MICROLOCAL ANALYTIC SINGULARITIES

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These notes approximately represent the contents of a course given at l'Université de Paris Sud in Spring 1981 and in an accelerated version at Lund University. They are based on about three years of work with analytic singularities. The author arrived at this subject through the study of the propagation of analytic singularities for boundary value problems. Among the three existing definitions of the analytic wavefront set, that of Bros-Iagolnitzer (see [8]; the other two are due to Hörmander [13] and Sato [27]) is revealed to be the most manageable, at least for an analyst with a certain predilection for Fourier integral operators with complex phases. The need to systematically develop the approach of Bros-Iagolnitzer is then already apparent in the boundaryless case. Some steps in this direction were made by G. Lebeau [22] and also by the author in an unpublished course at Stanford during Summer 1980. Next, thanks to numerous discussions with B. Lascar and inspired by an article of Schapira [29], it has become clear that the notion of phase function merges very profitably with the notion of plurisubharmonic weight function. Thus Sections 3, 7, and 9-16 were added after the course at Stanford, and Section 16 even after the course at Orsay.

These notes do not attempt to give a definitive treatment of the theory (which is still active) but only develop some techniques, to be modified according to the needs of each particular problem. (For example, for rather constructive problems one can eliminate the small parameter h, even if it means to work in unbounded domains.) We did not seek to be complete; we do not treat classical pseudodifferential operators or hyperfunctions, and we do not even treat the boundary value problems which are the origin of all this work.

Each reader (who is supposed to be already a bit familiar with Fourier integral operators, etc.) will judge for himself the interest and novelty of these notes. Perhaps the central point is Section 11, which contains a beginning of a systematic theory which someday, for the most part, will be developed in the C^∞ setting.

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1. H_φ, Analytic Symbols, and Formal Algebras of Pseudodifferential Operators

Let $\Omega \subset \mathbb{C}^n$ be open, and let $\varphi : \Omega \to \mathbb{R}$ be a continuous function. A function $u(z;h)$ on $\Omega \times \mathbb{R}_+$ belongs by definition to the space $H_\varphi^{\mathrm{loc}}(\Omega)$ if

$$u \text{ is holomorphic in } z \text{ for each } h > 0, \text{ and} \tag{1.1}$$

$$\begin{aligned} &\text{for each compact } K \subset \Omega \text{ and each } \epsilon > 0, \text{ there exists a constant } C_{K,\epsilon} > 0 \\ &\text{such that} \quad |u(z;h)| \le C_{K,\epsilon} e^{(\varphi(z)+\epsilon)/h} \quad \text{for } z \in K,\ 0 < h \le 1. \end{aligned} \tag{1.2}$$

Sometimes we say simply that u is of class H_φ on Ω. We will say that u is an analytic symbol if $u \in H_0^{\mathrm{loc}}(\Omega)$. In particular, u is a symbol of finite order $m \in \mathbb{R}$ if for every compact set $K \subset \Omega$ there exists $C_K > 0$ such that

$$|u(z;h)| \le C_K h^{-m}, \quad z \in K, \quad 0 < h \le 1. \tag{1.3}$$

In general we do not distinguish between two elements of $H_\varphi^{\mathrm{loc}}(\Omega)$ if the difference has exponential decrease compared to $e^{\varphi/h}$. More precisely, if $u, v \in H_\varphi^{\mathrm{loc}}(\Omega)$, we say that u and v are equivalent ($u \sim v$) if there exists a continuous function $\varphi_1 < \varphi$ on Ω such that

$$u - v \in H_{\varphi_1}^{\mathrm{loc}}(\Omega) \tag{1.4}$$

A *formal element* of H_φ on Ω is given by:

I. A covering $\Omega = \cup_{\alpha \in A} \Omega_\alpha$ where the $\Omega_\alpha \subset \Omega$ are open.
II. For each Ω_α an element $u_\alpha \in H_\varphi^{\mathrm{loc}}(\Omega_\alpha)$ (a local representative) such that $u_\alpha \sim u_\beta$ in $H_\varphi^{\mathrm{loc}}(\Omega_\alpha \cap \Omega_\beta)$ for each $\alpha, \beta \in A$.

In particular, for $\varphi = 0$ we arrive at the notion of *formal analytic symbol.* There is an obvious equivalence relation for the formal elements of class H_φ on Ω, and we do not distinguish between two equivalent elements.

Example 1.1. Let $a_k(z)$, $k = 0, 1, 2, \dots$ be a sequence of holomorphic functions on Ω such that for each $\tilde\Omega \subset\subset \Omega$ we have with $C = C_{\tilde\Omega} > 0$:

$$|a_k(z)| \le C^{k+1} k^k, \quad k = 0, 1, 2, \dots, \quad z \in \tilde\Omega. \tag{1.5}$$

Then modulo equivalence we can define a formal analytic symbol on Ω by the representatives

$$a_{\tilde\Omega}(z;h) = \sum_{0 \le k \le (eC_{\tilde\Omega} h)^{-1}} h^k a_k(z), \quad z \in \tilde\Omega.$$

We remark that $a_{\tilde\Omega}$ is an analytic symbol of order 0 on $\tilde\Omega$ since for $z \in \tilde\Omega$, $0 \le k \le \frac{1}{eC_{\tilde\Omega} h}$:

$$|h^k a_k(z)| \le C(Ckh)^k \le Ce^{-k}, \quad C = C_{\tilde\Omega}. \tag{1.6}$$

If $C_1 > C$ and $\frac{1}{eC_1h} < k \leq \frac{1}{eCh}$, $z \in \tilde{\Omega}$, then

$$|h^k a_k(z)| \leq Ce^{-\frac{1}{eC_1h}}, \tag{1.7}$$

which shows the equivalence of the local representatives in the regions $\tilde{\Omega}_\alpha \cap \tilde{\Omega}_\beta$. We write formally:

$$a(z;h) = \sum_0^\infty h^k a_k(z), \tag{1.8}$$

which we will call a formal classical analytic symbol.

