Worksheet 22
April 25th, 2008

1 9.4

1. Are the following vectors linearly independent?

$$\mathbf{v_1} = \begin{bmatrix} e^x \\ 0 \\ e^{2x} \end{bmatrix}, \mathbf{x_2} = \begin{bmatrix} e^{2x} \\ e^x \\ e^x \end{bmatrix}, \mathbf{x_2} = \begin{bmatrix} e^x \\ e^{2x} \\ e^{2x} \end{bmatrix}$$

2. Suppose you had the system $x' = Ax + f$, where

$$A = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$$

$$f = \begin{bmatrix} -t^2 - 2t + 3 \\ -4t^2 - t - 2 \end{bmatrix}$$

   (a) Verify that the following two vectors are linearly independent and solve the equation $y' = Ax$.

$$\mathbf{x_1} = \begin{bmatrix} e^{5t} \\ e^{5t} \end{bmatrix}, \mathbf{x_2} = \begin{bmatrix} e^{-3t} \\ -e^{-3t} \end{bmatrix}$$

   (b) One solution to $x' = Ax + f$ is $\begin{bmatrix} t^2 + 1 \\ t - 1 \end{bmatrix}$. Solve the initial value problem $y' = Ax + f$, $y(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

3. What would be 3 linearly independent solutions to the following system of equations? (You need not know the methods of section 9.5 to do this.)

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= -2x_1 + x_2 + 2x_3 \end{aligned}$$

4. For the system $x' = Ax$, where A is $n \times n$, suppose you had put the n fundamental solutions $\mathbf{x_1}(t), \dots, \mathbf{x_n}(t)$ into a matrix $X(t) = [\mathbf{x_1}(t) \dots \mathbf{x_n}(t)]$. Given that the initial point $\mathbf{x}(t_0) = \mathbf{x_0}$, find a nice expression for the solution $\mathbf{x}(t)$ in terms of X, $\mathbf{x_0}$.

2 10.1

1. The equation $u_{tt} = u_{xx}$ (that is, $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$) is effective at modeling the behavior of a one dimensional wave. (In higher dimensions it is $u_{tt} = \nabla^2 u$.) Suppose we have a string of length one, with endpoints fixed. (We can give it the convenient coordinate system $u(0,t) = u(1,t) = 0$.) If you've played with vibrating strings before, you've probably noticed 'standing waves' – waves which maintain their shape but change in magnitude. Thus, a standing wave would look like $u(x,t) = X(x)T(t)$. Substitute this into the PDE and see what the standing waves look like.