ON THE PSEUDOSPECTRUM OF THE ZAKHAROV-SHABAT SYSTEM

MICHAEL VANVALKENBURGH

1. Introduction

The Zakharov-Shabat system is the non-selfadjoint system of first-order differential equations given by

$$h\partial_x \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -i\lambda & A(x)e^{\frac{i}{h}S(x)} \\ -A(x)e^{-\frac{i}{h}S(x)} & i\lambda \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{1}$$

where $A > 0$ and S are real-valued functions, $h > 0$ is the semiclassical parameter, and λ is the (complex) spectral parameter. With the change of variables

$$v_1 = e^{-\frac{iS}{2h}}u_1, \qquad v_2 = e^{\frac{iS}{2h}}u_2,$$

we put the system (1) into a more familiar form:

$$\begin{pmatrix} -hD_x - \frac{1}{2}S'(x) & -iA(x) \\ -iA(x) & hD_x - \frac{1}{2}S'(x) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{2}$$

We denote the operator on the left-hand side by P, having principal symbol

$$p(x,\xi) = \begin{pmatrix} -\xi - \frac{1}{2}S'(x) & -iA(x) \\ -iA(x) & \xi - \frac{1}{2}S'(x) \end{pmatrix}.$$

In this paper we begin a study of the pseudospectrum of P—the set where the resolvent of P is large—by constructing quasimodes for particular choices of the spectral parameter λ. We do this by finding an asymptotic solution starting from a complex geometric optics ansatz. This method was famously used by Hörmander [6] and was rediscovered by Davies [3]. Extensions of this method may be found in the recent papers of Dencker, Sjöstrand, and Zworski [4], [5], which have strongly influenced our work presented here.

The main result of this paper is the following:

Theorem 1. *Let $S \in C^\infty(\mathbb{R};\mathbb{R})$, let $A \in \mathcal{S}(\mathbb{R};\mathbb{R})$, $A > 0$, and let $\lambda \in \mathbb{C}$ be such that for some $x_0 \in \mathbb{R}$ we have*

$$Re\,\lambda = -\frac{1}{2}S'(x_0) \quad \text{and} \quad 0 < |Im\,\lambda| < A(x_0).$$

Moreover, assume that $S^{(2k)}(x_0) \neq 0$ is the first nonvanishing derivative of S, at x_0, of order ≥ 2 (so $k \geq 1$). Then there exists $h_0 > 0$, and for any $N \in \mathbb{N}$ there exists $u_N \in C_0^\infty(\mathbb{R})$ with $||u_N||_{L^2} = 1$ and some constant $C_N > 0$ such that

$$||(P(x,hD_x) - \lambda I)u_N||_{L^2} \leq C_N h^N \qquad \forall h \in (0,h_0).$$

Here it is most practical to state the result in terms of derivatives of S. However, as emphasized by Dencker, Sjöstrand, and Zworski [4], [5], the underlying general mechanisms are the repeated Poisson brackets of the real and imaginary parts of $d(x,\xi)$, defined as

$$d(x,\xi) := \det(p(x,\xi)-\lambda I). \tag{3}$$

Indeed,

$$d(x,\xi) = -\xi^2 + (\frac{1}{2}S'(x)+\operatorname{Re}\lambda)^2 - (\operatorname{Im}\lambda)^2 + A(x)^2 + 2i(\frac{1}{2}S'(x)+\operatorname{Re}\lambda)\operatorname{Im}\lambda$$

and

$$\{\operatorname{Re} d, \operatorname{Im} d\} = -2\xi S''(x)\operatorname{Im}\lambda.$$

The general forms of higher Poisson brackets are quite messy, but the first nonvanishing Poisson bracket takes a simple form. Let $S^{(k+1)}$ be the first nonvanishing derivative of S of order greater than or equal to two ($k \geq 1$). Then

$$H^k_{\operatorname{Re} d}\operatorname{Im} d = (-2\xi)^k S^{(k+1)}(x)\operatorname{Im}\lambda$$

and all other Poisson brackets of order $\leq k$ are equal to zero.

In Sections 2 through 5 we prove Theorem 1, constructing quasimodes and thus proving blow-up of the resolvent. On the other hand, in Sections 6 and 7 we consider upper bounds for the resolvent. We prove that the genuine spectrum is discrete off the real line in Sections 8 and 9. Finally, in Section 10 we give two relevant examples appearing in the physics literature.

It remains to be seen what happens when the first nonvanishing derivative of S is an odd derivative. We would expect to have a subelliptic estimate; however, in general, subelliptic estimates seem very difficult and technical in the scalar case, while for systems they are largely unexplored. Hopefully in the future we can do something concrete and fairly simple for the case of the Zakharov-Shabat operator. It may also be interesting to more carefully study the boundary of the pseudospectrum. For both of these issues, we would welcome further physically significant examples from the physics community.

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2. The Geometric Optics Ansatz

We take the geometric optics ansatz,

$$e^{\frac{i\varphi(x)}{h}}\begin{pmatrix} a(x;h) \\ b(x;h)\end{pmatrix} \tag{4}$$

where the amplitudes

$$a(x;h) = a_0(x) + ha_1(x) + h^2a_2(x) + \cdots \qquad \text{and}$$
$$b(x;h) = b_0(x) + hb_1(x) + h^2b_2(x) + \cdots$$

are chosen so as to asymptotically solve (2).

