Additional Questions
April 13th, 2008

1. (7.1) Prove that if A is symmetric, and for every eigenvalue λ of A, $|\lambda| < 1$, then $\lim_{n\to\infty} A^n v = 0$.

   We use that $A : m \times m$ is symmetric to say that $\mathbb{R}^m$ has an orthonormal basis of eigenvectors of A. (In fact we only need that there is a basis of eigenvectors of A, the orthogonality is actually not necessary here. I.e. all we need is that A is guaranteed to be diagonalizable.)

   Then writing any $v \in \mathbb{R}^m$ as a sum of the eigenvectors,

$$v = a_1 v_1 + \ldots + a_m v_m$$
$$A^n v = a_1 \lambda_1^n v_1 + \ldots + a_m \lambda_m^n v_m$$
$$\lim_{n\to\infty} \lambda_i^n = 0$$

   Thus

$$\lim_{n\to\infty} A^n v = 0$$

2. (6.1) Find a basis of $W^\perp$, where $W = \text{span}\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}\}$

   $W^\perp$ consists of all vectors which are orthogonal to every vector in W. It is a theorem that any vector which is orthogonal to each of the vectors in a basis of W, will automatically be orthogonal to every vector in W. (This could be a proof you might want try.)

   So we want all vectors such that $(1,1,1,1)^T \cdot x = 0$ and $(1,0,1,0)^T \cdot x = 0$. In matrix form,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

   Find x.

3. (7.1) Prove that for a symmetric matrix, eigenvectors having different eigenvalues must be orthogonal. (Hint: Use the fact that $(Ax \cdot y) = (x \cdot A^T y)$.

   This was done in lecture. Let x and y be eigenvectors corresponding to distinct eigenvalues. $\lambda_1(x \cdot y) = (Ax \cdot y) = (x \cdot Ay) = \lambda_2(x \cdot y)$. Subtracting the leftmost and rightmost equations from each other, $(\lambda_1 - \lambda_2)(x \cdot y) = 0$. Well, the two eigenvalues are distinct, so $(x \cdot y) = 0$.

4. (Question 3 in the Practice side)

(a) Prove that if λ is an eigenvalue of an invertible matrix A, then $1/\lambda$ is an eigenvector of the inverse.

$$Av = \lambda v$$
$$v = A^{-1}\lambda v = \lambda A^{-1} v$$
$$\frac{1}{\lambda} v = A^{-1} v$$

And $\lambda \neq 0$ since A is invertible. (Invertible matrices do not have 0 eigenvalues.)

(b) $\text{rank}(AB) \leq \min\{\text{rank}(A)\text{rank}(B)\}$

First part. $\text{rank}(AB) \leq \text{rank}(A)$

We show that $\text{Col}(AB) \subseteq \text{Col}(A)$. Suppose $v \in \text{Col}(AB)$. Then $v = ABx$ for some x. But then $v = A(Bx)$, hence is a linear combinations of the rows of A. That means that if v is in the column span of AB then it is automatically in the column span of A.

Second part. $\text{rank}(AB) \leq \text{rank}(B)$

Let's prove a helper theorem. If w_{k+1} is in the span of $w_1, \ldots, w_k$, then Aw_{k+1} is in the span of $Aw_1, \ldots, Aw_k$. Why? Well, $w_{k+1} = a_1 w_1 + \ldots + a_k w_k$. Then $Aw_{k+1} = a_1 Aw_1 + \ldots + a_k Aw_k$. Note how the right hand side is already the span of what we wanted it to be.

Back to proving our statement. Let's write out the matrix multiplication. $AB = [Ab_1 Ab_2 \ldots Ab_n]$ where $b_1, \ldots, b_n$ are the columns of B. If B has rank k, then say there are k columns of B which are the basis of $\text{Col}(B)$. Call it $b_{i_1}, \ldots, b_{i_k}$ if you will. So indeed, any column of AB must be a linear combination of $Ab_{i_1} \ldots Ab_{i_k}$, by our helper theorem. That shows that the basis of columns of AB cannot be any more than k. (I.e. the column span of AB is spanned by k vectors, which are not necessarily linearly independent.)