The following result might only be of theoretical interest:

Proposition 1.2. *Let $\Omega \subset \mathbb{C}^n$ be open and pseudoconvex, let $\varphi : \Omega \to \mathbb{R}$ be continuous and plurisubharmonic, and let $u(z;h)$ be a formal element of H_φ on Ω. Then for each open set $\tilde{\Omega} \subset\subset \Omega$ there exists a representative $\tilde{u}(z;h) \in H_\varphi^{loc}(\tilde{\Omega})$ of u.*

Proof. This is a simple application of L. Hörmander's theorems about the $\bar{\partial}$ operator [14]. First of all, after extending $\tilde{\Omega}$ we may assume that $\tilde{\Omega}$ is pseudoconvex. Let $u_j(z;h)$ be the local representatives on Ω_j, $j = 1, 2, \ldots, N$, where $\overline{\tilde{\Omega}} \subset \cup_1^N \Omega_j$, and let $\chi_j \in C_0^\infty(\Omega_j)$ be such that $\sum_1^N \chi_j = 1$ on $\overline{\tilde{\Omega}}$. Let $v(z;h) = \sum_1^N \chi_j u_j$. Then $\bar{\partial} v = \sum_1^N u_j \bar{\partial} \chi_j$ is of exponential decrease in $L^2_\varphi(\tilde{\Omega}) = L^2(\tilde{\Omega}; e^{-2\varphi/h} L(dz))$ (where $L(dz)$ denotes Lebesgue measure on $\mathbb{C}^n$), so according to the results of Hörmander [14] we can find w of exponential decrease in $L^2_\varphi(\tilde{\Omega})$ such that $\bar{\partial} w = -\bar{\partial} v$. Hence $\tilde{u} = v + w$ is holomorphic and $\tilde{u} - u_j$ is of exponential decrease in $L^2_\varphi(\Omega_j \cap \tilde{\Omega})$. Using then that $\tilde{u} - u_j$ is holomorphic we deduce that $(\tilde{u} - u_j)e^{-\varphi(z)/h}$ is of uniform exponential decrease on each compact subset of Ω_j. $\square$

It will also be convenient to speak of germs of functions of class H_φ. If $x_0 \in \mathbb{C}^n$ and $\varphi(x)$ is a continuous real-valued function defined near x_0, then by definition an element u of H_{φ,x_0} is given by an element $\tilde{u} \in H_\varphi^{\mathrm{loc}}(\Omega)$ where $x_0 \in \Omega$. We say that $u, v \in H_{\varphi,x_0}$ are equivalent if $\tilde{u} \sim \tilde{v}$ in $H_\varphi^{\mathrm{loc}}(W)$ for a neighborhood W of x_0.

Let $p(x,\xi;h)$, $q(x,\xi;h)$ be analytic symbols defined in a neighborhood of $(x_0,\xi_0) \in \mathbb{C}^{2n}$. We then define the composed symbol in a neighborhood of (x_0,ξ_0) by

$$r = p \circ q = \sum_{0 \leq |\alpha| \leq (hC_0)^{-1}} \frac{h^{|\alpha|}}{\alpha!} i^{-|\alpha|} \frac{\partial^\alpha p}{\partial \xi^\alpha} \frac{\partial^\alpha q}{\partial x^\alpha} \tag{1.9}$$

with C_0 sufficiently large. By the Cauchy inequalities we have for each $\epsilon > 0$:

$$\left|\frac{\partial^\alpha p}{\partial \xi^\alpha}\right| + \left|\frac{\partial^\alpha q}{\partial x^\alpha}\right| \leq C_\epsilon e^{\epsilon/h} C^{|\alpha|} \alpha! \qquad \text{in a neighborhood of } (x_0,\xi_0),$$

where C is a geometric constant that does not depend on $\epsilon > 0$. It is then easy to see that (1.9) defines an analytic symbol near (x_0,ξ_0) for $C_0 > 0$ sufficiently large, and that the equivalence class does not change if we make C_0 larger still. Later we will define operators for which the symbol of a composition is given by (1.9). For the moment, we are especially interested in the problem of inversion of elliptic symbols in the framework of classical symbols. We let then

$$p=\sum_0^\infty h^k p_k(x,\xi)\,,\quad q=\sum_0^\infty h^k q_k(x,\xi)$$

be classical analytic symbols defined near (x_0,ξ_0), and we define the composed symbol by

$$r=p\circ q=\sum_{\alpha\in\mathbb{N}^n}\frac{h^{|\alpha|}}{\alpha!}i^{-|\alpha|}\frac{\partial^\alpha p}{\partial\xi^\alpha}\frac{\partial^\alpha q}{\partial x^\alpha} \tag{1.10}$$

where the infinite sum is taken in the sense of a finite sum in each degree of homogeneity with respect to h.

We leave as an exercise to show that r is in fact a classical analytic symbol and that if $\tilde{p}$, $\tilde{q}$ are local representatives of p, q and if $\tilde{r}=\tilde{p}\circ\tilde{q}$ is defined by (1.9), then $\tilde{r}$ is in fact a representative of r. In the rest of this section we will discuss only the formal classical analytic symbols while taking as a starting point the work of L. Boutet de Monvel and P. Kree [6] and of Boutet de Monvel [4]. If $p(x,\xi;h)$ is a classical analytic symbol of order 0 we associate to it the following differential operator of order infinity:

$$\begin{aligned}A(x,\xi,D_x;h)&=p(x,\xi+hD_x;h)\\&=\sum\frac{1}{\alpha!}p^{(\alpha)}(x,\xi;h)(hD_x)^\alpha\\&=\sum_0^\infty h^kA_k(x,\xi,D_x),\end{aligned}$$

where

$$A_k=\sum_{\nu+|\alpha|=k}\frac{1}{\alpha!}p_\nu^{(\alpha)}(x,\xi)D_x^\alpha. \tag{1.11}$$

From A we recover p by the formula:

$$p(x,\xi;h)=A(x,\xi,D_x;h)(1).$$

(In all the calculations here we manipulate the finite sums in each degree of homogeneity.) If q is another formal analytic symbol of order 0 and if B is its associated operator, then it is easy to verify that the operator associated to $r=p\circ q$ is $C=A\circ B=\sum_0^\infty h^kC_k$ (defined by $C_k=\sum_{\nu+\mu=k}A_\nu\circ B_\mu$). It follows that the composition of classical symbols is associative.

Let $\Omega_t\subset\mathbb{C}^{2n}$, $0\le t\le t_0$ be a family of small open neighborhoods of (x_0,ξ_0) such that $\Omega_s\subset\Omega_t$ for $s\le t$ and also such that

$$\begin{cases}(y,\xi)\in\Omega_s\\|x-y|\le t-s\end{cases}\Rightarrow(x,\xi)\in\Omega_t$$

for $s\le t$. Then for $0\le s<t\le t_0$, D_x^α is a bounded operator from the space $B(\Omega_t)$ of bounded holomorphic functions on Ω_t to the space $B(\Omega_s)$ of norm

$$||D_x^\alpha||_{t,s}\le\frac{C_0^{|\alpha|}|\alpha|^{|\alpha|}}{(t-s)^{|\alpha|}}$$

where C_0 is a universal constant depending only on n. (If we chose rather Ω_t to be a polydisc of radius *constant* $+ t$ then we would have $||D_x^\alpha||_{t,s} \leq (t-s)^{-|\alpha|}\alpha!$.)

