Worksheet 8
September 20th, 2007

1. If $f(3) = 4$, $g(3) = 2$, $f'(3) = -6$ and $g'(3) = 5$ find the following numbers.

   (a) $(f+g)'(3)$
   Solution.
   $$(f+g)'(3) = f'(3) + g'(3) = -6 + 5 = -1$$

   (b) $(fg)'(3)$
   Solution.
   $$(fg)'(3) = g(3)f'(3) + f(3)g'(3) = 2(-6) + 4(5) = 8$$

   (c) $(f/g)'(3)$
   $$\begin{aligned}(f/g)'(3) &= \frac{g(3)f'(3) - f(3)g'(3)}{g(3)^2} \\ &= \frac{2(-6) - 4(5)}{2^2} = -8\end{aligned}$$

2. Find the derivatives of the following:

   (a) $\tan(x)$

   (b) $\frac{3\sin(x)}{x^2+4x+4}$

   (c) $\frac{f(x)g(x)}{h(x)}$

3. A spring and mass system can be described by the following equation: $a(t) = -\frac{k}{m}x(t)$ where where k is called the spring constant, x is the displacement, and m is the mass. Let $k = m = 1$. Which of the following functions could describe the movement of the spring?

   Solutions. In general, find the second derivative of $x(t)$ and verify whether or not $x(t) = -x''(t)$.

   (a) $x(t) = \sin(t)$

(b) $x(t) = \cos(t)$

(c) $x(t) = \frac{1}{t^2}$

(d) $x(t) = \sin(t) + \cos(t)$

(e) $x(t) = \sin(t)\cos(t)$

(f) $x(t) = a\sin(t) + b\cos(t)$ where a and b are any real number

4. For the following statements determine if they are true or false. Justify your answer.

(a) If $\lim_{x\to a} g(x) = \infty$ and $\lim_{x\to a} f(x) = 0$ then $\lim_{x\to a} f(x)g(x) = 0$.
Solution. The limit laws, $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$ hold if each of the limits of f and g exist and are finite. Otherwise, anything could happen. For example, let $g(x) = \frac{1}{x^2}$, and $f(x) = x^2$. Then $\lim_{x\to 0} x^2 \cdot \frac{1}{x^2} = \lim_{x\to 0} 1 = 1$.
However, if $g(x) = \frac{1}{x^2}$ and $f(x) = x$, then their combined limit to 0 is actually ∞. And finally, let $g(x) = \frac{1}{x^2}$ and $f(x) = x^3$. Then their combined limit is 0.

False.

On a similar note, you should all be aware of **indeterminate forms** , like $0 \cdot \infty$, $\frac{\infty}{\infty}$, $\infty - \infty$, and $\frac{0}{0}$. (There are a few other examples as well, but I can't think of them.) These forms are 'indeterminate' because if you have any of them, you can't at that point determine what the limit is. As an exercise, for any given form, come up with two limits that match that form that have two different results.

(b) $\sqrt{x}$ is differentiable at 0.
If

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}$$

using the power rule, we have

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

We see that $f'(0)$ is undefined, so $\sqrt{x}$ is not differentiable at $x = 0$.
False.

(c) If $f(0) < g(0)$ and $f(1) > g(1)$ then there is a $c \in [0,1]$ where $f(c) = g(c)$

Solution. Since we made no mention of the continuity of f or g, we cannot use the Intermediate Value Theorem. So in general, the solution is **False.** However, if f and g were continuous, then let $h(x) = f(x) - g(x)$. Note that h is continuous, and that $h(0) < 0$ and $h(1) > 0$, and now you can apply the intermediate value theorem.

(d) The derivative of a continuous function is continuous.

Let $f(x) = |x|$. In section 1.5 you proved this was continuous.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The derivative is

$$|x| = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ ? & \text{if } x = 0 \end{cases}$$

which is not continuous at 0. **False.**

(e) If $f(x)$ differentiable at a then $f(x)$ is continuous at a.

True. Refer to section 2.2, Theorem 4.

(f) A function can be continuous at exactly one point.

Examine the following function.

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is not rational} \end{cases}$$

This (should) not be on the test.

One may verify that this function is continuous only at $x = 0$.

True.

(g) For $f(x) = \frac{1}{x^2-4}$, there is an interval where $f(x) > 4000$

Let's try to find an interval with this property. $f(x) < 0$ for x in $(-2, 2)$, so we could either look at $x > 2$ or $x < -2$. Since we only want one interval, let's just look at $x > 2$.

$$\frac{1}{x^2-4} > 4000$$

Since $x > 2$, $x^2 - 4 > 0$.

$$\frac{1}{4000} > x^2 - 4 > 0$$

$$\frac{1}{4000} + 4 > x^2 > 4$$

$$\sqrt{\frac{1}{4000} + 4} > x > 2$$

True.