We let

$$M(x) := \begin{pmatrix} \varphi' + \frac{1}{2}S' + \lambda & iA \\ iA & -\varphi' + \frac{1}{2}S' + \lambda \end{pmatrix}.$$

Then for the ansatz (4) to asymptotically solve (2), we want

$$M \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \text{and} \tag{5}$$

$$M \begin{pmatrix} a_{j+1} \\ b_{j+1} \end{pmatrix} = \begin{pmatrix} ia'_j \\ -ib'_j \end{pmatrix} \qquad \forall j \in \{0, 1, 2, 3, \cdots\}. \tag{6}$$

3. The Eikonal Equation

In order to have non-zero solutions to (5), of course we need to have $\det M = 0$; that is, we need φ to solve the (complex) "eikonal equation":

$$(\varphi')^2 = (\frac{1}{2}S' + \lambda)^2 + A^2.$$

We note that the "turning-point curve", defined to be the set where $\varphi' = 0$, is given parametrically by

$$\lambda(x) = -\frac{1}{2}S'(x) \pm iA(x). \tag{7}$$

While it is possible for our λ to lie on the turning point curve, we still have that $\varphi'(x_0) \neq 0$ by hypothesis.

Near the point x_0, where we choose to take $\varphi(x_0) = 0$, this has the solution

$$\varphi(x) = \pm \int_{x_0}^{x} \sqrt{(\frac{1}{2}S'(t) + \lambda)^2 + A(t)^2}\, dt.$$

Taking Taylor expansions and integrating, we get

$$\begin{aligned} \varphi(x) = \pm & \left[\sqrt{(\frac{1}{2}S'(x_0) + \lambda)^2 + A(x_0)^2} \right] (x - x_0) \\ & \pm \frac{1}{4} \left[\frac{S''(x_0)(\frac{1}{2}S'(x_0) + \lambda) + 2A(x_0)A'(x_0)}{\sqrt{(\frac{1}{2}S'(x_0) + \lambda)^2 + A(x_0)^2}} \right] (x - x_0)^2 + \mathcal{O}((x - x_0)^3). \end{aligned}$$

To prove the theorem in its full generality, we will need to expand φ to higher orders. But this is simplified by the fact that, in the final estimates, the important object is the *imaginary* part of the phase φ. For this we have the following lemma, where for convenience we let

$$\alpha := (\frac{1}{2}S'(x_0) + \lambda)^2 + A(x_0)^2 = A(x_0)^2 - (\operatorname{Im}\lambda)^2.$$

Lemma 1. *Let m be the order of the first nonvanishing derivative of S at the point x_0. Then* $Im\, \varphi(x) = \frac{\pm Im\, \lambda}{2(m!)\alpha^{\frac{1}{2}}} S^{(m)}(x_0)(x - x_0)^m + \mathcal{O}((x - x_0)^{m+1})$.

Proof. Let T_k^S denote the kth Taylor coefficient, centered at x_0, of $(\frac{1}{2}S'(x)+\lambda)^2$, and let T_k^A denote that of $A(x)^2$. We then have

$$\begin{aligned}\pm &\sqrt{(\frac{1}{2}S'(x)+\lambda)^2+A(x)^2}\\ &=\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}\alpha^{\frac{1}{2}-n}\left(\sum_{k=1}^{\infty}T_k^S(x-x_0)^k+\sum_{k=1}^{\infty}T_k^A(x-x_0)^k\right)^n .\end{aligned}$$

Let $S^{(j+1)}$ be the smallest nonvanishing derivative of S of order greater than or equal to two ($j\geq 1$). Then

$$\begin{aligned}(j!)T_j^S &= (\frac{1}{2}S'(x_0)+\lambda)S^{(j+1)}(x_0)\\ &\neq 0 \qquad \text{since } \operatorname{Im}\lambda\neq 0.\end{aligned}$$

Hence

$$\begin{aligned}\pm \operatorname{Im}&\sqrt{(\frac{1}{2}S'(x)+\lambda)^2+A(x)^2}\\ &=\operatorname{Im}\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}\alpha^{\frac{1}{2}-n}\left(\sum_{k=j}^{\infty}T_k^S(x-x_0)^k+\sum_{k=1}^{\infty}T_k^A(x-x_0)^k\right)^n\\ &=\operatorname{Im}\sum_{n=1}^{\infty}\binom{\frac{1}{2}}{n}\alpha^{\frac{1}{2}-n}\left[T_j^S(T_1^A)^{n-1}(x-x_0)^{n+j-1}+\mathcal{O}((x-x_0)^{n+j})\right]\\ &=\operatorname{Im}\left[\frac{1}{2}\alpha^{-\frac{1}{2}}T_j^S(x-x_0)^j+\mathcal{O}((x-x_0)^{j+1})\right]\\ &=\frac{\operatorname{Im}\lambda}{2\alpha^{\frac{1}{2}}(j!)}S^{(j+1)}(x_0)(x-x_0)^j+\mathcal{O}((x-x_0)^{j+1}).\end{aligned}$$

And so by integrating we finally get

$$\operatorname{Im}\varphi(x)=\frac{\pm\operatorname{Im}\lambda}{2\alpha^{\frac{1}{2}}(j+1)!}S^{(j+1)}(x_0)(x-x_0)^{j+1}+\mathcal{O}((x-x_0)^{j+2}),$$

which proves the lemma. □

4. THE TRANSPORT EQUATIONS

Since we are taking φ to solve the eikonal equation, the image of $M(x)$ is spanned by the eigenvector

$$\begin{pmatrix}\varphi'+\frac{1}{2}S'+\lambda\\ iA\end{pmatrix}$$

having eigenvalue $S'+2\lambda$. Since this eigenvalue is not zero and since $\varphi'+\frac{1}{2}S'+\lambda\neq 0$ (both because $\operatorname{Im}\lambda\neq 0$), we can diagonalize M as follows:

$$M=R\begin{pmatrix}S'+2\lambda & 0\\ 0 & 0\end{pmatrix}R^{-1}$$

where

$$R = \begin{pmatrix} \varphi' + \frac{1}{2}S' + \lambda & -iA \\ iA & \varphi' + \frac{1}{2}S' + \lambda \end{pmatrix}$$

and hence

$$R^{-1} = (S' + 2\lambda)^{-1} \begin{pmatrix} 1 & \frac{iA}{\varphi' + \frac{1}{2}S' + \lambda} \\ \frac{-iA}{\varphi' + \frac{1}{2}S' + \lambda} & 1 \end{pmatrix}.$$