If Ω_{t_0} is sufficiently small, then, on Ω_{t_0},

$$|p_\nu^{(\alpha)}| \leq C^{1+\nu+|\alpha|}\nu^\nu\alpha!$$

so that, with a new constant C_1,

$$\Big|\Big|\frac{1}{\alpha!}p_\nu^{(\alpha)}D_x^\alpha\Big|\Big|_{t,s} \leq C_1^{1+\nu+|\alpha|}\nu^\nu|\alpha|^{|\alpha|}(t-s)^{-|\alpha|}, \quad 0 \leq s < t \leq t_0.$$

There are at most $(1+k)^{n+1}$ terms in (1.11), so with a new constant C we obtain:

$$||A_k||_{t,s} \leq C^{k+1}k^k(t-s)^{-k}, \quad 0 \leq s < t \leq t_0. \tag{1.12}$$

Conversely, if $p = \sum_0^\infty h^k p_k(x,\xi)$ is a formal classical symbol in the sense that the p_k are holomorphic in a neighborhood of (x_0,ξ_0) independent of k but that the condition (1.5) is not necessarily satisfied, and if (1.12) is satisfied, then we deduce that p is an *analytic* classical symbol near (x_0,ξ_0). Indeed, $p_k = A_k(1)$, and we have with a new constant C:

$$\sup_{\Omega_{\frac{t_0}{2}}} |p_k| \leq C^{k+1}k^k. \tag{1.13}$$

If A satisfies (1.12) we associate to it the sequence $f(A) = (f_k(A))_{k=0}^\infty$ where $f_k = f_k(A)$ is the smallest number ≥ 0 such that:

$$||A_k||_{t,s} \leq f_k\, k^k(t-s)^{-k}, \quad 0 \leq s < t \leq t_0. \tag{1.14}$$

Hence (1.12) shows that $(f_k)_0^\infty$ is of at most exponential increase. We now let $B = \sum_0^\infty h^k B_k$ be another operator of the same type.

Lemma 1.3. *If $C = A \circ B = \sum_0^\infty h^k C_k$, then $f_k(C) \leq \sum_{\nu+\mu=k} f_\nu(A)f_\mu(B)$.*

Proof. We have $C_k = \sum_{\nu+\mu=k} A_\nu \circ B_\mu$ and hence for $0 \leq s < r < t \leq t_0$:

$$||A_\nu \circ B_\mu||_{t,s} \leq f_\nu(A)f_\mu(B)\nu^\nu(r-s)^{-\nu}\mu^\mu(t-r)^{-\mu}.$$

If s and t are given, we choose r such that

$$r - s = \frac{\nu}{\nu+\mu}(t-s), \quad t - r = \frac{\mu}{\mu+\nu}(t-s).$$

Then

$$||A_\nu \circ B_\mu||_{t,s} \leq f_\nu(A)f_\mu(B)(\nu+\mu)^{\nu+\mu}(t-s)^{-(\nu+\mu)}$$

which proves the lemma. □

For $\rho > 0$ we put

$$||A||_\rho = \sum_0^\infty \rho^k f_k(A).$$

Then (1.12) is satisfied for some $C > 0$ if and only if $||A||_\rho < \infty$ for some $\rho > 0$.

Lemma 1.4. *If $C = A \circ B$, $\rho > 0$, $||A||_\rho < \infty$, and $||B||_\rho < \infty$, then $||C||_\rho$ is finite and*

$$||C||_\rho \leq ||A||_\rho ||B||_\rho.$$

Proof.

$$\begin{aligned}||C||_\rho &= \sum_0^\infty \rho^k f_k(C)\\ &\le \sum\sum \rho^{\nu+\mu} f_\nu(A) f_\mu(B)\\ &= ||A||_\rho ||B||_\rho.\end{aligned}$$

□

If p is a formal classical symbol (in a neighborhood of $\overline{\Omega}_{t_0}$) we put $||p||_\rho = ||A||_\rho$ where A is the associated operator, and we know that $||A||_\rho < \infty$ for some $\rho > 0$ if p is an analytic symbol in a neighborhood of $\overline{\Omega}_{t_0}$. Conversely, if $||A||_\rho < \infty$ then p is an analytic symbol in Ω_{t_0}. It then becomes obvious that $p \circ q$ is a classical analytic symbol if p and q are.

Now we let $p = \sum_0^\infty h^k p_k$ be a classical analytic symbol. We suppose that p is elliptic; that is to say $p_0 \neq 0$ everywhere. It is then standard to construct the unique classical symbol, $q = \sum_0^\infty h^k q_k$, such that $p \circ q = q \circ p = 1$. (One proceeds by recurrence, starting with $q_0 = \frac{1}{p_0}$.) Thanks to pseudonorms we then show

Theorem 1.5. *(Boutet de Monvel-Kree* [6]*): The "inverse" symbol q is a classical analytic symbol.*

Proof. We fix a point (x_0, ξ_0) in the domain of definition of p and propose to show that q is analytic near (x_0, ξ_0). We then define the pseudonorms $||\ ||_\rho$ as above. Let $q_0 = \frac{1}{p_0}$. Then $p \circ q_0 = 1 - r$ where r is a classical analytic symbol of order -1. If $\rho > 0$ is sufficiently small we have $||r||_\rho < \frac{1}{2}$. Now $q = q_0 \circ (1 + r + r^2 + \ldots)$ (a finite sum in each degree of homogeneity), and q is analytic because $||1 + r + r^2 + \ldots||_\rho \le 2$. □

Our choice of pseudonorms $||\ ||_\rho$ (introduced in [30]) is different than those of [6] and [4]. It allows us to obtain the property of a Banach algebra without calculation.

2. The Method of Stationary Phase—The Saddle Point Method

In this section we adapt the method of Melin-Sjöstrand [23] (without doubt classical, at least conceptually) in the analytic case, following [30]. See also Lebeau [22]. We consider at first the case of a particular phase. Let $B \subset \mathbb{R}^n$ be the closed unit ball and $\tilde{B} = \{\lambda x;\, x \in B, \lambda \in \mathbb{C}, |\lambda| \le 1\}$.

