MATH 1A

Solutions to Selected Exercises of Worksheet 6

For the quiz on Tuesday, you **do not** have to worry over problems 3, Part (c) of 4, and 5.

1. Find the following slopes of the tangent lines at the given points.

   a) $\frac{1}{x}$, at $x = 3$. To find the slope of the tangent line to a function f at a point a, find the following limit:

   $$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

   In our case,

   $$\lim_{x \to 3} \frac{\frac{1}{x} - \frac{1}{3}}{x - 3}$$
   $$= \lim_{x \to 3} \frac{\frac{3}{3x} - \frac{x}{3x}}{x - 3}$$
   $$= \lim_{x \to 3} -\frac{1}{3x}$$
   $$= -\frac{1}{9}$$

   .

   b) x^2, at $x = 3$. Using the method above, you should get that the slope is 6.

   c) $\sin(x)$, at $x = 0$. Solution: 1.

2. Refer to problems 17 and 18 in Section 2.1 of the text. (I cannot reproduce them here.)

3. **(Difficult)** Find the derivative for x^n using the limit notation.

   There are two good ways to do this problem. Using the limit definition
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
   We have
$$f'(a) = \lim_{x \to a} \frac{x^n - a^n}{x - a}$$
   Substituting $x = a$ to the top, we see that a is a root of $x^n - a^n$. Thus we may factor $(x - a)$ out of the top. To complete the problem, use long or synthetic division to factor $x^n - a^n$.

   The alternate way uses the Binomial Theorem, if you remember it. Using the other limit definition
$$f'(a) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$
   Now, $-f(x) + f(x + h) =$
$$-x^n + (x+h)^n = -x^n + x^n + \binom{n}{1} x^{n-1} h + \binom{n}{2} x^{n-2} h^2 + \ldots + \binom{n}{n} x^0 h^n$$
   So
$$\frac{f(x + h) - f(x)}{h} = \binom{n}{1} x^{n-1} + \binom{n}{2} x^{n-2} h + \binom{n}{3} x^{n-3} h^2 + \ldots + \binom{n}{n} x^0 h^n$$
   Letting $h \to 0$ we see that only the first term remains.

4. Find the derivatives of Problem 1 for an arbitrary point, a. (On the worksheet, it said to find the derivative at a general point x, which is the same thing, but I figured it would be less ambiguous to ask for a, since it doesn't conflict with the equation's independent variable.)

   a) Solution: $-\frac{1}{a^2}, a \neq 0$

   b) x^2. Solution:
$$f'(a) = \lim_{x \to a} \frac{x^2 - a^2}{x - a}$$

$$= \lim_{x \to a} \frac{(x-a)(x+a)}{x-a}$$
$$= \lim_{x \to a} x + a = 2a$$

.

c) See page 100 in Stewart's textbook. (Section 2.3)

5. For now, I will only give hints on this problem. For both parts, use either limit definition on the function $g(x)$.

6. Prove using Definition 8 in Section 1.6 that

$$\lim_{x \to \infty} x^3 = \infty$$

.

We want to show that for any positive $M > 0$, every single x beyond a certain value makes x^3 bigger than M. (Why does this prove the limit? This just means f(x) gets 'bigger and bigger', and specifically that it gets larger than any number you can think of.)

So, for $x^3 > M$ to be true, we can take the cube root of both sides. So $x > \sqrt[3]{M}$.

Literally, if x^3 is larger than M, then x must be larger than $\sqrt[3]{M}$. Formally, we call $N = \sqrt[3]{M}$.

To do the 'verification' step, we start with the assumption: Suppose $x > N = \sqrt[3]{M}$. Then that means that, by cubing both sides, that $x^3 > M$.

Addendum. One point of confusion in the last problems is, when can you apply some function f to both sides of an inequality? Well, what do we want to happen? If $x < y$, then we want $f(x) < f(y)$. Visually, what does this mean?