Then, writing (5) and (6) in terms of this diagonalization, we want a and b to satisfy

$$a_j + \left(\frac{iA}{\varphi' + \frac{1}{2}S' + \lambda} \right) b_j = \begin{cases} 0 & \text{if } j = 0 \\ \frac{ia'_{j-1}}{S'+2\lambda} + \frac{Ab'_{j-1}}{(S'+2\lambda)(\varphi'+\frac{1}{2}S'+\lambda)} & \text{if } j \geq 1. \end{cases} \tag{8}$$

and

$$a'_j = \frac{i(\varphi' + \frac{1}{2}S' + \lambda)b'_j}{A} \qquad \forall j. \tag{9}$$

We will now construct a_0 and b_0 in detail. First of all, we want

$$\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \in \mathrm{Ker(M)} = \mathrm{Span}\left\{ \begin{pmatrix} -iA \\ \varphi' + \frac{1}{2}S' + \lambda \end{pmatrix} \right\}.$$

And secondly, we want $\begin{pmatrix} ia'_0 \\ -ib'_0 \end{pmatrix}$ to be in the image of M. But clearly $\mathrm{Im}(M) = (\mathrm{Ker}M^*)^\perp$, and we can easily compute

$$(\mathrm{Ker}M^*)^\perp = \mathrm{Span}\left\{ \begin{pmatrix} \varphi' + \frac{1}{2}S' + \lambda \\ iA \end{pmatrix} \right\}.$$

Therefore, we want both

$$\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \alpha(x) \begin{pmatrix} -iA \\ \varphi' + \frac{1}{2}S' + \lambda \end{pmatrix} \tag{10}$$

and

$$\begin{pmatrix} ia'_0 \\ -ib'_0 \end{pmatrix} = \beta(x) \begin{pmatrix} \varphi' + \frac{1}{2}S' + \lambda \\ iA \end{pmatrix}, \tag{11}$$

where the coefficients α and β are to be determined. But by a direct calculation, this is possible when

$$\alpha(x) = \left[(\varphi'(x) + \frac{1}{2}S'(x) + \lambda)^2 + A(x)^2 \right]^{-\frac{1}{2}}.$$

To solve for the remaining amplitudes, for $j \geq 1$ in (8) and (9), we let

$$\beta := \frac{iA}{\varphi' + \frac{1}{2}S' + \lambda}$$

and

$$c_{j-1} := \frac{ia'_{j-1}}{S' + 2\lambda} + \frac{Ab'_{j-1}}{(S' + 2\lambda)(\varphi' + \frac{1}{2}S' + \lambda)}.$$

(Note that $\frac{1}{\beta} - \beta = \frac{2\varphi'}{iA}$.) Then we are to solve the system

$$\begin{cases} a_j + \beta b_j = c_{j-1} \\ a_j' + \frac{1}{\beta} b_j' = 0. \end{cases}$$

But this is easily accomplished.

5. The Final Estimates

It is now time to complete the quasimode construction by estimating the error generated by taking only finitely many terms in (4), hence making rigorous the asymptotic series.

We take only finitely many amplitude terms:

$$\begin{aligned} a(x;h) &= a_0(x) + ha_1(x) + h^2 a_2(x) + \cdots + h^N a_N(x) \qquad \text{and} \\ b(x;h) &= b_0(x) + hb_1(x) + h^2 b_2(x) + \cdots + h^N b_N(x). \end{aligned} \tag{12}$$

Then

$$\begin{aligned} (P - \lambda I) \begin{pmatrix} e^{\frac{i\varphi}{h}} a \\ e^{\frac{i\varphi}{h}} b \end{pmatrix} &= e^{\frac{i\varphi}{h}} \left[ih^{N+1} \begin{pmatrix} a_N' \\ -b_N' \end{pmatrix} + \sum_{k=0}^{N-1} h^{k+1} \left(i \begin{pmatrix} a_k' \\ -b_k' \end{pmatrix} - M \begin{pmatrix} a_{k+1} \\ b_{k+1} \end{pmatrix} \right) - M \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \right] \\ &= ih^{N+1} e^{\frac{i\varphi}{h}} \begin{pmatrix} a_N' \\ -b_N' \end{pmatrix} \end{aligned}$$

where we have solved the eikonal and transport equations as above.

We now assume that $S^{(2k)}(x_0) \neq 0$ is the first nonvanishing derivative of S of order greater than or equal to two ($k \geq 1$). Then, using Lemma 1 with $m = 2k$, we choose the sign such that the leading term

$$\frac{\pm \operatorname{Im} \lambda}{2(2k)! \alpha^{\frac{1}{2}}} S^{(2k)}(x_0)(x - x_0)^{2k}$$

is a nonnegative quantity. Then there exists some $\gamma > 0$ such that, for x sufficiently close to x_0,

$$\gamma (x - x_0)^{2k} \leq \operatorname{Im} \varphi(x) \leq 3\gamma (x - x_0)^{2k}.$$