Theorem 2.1. *There exists a constant $C > 0$ depending only on the dimension n, such that for each $N \in \mathbb{N}$, each $h \in (0, \infty)$, and each holomorphic function u, defined in a neighborhood of $\tilde{B}$:*

$$\int_B e^{-\frac{x^2}{2h}} u(x)\, dx = \sum_{\nu=0}^{N-1} h^{\frac{n}{2}+\nu} (2\pi)^{\frac{n}{2}} \frac{1}{\nu!} \Big(\frac{\Delta}{2}\Big)^\nu u(0) + R_N(h), \tag{2.1}$$

where

$$\frac{|R_N(h)|}{\sup_{\tilde{B}}|u(z)|} \leq Ch^{\frac{n}{2}}(N+1)^{\frac{n}{2}}N!\,2^N h^N. \tag{2.2}$$

Proof. In the case of one variable, if $u(z)$ is holomorphic in $|z| \leq 1$ and $|u(z)| \leq 1$, then $|u^{(k)}(0)| \leq k!$ and hence

$$\Big|u(z) - \sum_0^{2N-1} \frac{u^{(k)}(0)z^k}{k!}\Big| \leq (2N+1).$$

By the maximum principle we obtain:

$$\Big|u(z) - \sum_0^{2N-1} \frac{u^{(k)}(0)z^k}{k!}\Big| \leq (2N+1)|z|^{2N}.$$

We also have

$$\Big|\frac{u^{(k)}(0)z^k}{k!}\Big| \leq |z|^k.$$

Now we let $u(z)$ be holomorphic in a neighborhood of $\tilde{B} \subset \mathbb{C}^n$ and be such that $\sup_{\tilde{B}}|u(z)| \leq 1$. We can apply the preceding inequalities to all the complex lines passing through 0, and we obtain for all $z \in \tilde{B}$:

$$\Big|\sum_{|\alpha|=k} \frac{u^{(\alpha)}(0)z^\alpha}{\alpha!}\Big| \leq |z|^k \tag{2.3}$$

$$\Big|u(z) - \sum_{|\alpha|\leq 2N-1} \frac{u^{(\alpha)}(0)z^\alpha}{\alpha!}\Big| \leq (2N+1)|z|^{2N}. \tag{2.4}$$

Lemma 2.2. *Let $p_{2k}(z)$ be a homogeneous polynomial of degree $2k$. If $2k$ is odd, then*

$$\int_{\mathbb{R}^n} e^{-\frac{x^2}{2}} p_{2k}(x)\,dx = 0,$$

and if $2k$ is even, then

$$\int_{\mathbb{R}^n} e^{-\frac{x^2}{2}} p_{2k}(x)\,dx = \frac{(2\pi)^{\frac{n}{2}}}{k!}\Big(\frac{\Delta}{2}\Big)^k p_{2k}. \tag{2.5}$$

(Here $(\frac{\Delta}{2})^k p_{2k}$ is a constant.)

Proof. The case when $2k$ is odd is trivial. Let then $2k$ be even. By the Euler homogeneity relations, we have

$$\frac{1}{2k}\sum_1^n x_j \frac{\partial}{\partial x_j}(p_{2k}) = p_{2k}.$$

By integrating by parts we obtain:

$$\begin{aligned}\int e^{-\frac{x^2}{2}}p_{2k}(x)\,dx &= \frac{1}{2k}\int \Big(\sum \frac{\partial}{\partial x_j}\Big(-x_j e^{-\frac{x^2}{2}}\Big)\Big)p_{2k}(x)\,dx\\ &= \frac{1}{k}\int \frac{\Delta}{2}(e^{-\frac{x^2}{2}})p_{2k}(x)\,dx\\ &= \frac{1}{k}\int e^{-\frac{x^2}{2}}\frac{\Delta}{2}(p_{2k})\,dx.\end{aligned}$$

By integration we obtain (2.5), since

$$\int e^{-\frac{x^2}{2}}\,dx = (2\pi)^{\frac{n}{2}}.$$

□

From (2.5) we obtain by a change of variables

$$\int e^{-\frac{x^2}{2h}}p_{2k}(x)\,dx = \frac{1}{k!}(2\pi h)^{\frac{n}{2}}h^k\Big(\frac{\Delta}{2}\Big)^k p_{2k}, \tag{2.6}$$

hence

$$\int e^{-\frac{x^2}{2h}}\Big(\sum_{|\alpha|=2k}\frac{u^{(\alpha)}(0)}{\alpha!}x^\alpha\Big)\,dx = \frac{1}{k!}(2\pi h)^{\frac{n}{2}}h^k\Big(\Big(\frac{\Delta}{2}\Big)^k u\Big)(0). \tag{2.7}$$

We then have (2.1) with

$$R_N(h) = I'(h) + I''(h),$$

where

$$I'(h) = \int_{|x|\le 1} e^{-\frac{x^2}{2h}}\Big(u(x) - \sum_{|\alpha|\le 2N-1}\frac{u^{(\alpha)}(0)}{\alpha!}x^\alpha\Big)\,dx,$$

$$I''(h) = -\int_{|x|\ge 1} e^{-\frac{x^2}{2h}}\sum_{|\alpha|\le 2N-1}\frac{u^{(\alpha)}(0)}{\alpha!}x^\alpha\Big)\,dx.$$

Using (2.3), (2.4) we obtain

$$\begin{aligned}|R_N(h)| &\le (2N+1)\int_{\mathbb{R}^n} e^{-\frac{x^2}{2h}}|x|^{2N}\,dx\\ &= (2N+1)h^{\frac{n}{2}+N}\int e^{-\frac{x^2}{2}}|x|^{2N}\,dx.\end{aligned}$$

One may calculate this last integral either by Lemma 2.2 or by passing to polar coordinates. This gives

$$|R_N(h)| \le (2N+1)h^{\frac{n}{2}+N}(2\pi)^{\frac{n}{2}}2^N(N-1+\frac{n}{2})(N-2+\frac{n}{2})\dots(\frac{n}{2}).$$

Here we remark that if $0 \le a \le C_1$ then there exists a constant C_2 such that

$$\frac{(n+a)(n-1+a)\dots(a+1)}{n(n-1)\dots 1} \le C_2 n^a.$$

Hence

$$|R_N(h)| \le C(n)(N+1)^{\frac{n}{2}}h^{\frac{n}{2}}N!\,2^N h^N.$$

□

Remark 2.3. We apply Stirling's formula:

$$N! = (2\pi)^{\frac{1}{2}} N^{N+\frac{1}{2}} e^{-N}\Big(1+\mathcal{O}\Big(\frac{1}{N}\Big)\Big)$$

and find that with a new constant

$$\frac{|R_N(h)|}{\sup_{\tilde{B}}|u|} \le C(N+1)^{\frac{n+1}{2}} h^{\frac{n}{2}} (2N)^N e^{-N} h^N, \quad N \ge 0, \quad h \in (0,\infty).$$

The optimal choice, to minimize the last factor, is $N = \frac{1}{2h}$. We can take the integer part, $N = [\frac{1}{2h}]$, and find that with a new constant that depends only on n:

$$\frac{|R_{[1/2h]}(h)|}{\sup_{\tilde{B}}|u|} \le C\Big(1+\frac{1}{h}\Big)^{\frac{1}{2}} e^{-\frac{1}{2h}}, \quad 0 < h \le 1.$$

The factor $e^{-\frac{1}{2h}}$ is to be regarded as $e^{-\frac{x^2}{2h}}\Big|_{\partial B}$.

