A SIMPLE PROOF OF UNIQUE CONTINUATION FOR J-HOLOMORPHIC CURVES

MICHAEL VANVALKENBURGH

Abstract. In this expository paper, we prove strong unique continuation for J-holomorphic curves by first giving a simple proof of Aronszajn's theorem in the special case of the two-dimensional flat Laplacian.

1. Introduction

In the study of J-holomorphic curves and symplectic topology as presented by McDuff and Salamon [7], a basic fact is the strong unique continuation property for J-holomorphic curves. In their book, the strong unique continuation property is a first step in a chain of events leading to the proof that, for a generic almost complex structure J, the moduli space $\mathcal{M}^*(A, \Sigma; J)$ of simple J-holomorphic A-curves is a smooth finite dimensional manifold, and from there to the construction of the Gromov-Witten invariants for a suitable class of symplectic manifolds (see pages 4 and 38 of [7] for the outline of this approach).

McDuff and Salamon give three proofs of the unique continuation property. The first proof is a few lines long but cites Aronszajn's theorem as proven in [2]. The second and third proofs are given self-contained treatments, and, moreover, the methods find further application in their book. The second proof uses the Hartman–Wintner theorem [6] (proven in McDuff and Salamon's Appendix E.4), which in fact implies the needed special case of Aronszajn's theorem, and the third proof uses the Carleman similarity principle and the Riemann-Roch theorem (proven in their Appendix C).

Here we return to the first method of proof, but give a simplified argument. The method is well known in certain branches of partial differential equations; it is the method of weighted integral estimates depending on a parameter. This is also the approach of Aronszajn [2], but it goes back even further, to Carleman [3]. For a general treatment with some historical comments, one may consult Sections 17.1 and 17.2 of Hörmander's book [9] or his corresponding paper [8]. However, all these references give much more than is needed for our application. Here we present only what is needed for J-holomorphic curves.

We give the full details for the case of C^∞ J-holomorphic curves; for J-holomorphic curves in Sobolev spaces with minimal assumptions, discussed in McDuff and Salamon's book, one may find the appropriate modifications in Sections 17.1 and 17.2 of Hörmander's book [9]. Here we focus on the C^∞ case, for ease of exposition and since the C^∞ case is sufficient for

Date: August 25, 2009.

many purposes; after all, in Gromov's original definition all J-holomorphic curves are C^∞ [5].

The weighted integral estimates will depend on a parameter $0 < h \ll 1$ which may be interpreted as "Planck's constant" as appearing in the correspondence principle of the old quantum theory, or, more generally, as appearing in semiclassical analysis [4]. The general idea is that as h tends to zero, asymptotic analysis reveals the classical mechanics of the operator's symbol, interpreted as a Hamiltonian function. Hence symplectic geometry plays a role beneath the surface.

We begin by recalling the basic definitions, so that our presentation is self-contained. Let (Σ, j) be a Riemann surface and (M, J) an almost complex manifold. A smooth function $u : \Sigma \to M$ is called a ***J*-holomorphic curve** if its differential du is a complex linear map with respect to j and J; that is, if

$$J \circ du = du \circ j,$$

or, equivalently,

$$\bar{\partial}_J(u) := \frac{1}{2}\left(du + J \circ du \circ j\right) = 0.$$

Unique continuation is a local problem, so for our purposes we may take the domain of u to be a connected neighborhood $X \subset \mathbb{C}$ of the origin, writing the elements of X as $x = x_1 + ix_2$, and we may take M to be $\mathbb{C}^n$. Hence we are interested in those $u \in C^\infty(X, \mathbb{C}^n)$ satisfying

$$\partial_{x_1} u + J(u)\partial_{x_2} u = 0, \tag{1}$$

where $J : \mathbb{C}^n \to \mathrm{GL}(2n, \mathbb{R})$ is smooth and satisfies $J^2 = -I$.