To conclude the quasimode construction, we let $\chi \in C_0^\infty(\mathbb{R})$ be $= 1$ for $|x - x_0| < \frac{1}{2}\delta$ and $= 0$ for $|x - x_0| > \delta$, where $\delta > 0$ is to be determined. Then we set

$$f(x) = e^{\frac{i\varphi}{h}} \begin{pmatrix} a \\ b \end{pmatrix}$$

with a and b as in (12). And we let

$$\tilde{f}(x) = \chi(x) f(x).$$

Then

$$||(P - \lambda I)\tilde{f}||_2 \leq \left|\left| \begin{pmatrix} -hD\chi & 0 \\ 0 & hD\chi \end{pmatrix} f \right|\right|_2 + ||\chi (P - \lambda I) f||_2.$$

As already noted,

$$||\chi(P-\lambda I)f||_2 = h^{N+1}\left|\left|\chi e^{\frac{i\varphi}{h}}\begin{pmatrix} a_N' \\ -b_N' \end{pmatrix}\right|\right|_2,$$

and then we compute

$$\begin{aligned}\left|\left|\chi e^{\frac{i\varphi}{h}}\begin{pmatrix} a_N' \\ -b_N' \end{pmatrix}\right|\right|_2^2 &= \int |\chi|^2 e^{-\frac{2\mathrm{Im}\,\varphi}{h}}[|a_N'|^2+|b_N'|^2]\,dx \\ &\leq C\int_{|x-x_0|\leq\delta} e^{-\frac{2\mathrm{Im}\,\varphi}{h}}\,dx \\ &\leq C\int_{|x-x_0|\leq\delta} e^{-\frac{2\gamma(x-x_0)^{2k}}{h}}\,dx \\ &= \mathcal{O}(h^{\frac{1}{2k}}).\end{aligned}$$

We also have

$$\begin{aligned}\left|\left|\begin{pmatrix} -hD\chi & 0 \\ 0 & hD\chi \end{pmatrix} f\right|\right|_2^2 &= \int |h\chi'|^2 e^{-\frac{2\mathrm{Im}\,\varphi}{h}}[|a|^2+|b|^2]\,dx \\ &\leq ch^2\int_{\frac{\delta}{2}<|x-x_0|<\delta} e^{-\frac{2\mathrm{Im}\,\varphi}{h}}\,dx \\ &\leq ch^2\int_{\frac{\delta}{2}<|x-x_0|<\delta} e^{-\frac{2\gamma(x-x_0)^{2k}}{h}}\,dx\end{aligned}$$

which decreases exponentially as $h \to 0$. Hence

$$||(P-\lambda I)\tilde{f}||_2 \leq \mathcal{O}(h^{N+1+\frac{1}{4k}}).$$

The last step is to bound $\tilde{f}$ from below:

$$\begin{aligned}||\tilde{f}||_2^2 &= \int \chi^2|f|^2\,dx \\ &\geq \int_{|x-x_0|\leq\frac{\delta}{2}} |f|^2\,dx \\ &= \int_{|x-x_0|\leq\frac{\delta}{2}} e^{-\frac{2\mathrm{Im}\,\varphi}{h}}[|a|^2+|b|^2]\,dx \\ &\geq c\int_{|x-x_0|\leq\frac{\delta}{2}} e^{-\frac{6\gamma(x-x_0)^{2k}}{h}}\,dx \\ &\geq c_0 h^{\frac{1}{2k}},\end{aligned}$$

where we have used the fact that we have non-zero solutions to the transport equations.

We have thus proved that

$$||(P-\lambda I)\tilde{f}||_2 \leq \mathcal{O}(h^{N+1+\frac{1}{4k}}) \leq \mathcal{O}(h^{N+1})||\tilde{f}||_2,$$

concluding the quasimode construction, and hence finally proving the theorem.

6. Upper Bounds for the Resolvent

To obtain upper bounds for the resolvent, we will use the semiclassical pseudodifferential calculus. In this and the following sections, we will restrict ourselves to $S \in C^\infty(\mathbb{R};\mathbb{R})$ such that $S' \in C_b^\infty(\mathbb{R};\mathbb{R})$, where

$$C_b^\infty := \{f \in C^\infty;\ \partial^\alpha f \in L^\infty\ \forall\alpha\}.$$

And we will take $A \in \mathcal{S}(\mathbb{R};\mathbb{R})$, $A > 0$, as before.

In studying our matrix-valued symbols, we might as well use the norm

$$||B|| = \max_{i,j} |b_{ij}|, \qquad \text{where } B = (b_{ij})_{1\le i,j\le n}.$$

Then for our symbol

$$p(x,\xi) = \begin{pmatrix} -\xi - \frac{1}{2}S'(x) & -iA(x) \\ -iA(x) & \xi - \frac{1}{2}S'(x) \end{pmatrix}$$

we have

$$\begin{aligned} &||p(x,\xi) - \lambda I|| \le C(1+|\xi|), \\ &||\partial_x^\alpha p(x,\xi)|| \le C_\alpha && \text{for } \alpha \ge 1, \\ &||\partial_\xi p(x,\xi)|| = 1, && \text{and} \\ &||\partial_x^\alpha \partial_\xi^\beta p(x,\xi)|| = 0 && \text{for } \beta \ge 1 \text{ and } \alpha+\beta \ge 2. \end{aligned}$$

So, with the admissible weight function $m(x,\xi) = 1 + |\xi|$, we have $p - \lambda I \in S(m)$.

We now prove an ellipticity result:

Lemma 2. *Suppose that $A \in \mathcal{S}(\mathbb{R};\mathbb{R})$, $A > 0$, and that S is such that $S' \in C_b^\infty(\mathbb{R};\mathbb{R})$. If $\lambda \in \mathbb{C}$ is such that $|d(x,\xi)| \ge \epsilon$ for all $(x,\xi) \in \mathbb{R}^2$, for some fixed $\epsilon > 0$, then we have*

$$||(P-\lambda I)^{-1}u||_{L^2} \le C(\epsilon,\lambda)||u||_{L^2}.$$

Proof. The hypothesis says precisely that

$$\begin{aligned} d(x,\xi) &\equiv |\det(p(x,\xi) - \lambda I)| \\ &= |-\xi^2 + (\frac{1}{2}S'(x) + \lambda)^2 + A(x)^2| \qquad (13)\\ &\ge \epsilon \qquad \forall (x,\xi), \qquad \text{for some } \epsilon > 0 \end{aligned}$$

(which requires $|\mathrm{Im}\,\lambda| \ne 0$; also see Section 7).