Exercise 2.4. Deduce the following version of the Cauchy inequalities from the proof of Theorem 2.1:

$$\frac{1}{k!}\Big|\Big(\frac{\Delta}{2}\Big)^k u(0)\Big| \le C(n)(k+1)^{\frac{n}{2}}(k-1)!2^k \sup_{\tilde{B}}|u|, \tag{2.8}$$

for $k \ge 1$.

Remark 2.5. Let $B_r = rB$, $\tilde{B}_r = r\tilde{B}$, $r > 0$. Let u be holomorphic near $\tilde{B}_r$. Then to expand $\int_{B_r} e^{-\frac{x^2}{2h}} u(x)\,dx$ we make the change of variables $x = ry$ and replace h by hr^{-2} in Theorem 2.1. We then get the same formula as in Theorem 2.1, where only the remainder is to be modified. The new remainder satisfies:

$$\frac{|R_N(h)|}{\sup_{\tilde{B}_r}|u|} \le C(n)(N+1)^{\frac{n}{2}} h^{\frac{n}{2}} N! 2^N r^{-2N} h^N.$$

With $N = [\frac{r^2}{2h}]$ we obtain as before

$$\frac{|R_N(h)|}{\sup_{\tilde{B}_r}|u|} \le C(n) r^n \Big(1+\frac{r^2}{h}\Big)^{\frac{1}{2}} e^{-\frac{r^2}{2h}}, \qquad 0 < h \le r^2. \tag{2.9}$$

Let $q(z)$ be a nondegenerate quadratic form on $\mathbb{C}^n$. Let $\Gamma \subset \mathbb{C}^n$ be a subspace that is totally real (that is to say $\Gamma \cap i\Gamma = \{0\}$) of maximal dimension n, and is such that $q|_\Gamma$ is real and positive definite. Let $B_r = \{z \in \Gamma;\ q(z) \le r\}$, $r > 0$, and $\tilde{B}_r = \{wz;\ z \in B_r, w \in \mathbb{C}, |w| \le 1\}$. With a choice of orientation on Γ, we consider

$$I(h) = \int_{B_r} e^{-\frac{q(z)}{h}} u(z)\,dz, \tag{2.10}$$

with u holomorphic in a neighborhood of $\tilde{B}_r$. After a linear change of coordinates $z \to \tilde{z}$ we may suppose that $\Gamma = \mathbb{R}^n_{\tilde{x}}$, $q = \frac{\tilde{z}^2}{2}$, $B_r = \{\tilde{x} \in \mathbb{R}^n; \frac{\tilde{x}^2}{2} \le r\}$. With $\tilde{u}(\tilde{z}) = u(z)$ we then obtain

$$I(h) = \det\left(\frac{dz}{d\tilde{z}}\right) \int_{\frac{\tilde{x}^2}{2} \le r} e^{-\frac{\tilde{x}^2}{2h}} \tilde{u}(\tilde{x})\, d\tilde{x}. \tag{2.11}$$

With $q(z) = \frac{1}{2}\langle Qz, z\rangle$, we have

$$\det\left(\frac{dz}{d\tilde{z}}\right) = \pm(\det Q)^{-\frac{1}{2}}. \tag{2.12}$$

Applying the preceding results to the integral (2.11) we obtain the expansion of Theorem 2.1 with the additional factor $\det(\frac{dz}{d\tilde{z}})$ and with Δ replaced by $\tilde{\Delta}$, the Laplacian in the $\tilde{z}$-coordinates. The estimate of the remainder becomes

$$\frac{|R_N(h)|}{\sup_{\tilde{B}_r} |u|} \le C(n)(N+1)^{\frac{n}{2}} h^{\frac{n}{2}+N} N! r^{-N} |\det Q|^{-\frac{1}{2}}. \tag{2.13}$$

Example 2.6. We consider the following contour integral on $\mathbb{C}^{2n}$:

$$\Im(h) = (2\pi h)^{-n} \iint_{\substack{|x| \le C_1 \\ \xi = -C_2 i\bar{x}}} e^{-\frac{ix\xi}{h}} u(x,\xi)\; dx \wedge d\xi.$$

Here Γ is given by $\xi = -C_2 i\bar{x}$, the quadratic form is $q(x,\xi) = ix\xi$, and $r = C_2 C_1^2$. The projections of B_r on $\mathbb{C}^n_x$ and $\mathbb{C}^n_\xi$ are respectively $|x| \le C_1$ and $|\xi| \le C_1 C_2$, so $\tilde{B}_r \subset \{(x,\xi); |x| \le C_1, |\xi| \le C_1 C_2\}$. As we shall see, $\Im(h)$ has a stationary phase expansion with remainder:

$$\frac{|R_N(h)|}{\sup_{|x| \le C_1, |\xi| \le C_1 C_2} |u(x,\xi)|} \le C(n)(N+1)^n \frac{h^N N!}{(C_1^2 C_2)^N}\ . \tag{2.14}$$

To calculate the terms of the expansion we parameterize the contour Γ by $x \in \mathbb{C}^n$:

$$dx \wedge d\xi = (-C_2 i)^n dx \wedge d\bar{x} = (2C_2)^n \frac{dx \wedge d\bar{x}}{(2i)^n} = \overset{+}{\underset{(-)}{}} (2C_2)^n L(dx)$$

where $L(dx)$ is Lebesgue measure on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ and where we choose the orientation such that we have the "+" sign. We thus obtain

$$\Im(h) = \left(\frac{2C_2}{2\pi h}\right)^n \int_{|x| \le C_1} e^{-\frac{C_2|x|^2}{h}} u(x, \frac{C_2}{i}\bar{x})\, L(dx).$$

Putting $\tilde{h} = \frac{h}{2C_2}$ we can then identify the k^{th} term in the expansion:

$$\left(\frac{h}{2C_2}\right)^k \frac{1}{k!} \Big((\frac{1}{2}\Delta_{x,\bar{x}})^k u(x, \frac{C_2}{i}\bar{x}) \Big)(0).$$

Here we remark that $\frac{1}{2}\Delta_{x,\bar{x}} = 2\sum_1^n \frac{\partial}{\partial x_j}\frac{\partial}{\partial \bar{x}_j}$. Hence

$$(\frac{1}{2}\Delta_{x,\bar{x}})^k u(x, \frac{C_2}{i}\bar{x}) = \Big((2C_2 \frac{1}{i} \sum_1^n \frac{\partial}{\partial x_j}\frac{\partial}{\partial \xi_j})^k u \Big)(x, \frac{C_2}{i}\bar{x}),$$

and we obtain

$$\begin{aligned}\Im(h) &= \sum_0^{N-1} \frac{1}{k!}\Big(\frac{h}{i}\sum_1^n \frac{\partial}{\partial x_j}\frac{\partial}{\partial \xi_j}\Big)^k u(0,0) + R_N(h)\\ &= \sum_{\substack{\alpha\in\mathbb{N}^n\\ |\alpha|\le N-1}} \frac{1}{\alpha!}\Big(\frac{h}{i}\Big)^{|\alpha|}(\partial_x^\alpha\partial_\xi^\alpha u)(0,0) + R_N(h),\end{aligned} \tag{2.15}$$

where $R_N(h)$ satisfies (2.14). When $u = u(x)$, we have $\Im(h) = u(0) + R_N(h)$ where $R_N(h)$ is bounded in terms of $\sup_{|x|\le C_1} |u(x;h)|$.