The main point of this paper is to give a simple, elementary proof of the following strong unique continuation result:

Theorem 1. *Let $X \subset \mathbb{C}$ be a connected neighborhood of* 0*, and suppose $u, v \in C^\infty(X, \mathbb{C}^n)$ satisfy (1) for some almost complex structure $J : \mathbb{C}^n \to GL(2n, \mathbb{R})$. If $u - v$ vanishes to infinite order at* 0*, then $u = v$ in X.*

Acknowledgements. I thank Dusa McDuff and Dietmar Salamon for pointing out that the Hartman–Wintner theorem implies the special case of Aronszajn's theorem. Moreover, it is a pleasure to thank them for their excellent books.

2. Proof of Unique Continuation

Let $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$ be the standard Laplacian. Since $(\partial_{x_1} J)J + J\partial_{x_1} J = 0$, we have that any solution u of (1) is also a solution of

$$\Delta u = (\partial_{x_2} J(u))\partial_{x_1} u - (\partial_{x_1} J(u))\partial_{x_2} u.$$

If v is another such function, then

$$\begin{aligned}\Delta(u-v) = (\partial_{x_2}J(u))\partial_{x_1}(u-v) + [\partial_{x_2}(J(u)-J(v))]\partial_{x_1}v \\ - (\partial_{x_1}J(u))\partial_{x_2}(u-v) - [\partial_{x_1}(J(u)-J(v))]\partial_{x_2}v.\end{aligned}$$

Also, of course,

$$J(u)-J(v) = \int_0^1 dJ(v+\tau(u-v))d\tau\cdot(u-v).$$

So, if $w := u-v$, then for some constant $C>0$ we have

$$|\Delta w| \leq C(|w| + |\partial_{x_1}w| + |\partial_{x_2}w|).$$

Since we are considering fixed functions u and v, the constant is allowed to depend on u, v, and their derivatives.

Thus Theorem 1 is a consequence of the following unique continuation result, a special case of Aronszajn's theorem [2]. (We follow the presentation of Theorem 17.2.6 in Hörmander's book [9].)

Theorem 2. *Let $X \subset \mathbb{R}^2$ be a connected neighborhood of 0, and let $u \in C^\infty(X, \mathbb{C}^n)$ be such that*

$$|\Delta u| \leq C(|u| + |\partial_{x_1}u| + |\partial_{x_2}u|). \tag{2}$$

If u vanishes to infinite order at 0, then $u = 0$ in X.

Proof. We first introduce conformal polar coordinates in $\mathbb{R}^2\backslash\{0\}$,

$$(x_1, x_2) = (e^t\cos\theta, e^t\sin\theta)$$

with $t \in \mathbb{R}$ and $\theta \in S^1$. Then, in these coordinates,

$$\begin{aligned}\partial_{x_1} &= e^{-t}\cos\theta\,\partial_t - e^{-t}\sin\theta\,\partial_\theta,\\ \partial_{x_2} &= e^{-t}\sin\theta\,\partial_t + e^{-t}\cos\theta\,\partial_\theta,\end{aligned}$$

and

$$\Delta = e^{-2t}(\partial_t^2 + \partial_\theta^2).$$

Next, we "convexify" the coordinates. Let $0<\epsilon<1$, and let T be such that

$$t = T + e^{\epsilon T}.$$

As noted by Hörmander [8], this change of coordinates comes from the work of Alinhac and Baouendi [1]. Then

$$\frac{\partial t}{\partial T} = 1 + \epsilon e^{\epsilon T} > 0,$$

and $T < t < T+1 < T/2$ when $T < -2$. In these coordinates,

$$\partial_t^2 + \partial_\theta^2 = (1+\epsilon e^{\epsilon T})^{-2}\partial_T^2 - \epsilon^2(1+\epsilon e^{\epsilon T})^{-3}e^{\epsilon T}\partial_T + \partial_\theta^2.$$

Multiplying by $(1+\epsilon e^{\epsilon T})^2$, we get the operator

$$Q := \partial_T^2 + c(T)\partial_T + (1+\epsilon e^{\epsilon T})^2\partial_\theta^2,$$

with

$$c(T) := -\epsilon^2(1+\epsilon e^{\epsilon T})^{-1}e^{\epsilon T}.$$

Our main tool is the following estimate:

Proposition 3. *For some $T_0 < 0$ and some $h_0 > 0$ we have*

$$h \iint \left(|U|^2 + |h\partial_T U|^2 + |h\partial_\theta U|^2 + |h^2\partial_T^2 U|^2 + |h^2\partial_{T,\theta}^2 U|^2 + |h^2\partial_\theta^2 U|^2\right) e^{-2T/h+\epsilon T} d\theta\, dT \\ \leq C \iint |h^2 QU|^2 e^{-2T/h} d\theta\, dT \tag{3}$$

for all $U \in C_0^\infty((-\infty, T_0) \times S^1)$, and for all $h \in (0, h_0)$. (The constant $C > 0$ is independent of h.)

Proof. (of the Proposition.) We set $U := e^{T/h}V$ and let

$$\tilde{Q} := h^2 e^{-T/h} \circ Q \circ e^{T/h}.$$

That is,

$$\tilde{Q} = (h\partial_T + 1)^2 + hc(T)(h\partial_T + 1) + (1+\epsilon e^{\epsilon T})^2 h^2 \partial_\theta^2.$$

Then the estimate (3) is equivalent to

$$h \iint \left(|V|^2 + |h\partial_T V|^2 + |h\partial_\theta V|^2 + |h^2\partial_T^2 V|^2 + |h^2\partial_{T,\theta}^2 V|^2 + |h^2\partial_\theta^2 V|^2\right) e^{\epsilon T} d\theta\, dT \\ \leq C \iint |\tilde{Q}V|^2 d\theta\, dT \tag{4}$$

for all $V \in C_0^\infty((-\infty, T_0) \times S^1)$.

For bookkeeping purposes, we write $\tilde{Q}$ as the sum of its symmetric and antisymmetric parts,

$$\tilde{Q} = A + B,$$

where

$$A = h^2\partial_T^2 + (1 + hc - \frac{1}{2}h^2 c') + (1+\epsilon e^{\epsilon T})^2 h^2 \partial_\theta^2,$$

and

$$B = (2 + hc)h\partial_T + \frac{1}{2}h^2 c'.$$

Hence, using the usual inner product notation on L^2, and with $[A, B] = AB - BA$ denoting the commutator,

$$\iint |\tilde{Q}V|^2 d\theta\, dT = ||AV||^2 + ||BV||^2 + \langle [A, B]V, V\rangle.$$

Repeated integration by parts gives

$$
\begin{aligned}
||AV||^2 = {} & ||h^2\partial_T^2 V||^2 \\
& + ||(1+hc-\frac{1}{2}h^2c')V||^2 && (5)\\
& + ||(1+\epsilon e^{\epsilon T})^2 h^2 \partial_\theta^2 V||^2 && (6)\\
& + h^3\langle V, (c''-\frac{1}{2}hc''')V\rangle \\
& - 2\langle h\partial_T V, (1+hc-\frac{1}{2}h^2c')h\partial_T V\rangle \\
& - 2\epsilon^3h^2\langle h\partial_\theta V, (1+2\epsilon e^{\epsilon T})e^{\epsilon T}h\partial_\theta V\rangle \\
& + 2\langle h^2\partial^2_{T,\theta}V, (1+\epsilon e^{\epsilon T})^2h^2\partial^2_{T,\theta}V\rangle \\
& - 2\langle h\partial_\theta V, (1+hc-\frac{1}{2}h^2c')(1+\epsilon e^{\epsilon T})^2 h\partial_\theta V\rangle, && (7)
\end{aligned}
$$

$$
||BV||^2 = ||(2+hc)h\partial_T V||^2 - \frac{1}{4}h^4||c'V||^2 - \frac{1}{2}h^4\langle V, c''cV\rangle - h^3\langle V, c''V\rangle,
$$

and

$$
\begin{aligned}
\langle [A,B]V,V\rangle = {} & -2h^2\langle c'h\partial_T V, h\partial_T V\rangle \\
& - 2h^2\langle c'V,V\rangle + h^3\langle (c''-cc')V,V\rangle + \frac{1}{2}h^4\langle (cc''+c''')V,V\rangle \\
& + 2h\epsilon^2\langle (2+hc)(1+\epsilon e^{\epsilon T})e^{\epsilon T}h\partial_\theta V, h\partial_\theta V\rangle. && (8)
\end{aligned}
$$