We first demonstrate the ellipticity of the symbol

$$(p(x,\xi) - \lambda I)^{-1} = \frac{1}{-\xi^2 + (\frac{1}{2}S'(x)+\lambda)^2 + A(x)^2} \begin{pmatrix} \xi - \frac{1}{2}S'(x) - \lambda & iA(x) \\ iA(x) & -\xi - \frac{1}{2}S'(x) - \lambda \end{pmatrix}.$$

That is, first we show that

$$||(p(x,\xi) - \lambda I)^{-1}|| \le C(\epsilon,\lambda)(1+|\xi|)^{-1} \qquad \forall (x,\xi).$$

For this we take $K >> 1$ to be determined. In fact, we immediately take K such that $|\lambda| \le \frac{1}{2}K$. If $|\xi| \le K$, then clearly

$$||(p(x,\xi) - \lambda I)^{-1}|| \le \frac{C(K)}{\epsilon}.$$

On the other hand, if $|\xi| \geq K$, then

$$\begin{aligned} |d(x,\xi)| &\geq \xi^2 - |\lambda|^2 - |S'(x)||\lambda| - (\frac{1}{2}S'(x))^2 - A(x)^2 \\ &\geq \frac{1}{2}\xi^2 + \frac{1}{2}K^2 - \frac{1}{4}K^2 - \frac{1}{2}|S'(x)|K - (\frac{1}{2}S'(x))^2 - A(x)^2 \\ &\geq \frac{1}{2}\xi^2 \qquad \text{when } K \text{ is sufficiently large.} \end{aligned}$$

Hence

$$||(p(x,\xi) - \lambda I)^{-1}|| \leq \frac{C}{|\xi|}.$$

Moreover, it is now easy to see that $(p(x,\xi) - \lambda I)^{-1} \in S(\frac{1}{m})$. Hence, using the pseudo-differential calculus,

$$(P - \lambda I)^{-1} : \quad L^2(\mathbb{R};\mathbb{C}^2) \to L^2(\mathbb{R};\mathbb{C}^2)$$

is a bounded operator; that is,

$$||(P - \lambda I)^{-1}u||_{L^2} \leq C(\epsilon,\lambda)||u||_{L^2}.$$

□

7. The Geometric Meaning of $|\det(p(x,\xi) - \lambda I)| \geq \epsilon$

In this section we give a geometric meaning to (13). We fix $\lambda \in \mathbb{C}$ and suppose that

$$\begin{aligned} |\det(p(x,\xi) - \lambda I)|^2 &= \left(-\xi^2 + (\frac{1}{2}S'(x) + \operatorname{Re}\lambda)^2 - (\operatorname{Im}\lambda)^2 + A(x)^2\right)^2 \\ &\qquad + 4(\operatorname{Im}\lambda)^2(\frac{1}{2}S'(x) + \operatorname{Re}\lambda)^2 \\ &\geq \epsilon^2 \qquad \forall (x,\xi), \qquad \text{for some } \epsilon > 0. \end{aligned} \tag{14}$$

Clearly for this to be true we need $\operatorname{Im}\lambda \neq 0$.

Lemma 3. *Suppose that $A \in \mathcal{S}(\mathbb{R};\mathbb{R})$, $A > 0$, and that S is such that $S' \in C_b^\infty(\mathbb{R};\mathbb{R})$. If $|\det(p(x,\xi) - \lambda I)| \geq \epsilon$ for all (x,ξ), for some $\epsilon > 0$, then $Im\,\lambda \neq 0$ and*

$$\inf |\lambda - \lambda_0| > 0$$

where the infimum is taken over all $\lambda_0 \in \mathbb{C}$ of the form

$$\lambda_0 = -\frac{1}{2}S'(x) + i Im\,\lambda_0$$

with $|Im\,\lambda_0| < A(x)$.

Proof. Case 1: There exists some $\delta > 0$ such that

$$(\frac{1}{2}S'(x) + \operatorname{Re}\lambda)^2 \geq \delta \qquad \forall x.$$

Case 2: For all $\delta > 0$ there exists some $x \in \mathbb{R}$ such that

$$(\frac{1}{2}S'(x) + \operatorname{Re}\lambda)^2 < \delta.$$

Hence there is a sequence x_k such that

$$\lim_{k\to\infty} -\frac{1}{2}S'(x_k) = \operatorname{Re}\lambda.$$

Suppose first that the x_k live in a compact set, so that there is a subsequence converging to some $x_0 \in \mathbb{R}$. Then we have

$$(\frac{1}{2}S'(x_0) + \operatorname{Re}\lambda)^2 = 0$$

and hence

$$|\xi^2 - (A(x_0)^2 - (\operatorname{Im}\lambda)^2)| \geq \epsilon \qquad \forall\xi.$$