We consider now the general case of a phase that is not necessarily quadratic, where we can reduce to the quadratic case with the help of Morse's Lemma. (See Hörmander [15].)

Lemma 2.7. *Let $\varphi(z)$ be a holomorphic function defined near $z_0 \in \mathbb{C}^n$. We suppose that z_0 is a nondegenerate critical point; that is, $\varphi'(z_0) = 0$ and $\det\varphi''(z_0) \neq 0$. Then we can choose holomorphic local coordinates centered at z_0 such that:*

$$\varphi(z) = \varphi(z_0) + \frac{1}{2}(\tilde z_1^2 + \ldots + \tilde z_n^2).$$

Proof. First of all, we reduce to the case $z_0 = 0$, $\varphi(z) = \frac{1}{2}(z_1^2 + \ldots + z_n^2) + \mathcal{O}(|z|^3)$. By Taylor's formula,

$$\begin{aligned}\varphi(z) &= \int_0^1 (1-t)\frac{\partial^2}{\partial t^2}[\varphi(tz)]\,dt\\ &= \frac{1}{2}\sum\sum q_{jk}(z) z_j z_k\\ &= \frac{1}{2}\langle Q(z)z, z\rangle,\end{aligned}$$

where

$$q_{jk}(z) = 2\int_0^1 (1-t)\frac{\partial^2\varphi}{\partial z_j \partial z_k}(tz)\,dt$$

so that $Q(0) = I$. We look for $\tilde z$ of the form $\tilde z = A(z)z$ with $A(0) = I$. It then suffices to solve ${}^tA(z)A(z) = Q(z)$, and we can take for example $A(z) = Q(z)^{\frac{1}{2}}$. The square root is well defined since $Q(z) = I +$ *small perturbation*. □

If $\varphi(w,z)$ is holomorphic near $(0,z_0) \in \mathbb{C}^k \times \mathbb{C}^n$, and if $\varphi(0,z)$ satisfies the hypotheses of Morse's Lemma, then for w small there exists a unique (nondegenerate) critical point $z(w)$ near z_0 for the function $z \mapsto \varphi(w,z)$. Moreover, $z(w)$ is a holomorphic function of w. The coordinates $\tilde z$, constructed in the proof of Morse's Lemma, depend holomorphically on w.

Theorem 2.8. *Let $U \subset \mathbb{C}^n$ be an open neighborhood of 0, and let φ be a function holomorphic on U with $z = 0$ as its only critical point. We suppose that $\varphi(0) = 0$ and $\det \varphi''(0) \neq 0$. Let $V \subset\subset U$ be an open neighborhood of 0, and suppose that $Re\, \varphi(x) \geq 0$ for each $x \in V_{\mathbb{R}} = V \cap \mathbb{R}^n$ and that $Re\, \varphi(x) > 0$ on $\partial V_{\mathbb{R}}$. Then there exist constants $C > 0$ and $\epsilon > 0$ such that for each bounded holomorphic function u on U we have*

$$\begin{aligned} I(h) &= \int_{V_{\mathbb{R}}} e^{-\frac{\varphi(x)}{h}} u(x)\, dx \\ &= \sum_{0 \leq k \leq \frac{1}{Ch}} (2\pi)^{\frac{n}{2}} \frac{1}{k!} h^{\frac{n}{2}+k} \Big(\frac{1}{2}\tilde{\Delta}\Big)^k \Big(\frac{u}{\Im}\Big)(0) + R(h) \end{aligned} \tag{2.16}$$

where

$$|R(h)| \leq \frac{1}{\epsilon} e^{-\frac{\epsilon}{h}} \sup_U |u(z)|, \quad 0 < h \leq 1, \quad \textit{and}$$

$$\tilde{\Delta} = \sum \frac{\partial^2}{\partial \tilde{z}_j^2}, \quad \Im = \pm \det \frac{d\tilde{z}}{dz}\, .$$

Here $\tilde{z}_1, ..., \tilde{z}_n$ are the coordinates as in Morse's Lemma, and $\Im(0) = (\det \varphi''(0))^{\frac{1}{2}}$, where we choose the branch which is continuously deformed into 1 by the homotopy $[0, 1] \ni s \mapsto (1 - s)\varphi''(0) + sI$.

Proof. ("The Saddle Point Method.") We first make a slight deformation. For $\delta > 0$ small we consider the contour $\Gamma_\delta : V_{\mathbb{R}} \ni x \mapsto x + \delta\, \overline{\varphi'(x)} = z$. Along (the image of) Γ_δ we have

$$\varphi(z) = \varphi(x) + \delta|\varphi'(x)|^2 + \mathcal{O}(\delta^2 |\varphi'(x)|^2)\, ,$$

and so, for $\delta > 0$ sufficiently small, $\mathrm{Re}\, \varphi(z) \geq C_\delta |z|^2$, where $C_\delta > 0$. If we fix $\delta > 0$ sufficiently small, then Stokes' formula and the hypothesis that $\mathrm{Re}\, \varphi > 0$ on $\partial V_{\mathbb{R}}$ give that, modulo an error $\mathcal{O}(\frac{1}{\epsilon} e^{-\frac{\epsilon}{h}} \sup_U |u|)$,

$$I(h) \equiv \int_\Gamma e^{-\frac{\varphi(z)}{h}} u(z)\, dz\ , \quad \Gamma = \Gamma_\delta. \tag{2.17}$$

Now it is clear that only a neighborhood of 0 will give a non-negligible contribution to (2.17), and we may assume that Morse's Lemma is applicable in a neighborhood of Γ. We then have

$$I(h) \equiv \int_{\tilde{\Gamma}} e^{-\frac{\tilde{z}^2}{2h}} \tilde{u}(\tilde{z})\, d\tilde{z}\ , \quad \tilde{u}(\tilde{z}) = u(z) \det \frac{dz}{d\tilde{z}}. \tag{2.18}$$

