Worksheet 7 solutions

1. $T(x_1, x_2, x_3) = |x_1 + x_2 + x_3|$ is not a linear transformation. Try $T(1, 0, 0) = 1$. Now, if it is a linear transformation, then $T(-1, 0, 0)$ should equal -1, but it doesn't; it also equals 1. You could also do something like $T(1, 0, 0) + T(-1, 0, 0) = 2$ should equal $T(0, 0, 0) = 0$, which it clearly does not.

2. The first column of a matrix is precisely what the vector

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

gets mapped to. This should be intuitive, since multiplying a matrix and a vector corresponds to taking a linear combination of the columns of the matrix. So if you want one particular column, multiply the matrix by a vector with a single 1 and the rest 0's.

Anyway, so our first matrix will just be to place vectors as the columns of A:

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$$

For the second matrix, our job is not as easy. If we knew what

$$T(\begin{bmatrix} 1 \\ 0 \end{bmatrix})T(\begin{bmatrix} 0 \\ 1 \end{bmatrix})$$

were, then we could have it easy. But notice!

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}(\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix})$$

So

$$T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \frac{1}{2}[T(\begin{bmatrix} 1 \\ -1 \end{bmatrix}) + T(\begin{bmatrix} 1 \\ 1 \end{bmatrix})] = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

Do the same for $T(0, 1)$.

3. Subspaces

   (a) Points in the first quadrant. Clearly adding up positive numbers will give you positive numbers. But you can't multiply anything by -1.

(b) Two lines through the origin. For each point on a line, if you multiply it by a scalar you remain on the line. But adding points from two different lines will give you a point not on either of them.

(c) The line through $(3, 4)$ and $(0, 0)$. Also you can do the whole space.

4. Think of the matrix of T, call it A. $A(x - y) = 0$, but $x - y \neq 0$, so A has a nontrivial null space. By the invertible matrix theorem A cannot span $\mathbb{R}^n$. Which is exactly what being onto means.

5.
$$A = \begin{bmatrix} 1 & 2 & -1 \\ -5 & 2 & 3 \\ 3 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{2}{3} \\ 0 & 1 & -\frac{1}{6} \\ 0 & 0 & 0 \end{bmatrix}$$

A basis of Col(A) is the first two columns of A (the ones that ended up having pivots in them when reduced), that is, $(1, -5, 3)$ and $(2, 2, 0)$. For Nul(A), we solve $Ax = 0$. Using our reduced echelon matrix we have x_3 free, $x_1 = \frac{2}{3}x_3$ and $x_2 = \frac{1}{6}x_3$. In other words, $(x_1, x_2, x_3) = x_3(\frac{2}{3}, \frac{1}{6}, 1)$ So our basis of Nul(A) is $(\frac{2}{3}, \frac{1}{6}, 1)$.

6. Turn it into a question about finding the basis of a column space. I.e. make a matrix with all of the vectors as the columns of this matrix. Then use that technique used in the previous problem.