This implies that

$$(\operatorname{Im}\lambda)^2 > A(x_0)^2,$$

that

$$\xi^2 + (\operatorname{Im}\lambda)^2 - A(x_0)^2 \geq \epsilon \qquad \forall\xi,$$

and hence that

$$(\operatorname{Im}\lambda)^2 \geq A(x_0)^2 + \epsilon.$$

On the other hand, if the sequence x_k is unbounded, we take a subsequence, also denoted by x_k, such that $|x_k| \to \infty$ as $k \to \infty$. We still have that $\lim_{k\to\infty} -\frac{1}{2}S'(x_k) = \operatorname{Re}\lambda$, but now we may assume that

$$-\frac{1}{2}S'(x) \neq \operatorname{Re}\lambda \qquad \forall x \in \mathbb{R},$$

as otherwise the previous argument would work. Then from (14) we have that

$$\left(-\xi^2 + (\frac{1}{2}S'(x_k) + \operatorname{Re}\lambda)^2 - (\operatorname{Im}\lambda)^2 + A(x_k)^2\right)^2 + 4(\operatorname{Im}\lambda)^2(\frac{1}{2}S'(x_k) + \operatorname{Re}\lambda)^2 \geq \epsilon^2 \quad \forall\xi \text{ and } \forall k.$$

Since A is a Schwartz function, this implies that

$$\xi^2 + (\operatorname{Im}\lambda)^2 \geq \epsilon \quad \forall\xi$$

and hence that

$$|\operatorname{Im}\lambda| \geq \sqrt{\epsilon}.$$

We take $R > 0$ such that $|x| \geq R$ implies that

$$A(x) \leq \frac{1}{2}\sqrt{\epsilon}.$$

We then also have some $d > 0$ such that $|x| \leq R$ implies that

$$|\operatorname{Re}\lambda + \frac{1}{2}S'(x)| \geq d.$$

If $\lambda_0 \in \mathbb{C}$ is such that

$$\lambda_0 = -\frac{1}{2}S'(x) + i\operatorname{Im}\lambda_0, \qquad \text{where } |\operatorname{Im}\lambda_0| < A(x),$$

then

$$|\lambda - \lambda_0|^2 = (\operatorname{Re}\lambda + \frac{1}{2}S'(x))^2 + (\operatorname{Im}\lambda - \operatorname{Im}\lambda_0)^2.$$

So if $|x| \leq R$, then we have

$$|\lambda - \lambda_0| \geq d,$$

and if $|x| \geq R$, then we have

$$\begin{aligned} |\lambda - \lambda_0| &\geq |\operatorname{Im}\lambda| - |\operatorname{Im}\lambda_0| \\ &\geq \frac{1}{2}\sqrt{\epsilon}. \end{aligned}$$

In all cases we have thus proven the result. □

8. $\Sigma(p)$ AND $\Sigma_\infty(p)$

In the next sections we use the methods of Dencker, Sjöstrand, and Zworski [5], [4], to prove the discreteness of the spectrum off the real axis.

We begin with two central definitions.

Definition 1. *Let $p \in C^\infty(T^*\mathbb{R}^n, \mathcal{L}(\mathbb{C}^N, \mathbb{C}^N))$. We denote the closure of the set of eigenvalues of p by*

$$\Sigma(p) = \overline{\{\lambda \in \mathbb{C};\, \exists w \in T^*\mathbb{R}^n,\, |p(w) - \lambda Id_N| = 0\}} \tag{15}$$

and the values at infinity:

$$\Sigma_\infty(p) = \{\lambda \in \mathbb{C};\, \exists w_j \to \infty,\, \exists u_j \in \mathbb{C}^N \backslash 0;\, |p(w_j)u_j - \lambda u_j|/|u_j| \to 0,\, j \to \infty\}, \tag{16}$$

which is closed in $\mathbb{C}$.

The statement that $\lambda(x, \xi)$ is an eigenvalue of the matrix

$$p(x, \xi) = \begin{pmatrix} -\xi - \frac{1}{2}S'(x) & -iA(x) \\ -iA(x) & \xi - \frac{1}{2}S'(x) \end{pmatrix} \tag{17}$$

is equivalent to the statement that

$$\begin{cases} \operatorname{Im}\lambda = 0 & \text{and} \quad \xi^2 = (\frac{1}{2}S'(x) + \operatorname{Re}\lambda)^2 + A(x)^2 \\ \text{OR} & \\ \frac{1}{2}S'(x) + \operatorname{Re}\lambda = 0 & \text{and} \quad \xi^2 = A(x)^2 - (\operatorname{Im}\lambda)^2 \end{cases} \tag{18}$$

Hence $\Sigma(p)$ is precisely the set

$$\{\lambda \in \mathbb{C};\, \operatorname{Im}\lambda = 0\} \cup \{\lambda \in \mathbb{C};\, \exists x \in \mathbb{R} \text{ s.t. } \operatorname{Re}\lambda = -\frac{1}{2}S'(x) \text{ and } |\operatorname{Im}\lambda| \leq A(x)\}. \tag{19}$$

We now turn to $\Sigma_\infty(p)$ and prove that $\Sigma_\infty(p) \subset \mathbb{R}$. Let $\lambda \in \mathbb{C}$ such that $\operatorname{Im}\lambda \neq 0$. In the following calculations, we use the expression of $p(x, \xi) - \lambda I$ as a sum of a selfadjoint matrix and an anti-selfadjoint matrix:

$$p(x, \xi) - \lambda I = A + B$$

where

$$A = \begin{pmatrix} -\xi - \frac{1}{2}S'(x) - \operatorname{Re}\lambda & 0 \\ 0 & \xi - \frac{1}{2}S'(x) - \operatorname{Re}\lambda \end{pmatrix},$$

$$B = -i \begin{pmatrix} \operatorname{Im}\lambda & A(x) \\ A(x) & \operatorname{Im}\lambda \end{pmatrix},$$

and where the commutator is

$$[A, B] = 2i\xi A(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Correspondingly, if we write $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{C}^2$, then

$$\begin{aligned} |A\vec{u}|^2 &= (\xi + \frac{1}{2}S'(x) + \operatorname{Re}\lambda)^2|u_1|^2 + (-\xi + \frac{1}{2}S'(x) + \operatorname{Re}\lambda)^2|u_2|^2, \\ |B\vec{u}|^2 &= [A(x)^2 + (\operatorname{Im}\lambda)^2][|u_1|^2 + |u_2|^2] + 4A(x)(\operatorname{Im}\lambda)\operatorname{Re}(\overline{u_1}u_2), \end{aligned}$$

and[1]

$$\langle [A, B]\vec{u}, \vec{u}\rangle = -4\xi A(x)\operatorname{Im}(\overline{u_1}u_2).$$

To prove that $\lambda \notin \Sigma_\infty(p)$, we consider

$$\{(x, \xi);\ |x| \geq C\} \cup \{(x, \xi);\ |\xi| \geq C\}, \tag{20}$$

for C to be determined.