Also, we recall that

$$c(T) = -\epsilon^2 e^{\epsilon T}(1+\epsilon e^{\epsilon T})^{-1}$$

so that

$$c'(T) = -\epsilon^3 e^{\epsilon T}(1+\epsilon e^{\epsilon T})^{-2}$$

is also a negative quantity.

Most of the terms in the above expansions may be absorbed into other terms when we take $0 < h$ to be sufficiently small. It is only the term (7) that gives some difficulty. We write (7) as

$$
(7') \quad -2\langle h\partial_\theta V, (1+\lambda hc)(1+\epsilon e^{\epsilon T})^2 h\partial_\theta V\rangle - 2\langle h\partial_\theta V, ((1-\lambda)hc - \frac{1}{2}h^2c')(1+\epsilon e^{\epsilon T})^2h\partial_\theta V\rangle.
$$

Here $\lambda \in \mathbb{R}$ is to be determined; as we will see, any $2 < \lambda < 3$ will suffice.

For the first term of (7′), we use the elementary inequality

$$2\langle(1+\lambda hc)^{1/2}V,(1+\lambda hc)^{1/2}(1+\epsilon e^{\epsilon T})^2h^2\partial_\theta^2 V\rangle$$

$$\geq -\langle(1+\lambda hc)V,V\rangle \tag{9}$$

$$-\langle(1+\lambda hc)(1+\epsilon e^{\epsilon T})^2h^2\partial_\theta^2 V,(1+\epsilon e^{\epsilon T})^2h^2\partial_\theta^2 V\rangle. \tag{10}$$

Now (9) is absorbed into (5) when $\lambda > 2$, and (10) may be absorbed into (6) when $\lambda > 0$ (in both cases we are left with an order h term).

As for the second term in (7′), it may be absorbed into (8) as long as $\lambda < 3$. All the terms are thus accounted for, completing the proof of (4) and of the proposition. □

End of proof of Theorem 2. We write

$$U(T,\theta) = (U_1(T,\theta),\ldots,U_n(T,\theta)) := u(x_1,x_2).$$

Since we are only considering $T < T_0(\ll 0)$, our hypothesized upper bound (2) gives

$$|QU| \leq Ce^T(|U| + |\partial_T U| + |\partial_\theta U|).$$

Now we let $\psi \in C^\infty(\mathbb{R})$ be such that

$$\begin{cases}\psi = 1 & \text{in } (-\infty, T_0-1)\\ \psi = 0 & \text{in } (T_0,\infty),\end{cases}$$

and we set

$$U^\psi(T,\theta) = \psi(T)U(T,\theta).$$

The vanishing hypothesis on u says that for every N there exists a constant C_N such that

$$|u_j(x)| \leq C_N|x|^N \qquad j = 1,\ldots,n$$

in a neighborhood of the origin, so that, in the new coordinates, for any N we have

$$|U_j(T,\theta)| \leq C_N e^{NT} \qquad j = 1,\ldots,n$$

for T in a neighborhood of $-\infty$. Therefore

$$\iint |U_j^\psi|^2 e^{-NT} d\theta\, dT < \infty \qquad j = 1,\ldots,n$$

for any N. The same argument holds for all derivatives of U_j^ψ. We then let $\chi \in C^\infty(\mathbb{R})$ be such that

$$\begin{cases}\chi = 0 & \text{in } (-\infty,-2)\\ \chi = 1 & \text{in } (-1,\infty),\end{cases}$$

and for $R > 0$ we let $\chi_R(T) = \chi(T/R)$. We may apply Proposition 3 to each $\chi_R(T)U_j^\psi(T,\theta)$ and take the limit as $R \to \infty$; by the Dominated Convergence Theorem, Proposition 3 thus holds for U_j^ψ, and summing both sides over j we get equation (3) for our vector-valued function U^ψ.