By construction, we have on $\tilde{\Gamma}$

$$\mathrm{Re}\, \frac{\tilde{z}^2}{2} \geq C|\tilde{z}|^2, \quad C > 0. \tag{2.19}$$

Writing $\tilde{z} = x+iy$, we then have that $|t_y|^2 \leq |t_x|^2$ for each $(t_x, t_y) \in T_0(\tilde{\Gamma})$, and the Implicit Function Theorem and (2.19) show that $\tilde{\Gamma}$ (at least on a neighborhood of 0) is of the form $y = H(x)$, $x \in W$, where $H(x)$ is a real-analytic function satisfying $|H(x)| \leq \theta|x|$, $\theta < 1$. Let

$$\tilde{\Gamma}_t : W \ni x \mapsto x + itH(x) \in \mathbb{C}^n.$$

Stokes' formula then shows that, with an exponentially small error, we have

$$I(h) \equiv \pm \int_W e^{-\frac{x^2}{2h}} \tilde{u}(x)\, dx. \tag{2.20}$$

We have ${}^t(\frac{d\tilde{z}}{dz}) \circ (\frac{d\tilde{z}}{dz}) = \varphi''(0)$ at 0, and, modulo the sign, the theorem now results from Theorem 2.1 and from Remark 2.5. To finally determine the sign, we continuously deform $\varphi(x)$ to $\frac{x^2}{2}$. ☐

Remark 2.9. The hypothesis "$\operatorname{Re}\varphi|_{\partial V_{\mathbb{R}}} > 0$" can be weakened. For example we can suppose only that $\partial V_{\mathbb{R}}$ is a real-analytic hypersurface near each connected component of $K = \{x \in \partial V_{\mathbb{R}}; \operatorname{Re}\varphi(x) = 0\}$, and that $d\varphi|_{\partial V_{\mathbb{R}}} \neq 0$ on K. There exist even weaker conditions.

Remark 2.10. We can introduce parameters into Theorem 2.8. For example, let $W \subset \mathbb{C}^k$ be a neighborhood of 0, and let $\varphi(w, z)$ be a holomorphic function on $W \times U$ such that $\varphi(0, z)$, U, V satisfy the hypotheses of Theorem 2.8. After shrinking W around 0 we obtain for each bounded holomorphic function $u(z)$ on U:

$$e^{\frac{\varphi(w,z(w))}{h}} \int_{V_{\mathbb{R}}} e^{-\frac{\varphi(w,x)}{h}} u(x)\, dx =$$
$$= \sum_{0 \le k \le \frac{1}{Ch}} (2\pi)^{\frac{n}{2}} \frac{1}{k!} h^{\frac{n}{2}+k} \Big(\frac{1}{2}\tilde{\Delta}_W\Big)^k \Big(\frac{u}{\Im_W}\Big)(z(w)) + R(w; h),$$

where $\tilde{\Delta}_W$, $\Im_W$, and $z(w)$ depend holomorphically on w, and

$$|R(w; h)| \le \frac{1}{\epsilon} e^{-\frac{\epsilon}{h}} \sup_U |u|.$$

We leave it as an exercise to make the necessary modifications in the proof of Theorem 2.8.

3. "The Fundamental Lemma" and the Fourier Transform in the Complex Domain

We begin by discussing the real(-valued) quadratic forms on $\mathbb{C}^n$. We let q be one such form, and let $\Im q(x) = q(ix)$. Then $\Im^2 = I$. Clearly q is pluriharmonic (that is, harmonic on every complex line) if and only if $\Im q = -q$. We say that q is of Levi type if $\Im q = q$. If q is an arbitrary real quadratic form, then we have a unique decomposition $q = \mathfrak{h} + \ell$, where $\mathfrak{h}$ is pluriharmonic and ℓ is of Levi type; moreover, $\ell = \frac{1}{2}(q + \Im q)$, $\mathfrak{h} = \frac{1}{2}(q - \Im q)$. We have also

$$\mathfrak{h}(z) = \operatorname{Re}\langle z, Az\rangle, \quad \ell(z) = \langle z, \mathcal{L}\bar{z}\rangle$$

where $A = (\frac{\partial^2 q}{\partial z_j \partial z_k})$, $\mathcal{L} = (\frac{\partial^2 q}{\partial z_j \partial \bar{z}_k})$. Then q is plurisubharmonic (pl.s.h.) if and only if $\ell \ge 0$.

We recall that $\mathrm{sign}(q) = m_+ - m_-$, where m_+ (m_-) is the number of strictly positive (negative) eigenvalues for a diagonalization of q, and that $m_\pm = \max \dim_{\mathbb{R}} L$, where L ranges over the real-linear subspaces such that $\pm q|_L > 0$. q is non-degenerate if and only if $m_+ + m_- = 2n$.

Proposition 3.1. *Let $q(z)$ be a plurisubharmonic quadratic form on $\mathbb{C}^n$. Then*

- ***(i)*** *$\mathrm{sign}(q) \geq 0$,*
- ***(ii)*** *If in addition q is non-degenerate of signature 0 (that is, $m_-(q) = n$), then the same thing is true for each quadratic form $\tilde{q} \leq q$ which is plurisubharmonic.*

Proof. (i) Let $L \subset \mathbb{C}^n$ be a real-linear subspace of dimension m_- such that $q|_L < 0$. Then if we decompose $q = \mathfrak{h} + \ell$ we find, for every $0 \neq x \in L$, that $\mathfrak{h}(x) = q(x) - \ell(x) \leq q(x) < 0$, and hence $q(ix) = -\mathfrak{h}(x) + \ell(x) > \ell(x) \geq 0$. Hence $q|_{iL} > 0$ and $m_+ \geq m_-$. (L is totally real.) Part (ii) is then obvious. □

We can now establish "the fundamental lemma":

Lemma 3.2. *Let $\varphi(x, y)$ be a real plurisubharmonic function of class C^∞, defined in a neighborhood of $(0, 0) \in \mathbb{C}^{n+k}$. We suppose that $\nabla_y \varphi(0, 0) = 0$ and that $\nabla_y^2 \varphi(0, 0)$ is non-degenerate of signature 0. For x in a small neighborhood of 0 in $\mathbb{C}^n$, let $y = y(x)$ be the (unique) critical point of $y \mapsto \varphi(x, y)$ which is close to 0 (hence $y(x)$ is a C^∞ function of x). Then the function $\Phi(x) = \varphi(x, y(x))$ is pl.s.h. If $\tilde{\varphi} \leq \varphi$ is another pl.s.h. function with $\tilde{\varphi}(0, 0) = \varphi(0, 0)$, then $\nabla_y^2 \tilde{\varphi}(0, 0)$ is also non-degenerate of signature 0, and in a neighborhood of $0 \in \mathbb{C}^n$ we have $\tilde{\Phi} \leq \Phi$, where $\tilde{\Phi}$ is defined for $\tilde{\varphi}$ as Φ is for φ.*

