 0 : 1]$, where x is a simultaneous solution of the equations

$$X^3 - AX - B = 0, \ -3X^2 + A = 0, \ 2AX + 3B = 0.$$

Now if these equations are satisfied, then we get that

$$-3B = 2AX$$

which upon squaring gives

$$9B^2 = 4A^2X^2.$$

Also from the second equation we get

$$A = 3X^2,$$

so we get that

$$27B^2 = 4A^3,$$

which contradicts our assumption that $\Delta \neq 0$.

Exercise 4. Let k be a field and let L_1 and L_2 be two distinct lines in $\mathbb{P}^2(k)$. What are the possibilities for the number of points in the intersection $L_1 \cap L_2$?

Exercise 5. The Euclidian algorithm for polynomials works over any field. In this exercise verify this and some of the basic consequences as follows. Throughout k is a field.

(a) Show that if f and g are polynomials, then there exist unique polynomials h and r with $\deg(r) < \deg(g)$ such that

$$f = gh + r.$$

(b) Show that if $a \in k$ is an element and f is a polynomial with $f(a) = 0$, then

$$f = (x - a)g$$

for some polynomial g.

(c) Let f be a polynomial of degree d, and suppose f has $d-1$ roots $a_1, \ldots, a_{d-1}$ in k. Show that then in fact f has d roots in k and f factors into a product of linear factors.

(d) Let f be a polynomial of degree d. Show that f has at most d roots in k.

Exercise 6. Let s be a complex variable (but if you have not taken any complex analysis you can just assume s is a real variable), and let

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}.$$

(a) Show that $\zeta(s)$ converges for s whose real part $\mathrm{Re}(s)$ is > 1.

(b) Makes sense of the infinite product (taken over all primes p)

$$\prod_p (1 - (1/p)^s)^{-1},$$

and show in particular that this infinite product converges for $\mathrm{Re}(s) > 1$.

(c) Show that for $\mathrm{Re}(s) > 1$ we have

$$\zeta(s) = \prod_p (1 - (1/p)^s)^{-1}.$$