MATH 115: HW 11 SOLUTIONS

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Section 7.8, exercise 6. Notice that we have

$$n^2 + (n+1)^2 = x^2$$

if and only if

$$(2n+1)^2 - 2x^2 = -1.$$

Therefore to show the exercise it suffices to show that the Pell-equation

$$u^2 - 2v^2 = -1$$

has infinitely many integer solutions with u odd. Notice that this last condition that u is odd is automatic since the right side of the equation is odd. Therefore it suffices to show that

$$u^2 - 2v^2 = -1$$

is infinitely many integer solutions. By corollary 7.23 it therefore suffices to show that the period of the continued fraction expansion of $\sqrt{2}$ is odd. By exercise 7.3 (a) we have

$$1 + \sqrt{2} = \langle 2, 2, 2, 2, \dots \rangle$$

which shows that the period is 1.

Section 7.8, problem 7. Notice that if k is an integer, then we have

$$(|k|)^2 - (k^2 - 1)1^2 = 1,$$

so the equation

$$x^2 - (k^2 - 1)y^2 = 1$$

has a positive solution $(|k|, 1)$. This solution is necessarily minimal among the (h_n, k_n)'s since the second coordinate is 1 and the k_n are increasing. Since the first solution to the equation

$$x^2 - (k^2 - 1)y^2 = -1$$

among the (h_n, k_n) must occur before the first solution to

$$x^2 - (k^2 - 1)y^2 = 1,$$

this shows that the equation

$$x^2 - (k^2 - 1)y^2 = -1$$

has no positive solutions, and therefore no integer solutions.

Section 7.8, problem 8. Notice that the period in this case is 2 so by theorem 7.25 the equation

$$x^2 - 18y^2 = -1$$

has no solutions, and the first solution to the equation

$$x^2 - 18y^2 = 1$$

is given by (h_1, k_1). Computing we have

$$h_0 = a_0 = 4, \quad k_0 = 1,$$

$$h_1 = a_1 h_0 + h_{-1} = 16 + 1 = 17, k_1 = a_1 k_0 + k_{-1} = 4.$$

Therefore the first solution is $(17, 4)$. Note that this really does satisfy the equation

$$(17)^2 - 18(4)^2 = 289 - 18 \cdot 16 = 289 - 288 = 1.$$

Section 7.9, problem 12. First note that if $p \equiv 1 \pmod 4$ and we have an integer solution

$$x^2 - py^2 = 1,$$

then x must be odd and y must be even. Indeed if x is even then reducing the equation modulo 4 we get

$$-y^2 \equiv 1 \pmod 4,$$

and since y^2 is either 0 or 1 mod 4 this is impossible. Therefore x is odd. Now in this case we $x^2 \equiv 1 \pmod 4$ so we get that

$$-py^2 \equiv 0 \pmod 4.$$

This implies that y must be even.

Now suppose we have

$$x_0^2 - py_0^2 = 1$$

with y_0 minimal. Then we have x_0 odd by the previous observation, so 2 divides both $x_0 + 1$ and $x_0 - 1$. On the other hand, we can write 2 as

$$2 = (x_0 + 1) - (x_0 - 1)$$

so 2 must be the greatest common divisor of $x_0 + 1$ and $x_0 - 1$. We therefore get

$$(x_0 + 1)(x_0 - 1) = x_0^2 - 1 = py_0^2.$$

Therefore we find that $2p$ divides either $x_0 + 1$ or $x_0 - 1$, and that

$$gcd((x_0 + 1)/2, (x_0 - 1)/2) = 1.$$

We conclude that we are in one of the two cases indicated in the exercise.

Case 1. This is the case when $x_0 - 1 = 2pu^2$ and $x_0 + 1 = 2v^2$ for some integers u and v (so $y_0 = uv$). Notice that $y_0 = uv$ implies that $|u| < y_0$, since we can't have $v = 1$ as this would give $x_0 = 0$. Therefore we find that

$$2v^2 - 2pu^2 = (x_0 + 1) - (x_0 - 1) = 2.$$

Dividing both sides by 2 we get

$$v^2 - pu^2 = 1,$$

which gives a strictly smaller solution that our original (x_0, y_0). Therefore this case is impossible.

Case 2. In this case we have $x_0 - 1 = 2u^2$ and $x_0 + 1 = 2pv^2$, and we get

$$2u^2 - 2pv^2 = (x_0 - 1) - (x_0 + 1) = -2,$$

which gives

$$u^2 - pv^2 = -1.$$

Therefore if $p \equiv 1 \pmod 4$ then the equation

$$x^2 - py^2 = -1$$

has a solution.

section 7.8, problem 13.

First note that

$$x^2 - 34y^2 = 1$$

has a solution given by $(35, 6)$. Indeed we have

$$(35)^2 - 34(6)^2 = 1225 - 34 \cdot 36 = 1225 - 1224 = 1.$$

To see that it is minimal it suffices to observe that the numbers

$$1 + 34y^2, \quad y = 1, 2, 3, 4, 5$$

are not perfect squares. These numbers are

$$35, 137, 307, 545, 851$$

and this can be checked by calculator (or directly). Since a minimal solution of

$$x^2 - 34y^2 = -1$$

would have y smaller than 6, to prove that we have no such solution it suffices to verify that the numbers

$$-1 + 34y^2, \quad y = 1, 2, 3, 4, 5$$

are not perfect squares. This list is

$$33, 135, 305, 543, 849,$$

and again a simple check with a calculator gives the result.

On the other hand, we have

$$(5/3)^2 - 34(1/3)^2 = (25 - 34)/9 = -1,$$

and

$$(3/5)^2 - 34(1/5)^2 = (9 - 34)/25 = -1.$$

Now observe that for any integer m with $3 \neq m$ it makes sense to reduce the solution $(5/3, 1/3)$ modulo m so this gives that we have a solution modulo all m prime m. Similary we get a solution modulo m from $(3/5, 1/5)$ for all m prime to 5. Now for any integer m, we can write $m = m_1 \cdot m_2$ with $(3, m_1) = 1$ and $(5, m_2) = 1$. By the chinese remainder theorem we can find integers a and b such that

$$a \equiv 5/3 \pmod{m_1}, \quad a \equiv 3/5 \pmod{m_2},$$
$$b \equiv 1/3 \pmod{m_1}, \quad b \equiv 1/5 \pmod{m_2}.$$

The pair (a, b) then gives a solution to

$$x^2 - 34y^2 = -1$$

modulo m.