(h) $\lim_{t\to\infty} \frac{\cos(t)-1}{t}$ does not exist.

$\lim_{t\to\infty} \frac{\cos(t)-1}{t}$ Using the Squeeze theorem we can show that both $\frac{\cos(t)}{t}$ and $\frac{1}{t}$ converge to the finite value 0.

In my discussion section, I said that the limit I intended for the worksheet was $\lim_{t\to 0} \frac{\cos(t)-1}{t}$ This limit also exists; it represents the derivative of $\cos x$ at $x = 0$.

False.

(i) $\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

It is true, but only if the limits of f and g both exist and are finite. So, if those conditions are not met, then **False.**

(j) If $p(x)$ is a polynomial then $\lim_{x\to b} p(x) = p(b)$

True.

To actually see why, you may prove it using ϵ-δ, and some background knowledge of polynomials.

Let $p(x)$ be the polynomial, and $L = p(b)$. Given any $\epsilon > 0$, we want

$$|p(x) - L| < \epsilon$$

if $|x - b| < \delta$, for some δ depending on ϵ. Since $p(x) - p(b) = 0$ when $x = b$, the polynomial $q(x) = p(x) - p(b)$ will factor into

$$q(x) = (x - b)r(x)$$

where $r(x)$ is another polynomial. Add the contraint $\delta < 1$. Since polynomials have no asymptotes, it will always be possible to find m and M such that $m < r(x) < M$ if $b - 1 < x < b + 1$. Let $N = \max\{|m|, |M|\}$. Then

$$|p(x) - L| = |x - b||r(x)| \leq N|x - b| < \epsilon$$

and you may use

$$\delta = \min\{\frac{\epsilon}{N}, 1\}$$

(k) If $f(x) > 1$ for all x except 0, then $\lim_{x\to 0} f(x) > 1$.

Examine the function $f(x) = \frac{x}{\sin(x)}$. Since $|\sin(x)| < |x|$, we have that $\frac{x}{\sin(x)} > 1$ for all $x \neq 0$, but $\lim_{x\to 0} \frac{x}{\sin(x)} = 1$. (So it is not strictly greater than 1, it could be equal to 1 also.)

False.

(l) If the line $x = 1$ is a vertical asymptote of $y = f(x)$ then f is not defined at 1.

This is a silly problem, because we could just define $f(x)$ to have a value. **False.**

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

It's not continuous or anything, but it still is defined.

(m) $\frac{d}{dx}[f(x)g(x)] = f'(x)g'(x)$

False.

Suppose $f(x) = x$ and $g(x) = x$. Then $f(x)g(x) = x^2$, whose derivative is $2x$, which is not equal to the product of f' and g', which would be 1.

Remember the product rule!

5. The gas law for an idea gas at absolute temperature T, pressure P (in atmospheres), and volume V is $PV = nRT$ where n is the number of moles of the gas and $R = 0.0821$ is the gas constant. Suppose that at a certain instant $T = 300$K $V = 10$L, and $n = 10$. The volume is changing at .5 L/min and the temperature is changing at 13 K/min. How fast is the pressure changing?

Solution. Rewrite $PV = nRT$ in terms of P: $P = nR\frac{T}{V}$. Since n and R are constants in our equations, finding the time derivative of P we have

$$P'(t) = nR\frac{V(t)T'(t) - T(t)V'(t)}{V(t)^2}$$

Now we may, along the lines of problem 1, substitute all given values into the expression for $P'(t)$.

6. Prove $\lim_{x\to 0^+} \frac{1}{\sqrt{x}} = \infty$

   Since the limit is $+\infty$, and x approaches a finite point, we must use δ and M proof.

   So, let $M > 0$ be given. We want to show for this value of M, we can find a δ such that if

   $0 < x - 0 < \delta$, then $\frac{1}{\sqrt{x}} > M$.

   (We have $0 < x - 0$ because this is a one-sided limit.)

   So, for

$$\frac{1}{\sqrt{x}} > M > 0$$
$$1 > M\sqrt{x} > 0$$
$$\frac{1}{M} > \sqrt{x} > 0$$
$$\frac{1}{M^2} > x > 0$$

   Let $\delta = \frac{1}{M^2}$.

   Now, to complete the proof, we show that if

$$0 < x < \delta = \frac{1}{M^2}$$

   then

$$\frac{1}{\sqrt{x}} > M$$

So, if $0 < x < \delta = \frac{1}{M^2}$, then

$$0 < \sqrt{x} < \frac{1}{M}$$

$$0 < M < \frac{1}{\sqrt{x}}$$