In the first case, take $|\xi| \geq R$, where R is to be determined, depending only on $||S'||_\infty$, $||A||_\infty$, and λ. We then have

$$| \pm \xi + \frac{1}{2}S'(x) + \operatorname{Re}\lambda| \geq |\xi| - |\frac{1}{2}S'(x) + \operatorname{Re}\lambda|.$$

Hence

$$\begin{aligned} |A\vec{u}|^2 &\geq \left[|\xi| - |\frac{1}{2}S'(x) + \operatorname{Re}\lambda|\right]^2 |\vec{u}|^2 \\ &\geq \left[\frac{1}{2}|\xi| + \frac{1}{2}R - |\frac{1}{2}S'(x) + \operatorname{Re}\lambda|\right]^2 |\vec{u}|^2 \\ &\geq \frac{1}{4}|\xi|^2|\vec{u}|^2 \end{aligned}$$

when R is large enough. Also,

$$\langle [A, B]\vec{u}, \vec{u}\rangle \geq -2|\xi|A(x)|\vec{u}|^2,$$

so that

$$\begin{aligned} |A\vec{u}|^2 + \langle [A, B]\vec{u}, \vec{u}\rangle &\geq |\xi| \left[\frac{1}{4}|\xi| - 2A(x)\right] |\vec{u}|^2 \\ &= R\left[\frac{1}{4}R - 2A(x)\right] |\vec{u}|^2. \end{aligned}$$

Taking $R \geq 4 + 8||A||_\infty$, we have

$$|(p - \lambda I)\vec{u}|^2 \geq R|\vec{u}|^2.$$

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[1] Note that $\operatorname{Im}(\overline{u}v)$ is a symplectic form.

In the second case, ξ is bounded: $|\xi| \leq R$. Let

$$\epsilon = \min\{\frac{1}{8}|\mathrm{Im}\,\lambda|,\ \frac{1}{8R}|\mathrm{Im}\,\lambda|^2\}.$$

We then take $C > 0$ to be so large that $A(x) \leq \epsilon$ for all $|x| \geq C$. Then we have

$$\begin{aligned}
|(p - \lambda I)\vec{u}|^2 &\geq |B\vec{u}|^2 + \langle [A, B]\vec{u}, \vec{u}\rangle \\
&\geq \left[(\mathrm{Im}\,\lambda)^2 - 2\epsilon(|\mathrm{Im}\,\lambda| + |\xi|)\right] |\vec{u}|^2 \\
&\geq \left[(\mathrm{Im}\,\lambda)^2 - \frac{1}{4}(\mathrm{Im}\,\lambda)^2 - \frac{1}{4R}(\mathrm{Im}\,\lambda)^2|\xi|\right] |\vec{u}|^2 \\
&\geq \frac{1}{2}(\mathrm{Im}\,\lambda)^2|\vec{u}|^2.
\end{aligned}$$

So in all cases we are done: $\Sigma_\infty(p) \subset \mathbb{R}$.

9. Discreteness of the Spectrum Away From $\mathbb{R}$

Here we use analytic Fredholm theory to prove discreteness of the spectrum away from the real line. The argument generally follows Dencker, Sjöstrand, and Zworski [4], [5].

Proposition 1. *Suppose that $A \in \mathcal{S}(\mathbb{R}; \mathbb{R})$, $A > 0$, and that S is such that $S' \in C_b^\infty(\mathbb{R}; \mathbb{R})$. Let $\Omega \subset \mathbb{C}$ be an open, connected, and bounded set such that*

$$\overline{\Omega} \cap \Sigma_\infty(p) = \emptyset \quad \textit{and} \quad \Omega \cap \complement\Sigma(p) \neq \emptyset.$$

Then

$$(P(h) - zI)^{-1}, \qquad 0 < h << 1,\ z \in \Omega,$$

is a meromorphic family of operators with poles of finite rank. In particular, for h sufficiently small, the spectrum of $P(h) := P(x, hD)$ is discrete in any such set. When $\Omega \cap \Sigma(p) = \emptyset$ we find that Ω contains no spectrum of $P(h)$.

Proof. We first claim that $\exists C > 0$ such that

$$|(p(w) - zI)^{-1}| \leq C \qquad \text{if } z \in \Omega \text{ and } |w| > C. \tag{21}$$

Suppose not. Then $\exists w_j \to \infty$ and $z_j \in \Omega$ such that

$$|(p(w_j) - z_j I)^{-1}| \to \infty \quad \text{as } j \to \infty.$$

Thus $\exists u_j \in \mathbb{C}^2$ with $|u_j| = 1$ such that

$$|(p(w_j) - z_j I)u_j| \to 0.$$

Since Ω is bounded, we may take a subsequence such that

$$z_j \to z \in \overline{\Omega} \cap \Sigma_\infty(p) = \emptyset$$

which of course is impossible.