The righthand side of (3) is then

$$\iint |h^2 QU^\psi|^2 e^{-2T/h} d\theta\, dT = h^4 \iint |\psi QU + \psi'' U + 2\psi' \partial_T U + c\psi' U|^2 e^{-2T/h} d\theta\, dT$$

$$\leq Ch^4 \iint e^{2T}(|U^\psi|^2 + |\partial_T U^\psi|^2 + |\partial_\theta U^\psi|^2) e^{-2T/h} d\theta\, dT \tag{11}$$

$$+ Ch^4 \iint_{T_0-1}^{T_0} (|U|^2 + |\partial_T U|^2) e^{-2T/h} d\theta\, dT. \tag{12}$$

Since $2T < \epsilon T$, the term (11) is bounded by

$$Ch^2 \iint (|U^\psi|^2 + |h\partial_T U^\psi|^2 + |h\partial_\theta U^\psi|^2) e^{-2T/h+\epsilon T} d\theta\, dT,$$

and hence can be absorbed into the lefthand side of (3) when $h > 0$ is sufficiently small.

Since U and $\partial_T U$ are bounded, the term (12) is bounded by

$$Ch^5 e^{-2(T_0-1)/h}.$$

Hence we have

$$h \iint \left(|U^\psi|^2 + |h\partial_T U^\psi|^2 + |h\partial_\theta U^\psi|^2 + |h^2 \partial_T^2 U^\psi|^2 + |h^2 \partial_{T,\theta}^2 U^\psi|^2 + |h^2 \partial_\theta^2 U^\psi|^2\right) e^{-2T/h+\epsilon T} d\theta\, dT$$
$$\leq Ch^5 e^{-2(T_0-1)/h}.$$

Letting $h \to 0$, we see that $U = 0$ when $T < T_0 - 1$, as otherwise the left side grows faster than the right side. Hence the original function u vanishes in a neighborhood of the origin.

We have thus shown that the set of points where u vanishes to infinite order is an open set. The complement is obviously also an open set, so by the connectedness of X we have that $u = 0$ in X. This concludes the proof of the theorem. □

References

[1] S. Alinhac and M. S. Baouendi. Uniqueness for the characteristic Cauchy problem and strong unique continuation for higher order partial differential inequalities. Amer. J. Math. 102 (1980), no. 1, 179–217.

[2] N. Aronszajn. A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order. J. Math. Pures Appl. (9) 36 (1957), 235–249.

[3] T. Carleman. Sur un problème d'unicité pur les systèmes d'équations aux dérivées partielles à deux variables indépendantes. (French) Ark. Mat., Astr. Fys. 26, (1939). no. 17, 9 pp.

[4] L. C. Evans and M. Zworski. Semi-classical analysis, Edition 0.3. www.math.berkeley.edu/∼zworski/semiclassical.pdf, 2007.

[5] M. Gromov. Pseudoholomorphic curves in symplectic manifolds. Invent. Math. 82 (1985), no. 2, 307–347.

[6] P. Hartman and A. Wintner. On the local behavior of solutions of non-parabolic partial differential equations. Amer. J. Math. 75 (1953), 449–476.

[7] D. McDuff and D. Salamon. J-holomorphic curves and symplectic topology. American Mathematical Society Colloquium Publications, 52. American Mathematical Society, Providence, RI, 2004.

[8] L. Hörmander. Uniqueness theorems for second order elliptic differential equations. Comm. Partial Differential Equations 8 (1983), no. 1, 21–64.

[9] L. Hörmander. The analysis of linear partial differential operators. III. Pseudodifferential operators. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 274. Springer-Verlag, Berlin, 1985.

UCLA Department of Mathematics, Los Angeles, CA 90095-1555, USA
E-mail address: mvanvalk@ucla.edu