

Worksheet 11

March 3rd, 2008

1. True or False

- (a) If B is invertible then $\text{Col}(AB) = \text{Col}(A)$.

True. When we say two sets are equal, we are really saying two things: that every v in the left side of the equal sign can be found on the right hand side, and vice versa. So we are saying that if $v \in \text{Col}(AB)$, then $v \in \text{Col}(A)$, AND that if $v \in \text{Col}(A)$, then $v \in \text{Col}(AB)$.

So suppose $v \in \text{Col}(AB)$. So $v = ABx$ for some x . But this means that $v = A(Bx)$, so to get v , just multiply A by the vector Bx . This means that any v in $\text{Col}(AB)$ is also in $\text{Col}(A)$.

Now, suppose we have a $v \in \text{Col}(A)$. This means that $v = Ax$ for some x . Let $y = B^{-1}x$, since B is invertible. Then we see that $ABy = v$, so indeed, v is also within the column span of AB .

- (b) If B is invertible then $\text{Nul}(AB) = \text{Nul}(A)$.

False. So what this statement is claiming is that if $ABx = 0$, then $Ax = 0$, and vice versa (if $Ax = 0$ then $ABx = 0$). Now, there's nothing blatantly false about this (that I can see). But it seems unnecessarily restrictive. So let's try to create a transformation for which this is not true.

Let's go for if $Ax = 0$ then $ABx = 0$. Well, make it so $Ae_1 = 0$, but $Ae_i \neq 0$ for $i > 1$, where e_i is the i th standard basis vector. So all we need to do is make it so that $Ae_1 = 0$, but $ABe_1 \neq 0$. So create an invertible matrix B so that e_1 gets sent to e_2 and e_2 gets sent to e_1 . Then $ABe_1 \neq 0$ but $ABe_2 = 0$.

- (c) If A is invertible then $\text{Col}(AB) = \text{Col}(B)$.

False. Similar story. Try

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Then

$$\text{Col}(B) = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$$

But

$$AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \text{Col}(AB) = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$$

- (d) If $\det(A) = 0$ and $\det(B) = 0$ then $\det(A + B) = 0$.

False. Example:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and the two right hand size matrices have determinant 0.

- (e) If A and B are square matrices then it is possible to have AB be invertible and yet have A not invertible.

False. If A is $n \times n$, then if A is not invertible then $\text{Col}(A)$ is not all of \mathbb{R}^n . Hence since $\text{Col}(AB) \subset \text{Col}(A)$, so $\text{Col}(AB)$ is smaller than \mathbb{R}^n so AB would not be invertible.

- (f) The set of all polynomials $p \in \mathbb{P}^3$ so that $p(2) = 2$ is a subspace.

False. The zero vector is not a member of this space.

- (g) The set of all polynomials $p \in \mathbb{P}^3$ so that $p(2x) = 4p(x)$ is a subspace.

True. (i) The zero vector satisfies the property; (ii) Suppose p satisfies the property. Then does $cp(x)$? Well, $cp(2x) = c \cdot 4p(x) = 4cp(x)$.

(iii) Suppose p and q satisfy the property. Does $p + q$? $(p + q)(2x) = p(2x) + q(2x) = 4p(x) + 4q(x) = 4(p + q)(x)$.

Indeed, you may solve for the subspace and find that it is simply $\text{span}\{x^2\}$.

2. In \mathbb{P}^3 , given

$$p(x) = x^3 + x + 1$$

write x with respect to the basis

$$\mathcal{B} = \{1 + x, 1 + x^2, x + x^2, x^3 - 3x\}$$

Write all of them in vector form.

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Now just solve this system of equations.

3. Show that e^x, e^{2x}, e^{3x} are linearly independent.

Well, let's look at the following equation:

$$c_1 e^x + c_2 e^{2x} + c_3 e^{3x} = 0$$

We want to show that $c_1 = c_2 = c_3 = 0$.

Well, for $x = 0$, we have

$$c_1 + c_2 + c_3 = 0$$

For $x = 1$ we get another equation:

$$ec_1 + e^2 c_2 + e^3 c_3 = 0$$

Now $x = 2$ we get

$$e^2 c_1 + e^4 c_2 + e^6 c_3 = 0$$

Now we have a system of equations $Ax = 0$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ e & e^2 & e^3 \\ e^2 & e^4 & e^6 \end{bmatrix}$$

When you row reduce this matrix you will see only the trivial solution works.

4. Is there a linear transformation T such that

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

finally,

$$T\left(\begin{bmatrix} 3 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

No; the observation is as follows.

$$2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

but, applying T to both sides gives us

$$2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

5. Find the inverse of the following matrix.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$