Next we show that $\exists \lambda_0 \in \Omega$ such that $(p(w) - \lambda_0 I)^{-1} \in C_b^\infty$. In all of this, we may take Ω to be

$$\Omega_{R,\epsilon} := \{z \in \mathbb{C};\ \mathrm{Im}\, z > \epsilon \text{ and } |z| < R\},$$

where we will take $R > 0$ to be sufficiently large. First of all, $(p - \lambda_0 I)^{-1}$ is bounded when $\lambda_0 \in \Omega_{R,\epsilon}$ and $|\lambda_0|$ is sufficiently large. Indeed, consider two cases of

$$(p(x,\xi) - \lambda_0 I)^{-1} = \frac{1}{-\xi^2 + (\frac{1}{2}S'(x) + \lambda_0)^2 + A(x)^2} \begin{pmatrix} \xi - \frac{1}{2}S'(x) - \lambda_0 & iA(x) \\ iA(x) & -\xi - \frac{1}{2}S'(x) - \lambda_0 \end{pmatrix}. \tag{22}$$

If $|\xi| \leq C$, then the matrix is bounded and

$$| - \xi^2 + (\frac{1}{2}S'(x) + \lambda_0)^2 + A(x)^2|^{-2} \leq C$$

when $|\lambda_0|$ is sufficiently large. On the other hand, if $|\xi| \geq C$ for $C >> 1$, then

$$|(p - \lambda_0 I)^{-1}| \leq \frac{C}{|\xi|} \leq C.$$

To show that the derivatives of $(p - \lambda_0 I)^{-1}$ are bounded, we need only to look at the expression (22).

To complete the proof, we closely follow Dencker, Sjöstrand, and Zworski [4], [5]. Let $\chi \in C_0^\infty(T^*\mathbb{R})$, $0 \leq \chi(w) \leq 1$, and $\chi(w) = 1$ when $|w| \leq C$, where C is given by (21). Let

$$R(w,z) = \chi(w)(p(w) - \lambda_0 I)^{-1} + (1 - \chi(w))(p(w) - zI)^{-1}$$

for $z \in \Omega$, which, by our previous arguments, is in C_b^∞. The symbol calculus then gives

$$R^w(x, hD, z)(P(h) - zI) = I + hB_1(h,z) + K_1(h,z)$$

and

$$(P(h) - zI)R^w(x, hD, z) = I + hB_2(h,z) + K_2(h,z),$$

where $K_j(h,z)$ are compact operators on $L^2(\mathbb{R})$ depending holomorphically on z, vanishing for $z = z_0$, and where the $B_j(h,z)$ are bounded on $L^2(\mathbb{R})$, $j = 1,2$. By the analytic Fredholm theory we then have that $(P(h) - zI)^{-1}$ is meromorphic in $z \in \Omega$ for h sufficiently small. When $\Omega \cap \Sigma(p) = \emptyset$ we may take $R(w,z) = (p(w) - zI)^{-1}$. Then $K_j \equiv 0$ and $P(h) - zI$ is invertible for small enough h. □

10. EXAMPLES

One special case of considerable interest occurs when

$$A(x) = \operatorname{sech}(2x) \quad \text{and} \quad S(x) = \operatorname{sech}(2x).$$

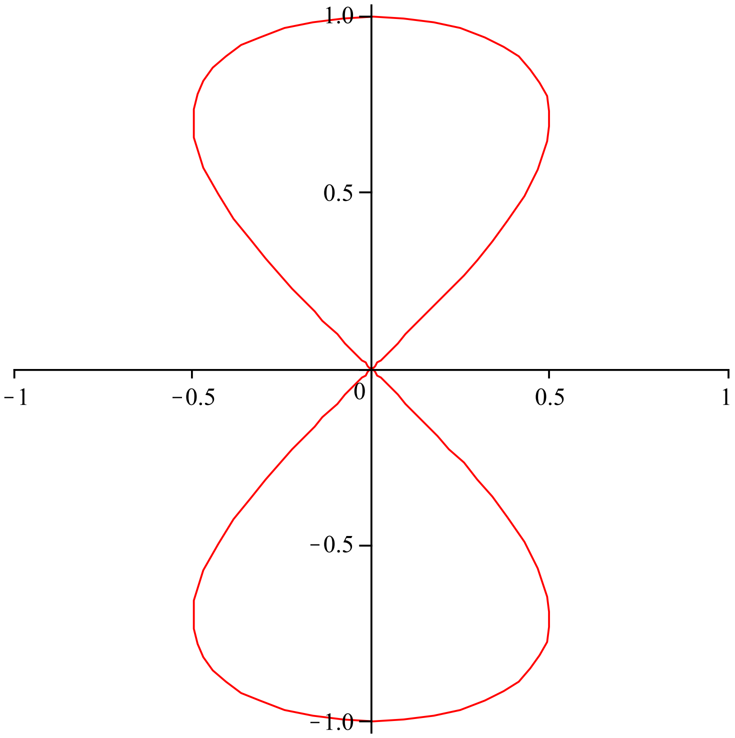
Then the turning point curve is the "figure eight" in the figure. Numerical studies of the eigenvalues of the Zakharov-Shabat system in this case can be found in the works of Bronski and Miller [1], [2], [7].

Another interesting special case occurs when

$$A(x) = -\operatorname{sech} x \quad \text{and} \quad S'(x) = -\mu \tanh x,$$

where μ is a real parameter. The semiclassical limit of the Zakharov-Shabat eigenvalue problem in this case was studied by Tovbis and Venakides, who found an explicit solution [8]. The turning point curve in this case is simply the ellipse given in the (x,y) plane by

$$\left(\frac{2x}{\mu}\right)^2 + y^2 = 1.$$



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UCLA Department of Mathematics, Los Angeles, CA 90095-1555, USA
E-mail address: **mvanvalk@ucla.edu**