Proof. Let $\Gamma_0 \subset \mathbb{C}^k$ be a subspace of real dimension $= k$ such that $\nabla_y^2 \varphi(0, 0)|_{\Gamma_0} < 0$. After a complex-linear change of variables, we may assume that $\Gamma_0 = \mathbb{R}^k$. We extend Γ_0 to a family of subspaces filling $\mathbb{C}^k$, putting $\Gamma_t = \{y \in \mathbb{C}^k;\ \mathrm{Im}\, y = t\}$, $t \in \mathbb{R}^k$. Here we remark that if $\psi(t, s)$ is a real C^∞ function defined near 0 in $\mathbb{R}^{n'+n''}$, with a critical point at 0 and such that $\nabla_s^2 \psi(0, 0)$ is non-degenerate, then, if we introduce the critical point $s(t)$ of $s \mapsto \psi(t, s)$, the function $f(t) = \psi(t, s(t))$ has a critical point at 0. Moreover, $\nabla_t^2 f(0)$ is non-degenerate if and only if $\nabla_{(t,s)}^2 \psi(0, 0)$ is non-degenerate, and $\mathrm{sign} \nabla_t^2 f(0) + \mathrm{sign} \nabla_s^2 \psi(0, 0) = \mathrm{sign} \nabla_{(t,s)}^2 \psi(0, 0)$. All of this becomes clear if we make a change of variables in s, $s \mapsto s - s(t)$. Then in the new coordinates $s(t) = 0$ and $\nabla_t \nabla_s \psi(0, 0) = 0$.

In the special case where ψ'' is of signature 0, $n' = n''$, and $\nabla_s^2 \psi < 0$, we have $\nabla_t^2 f > 0$ and we can express the critical point by a local min-max formula:

$$\psi(0, 0) = \inf_t \sup_s \psi(t, s).$$

Here the "inf" and "sup" are taken locally on suitable neighborhoods of 0.

Applying these remarks to $y \mapsto \varphi(x, y)$, $y = s + it$, we have

$$\Phi(x) = \inf_t \sup_{y \in \Gamma_t} \varphi(x, y)$$

and the analogous formula for $\tilde{\Phi}$. It is then clear that $\tilde{\Phi} \leq \Phi$. If $\tilde{\varphi}$ is pluriharmonic (pl.h.), and hence the real part of a holomorphic function, then it is known that $\tilde{\Phi}$ is the real part of a holomorphic function, hence pl.h. To verify that Φ is pl.s.h. at a point x_0, we can, without changing $\Phi''(x_0)$, replace φ by $\frac{1}{2}\langle (x-x_0, y-y_0), \nabla^2\varphi(x_0,y_0)(x-x_0,y-y_0)\rangle_{\mathbb{R}^{2n}}$ if $y_0 = y(x_0)$. We are then brought back to the case of quadratic forms, and we can find $\tilde{\varphi} \leq \varphi$ pl.h. with $\tilde{\varphi}(x_0,y_0) = \varphi(x_0,y_0)$. Hence $\tilde{\Phi} \leq \Phi$ with equality at x_0, and we deduce that $\Phi''(x_0)$ is pl.s.h. □

We now let $\varphi(x)$ be a C^∞ pl.s.h. function defined near $x_0 \in \mathbb{C}^n$ with $\varphi''(x_0)$ non-degenerate of signature 0. For $\xi \in \mathbb{C}^n$ near $\xi_0 = \frac{2}{i}\frac{\partial\varphi}{\partial x}(x_0)$, we define $\varphi^*(\xi) =$ the critical value of $x \mapsto \varphi(x) + \operatorname{Im} x\cdot\xi = \text{v.c.}_x(\varphi + \operatorname{Im} x\cdot\xi)$. For $f(x)$ holomorphic we have $\frac{\partial}{\partial x}\operatorname{Im} f = \frac{1}{2i}\frac{\partial f}{\partial x}$. Hence the critical point of $x \mapsto \varphi(x) + \operatorname{Im} x\cdot\xi$ is given by the equation $\xi = \frac{2}{i}\frac{\partial\varphi}{\partial x}(x)$. By the fundamental lemma, we know that $\varphi^*(\xi)$ is pl.s.h. If we define $\Lambda_\varphi \subset \mathbb{C}^n_x \times \mathbb{C}^n_\xi$ by $\xi = \frac{2}{i}\frac{\partial\varphi}{\partial x}$, then the projections of Λ_φ onto $\mathbb{C}^n_x$ and $\mathbb{C}^n_\xi$ are local diffeomorphisms. Λ_φ is also given by $x = x(\xi)$, where $x(\xi)$ is the critical point of $x \mapsto \varphi(x) + \operatorname{Im} x\cdot\xi$. We have

$$\frac{\partial\varphi^*}{\partial\xi} = \Big(\frac{\partial}{\partial\xi}(\varphi(x) + \operatorname{Im}\langle x,\xi\rangle)\Big)(x(\xi),\xi) = \frac{1}{2i}x(\xi).$$

That is, Λ_φ is also given by: $-x = \frac{2}{i}\frac{\partial\varphi^*}{\partial\xi}(\xi)$. The projection $\Lambda_\varphi \to \mathbb{C}^n_x$ being a diffeomorphism, we know that $(\varphi^*)''$ is non-degenerate. The signature is 0 because, as in the proof of the fundamental lemma, we can decrease $\varphi''(x_0)$ until it is a pluriharmonic form (while remaining non-degenerate), and then $(\varphi^*)''(\xi_0)$ decreases also as a non-degenerate form until it is a pluriharmonic form which is necessarily of signature 0. Finally, let us note that we can recover $\varphi(x)$ by

$$\varphi(x) = \text{v.c.}_\xi(\varphi^*(\xi) - \operatorname{Im} x\cdot\xi).$$

Of course, φ^* is essentially the Legendre transform of φ.

Before continuing to the Fourier transform, we make some general remarks. We initially let $\varphi(y)$ be a real C^∞ function defined near $0 \in \mathbb{C}^k$ and with a saddle point at 0. (A non-degenerate critical point of signature 0 will from now on be called a "saddle point," or simply a "saddle.") We define a *contour of integration* to be an injective mapping with injective differential $\Gamma : W \to \mathbb{C}^k$, where $W \subset\subset \mathbb{R}^k$ is open and Γ is C^∞ in a neighborhood of $\overline{W}$. We now let Γ be a contour of integration in $\mathbb{C}^k$ which passes through 0 and is such that $\varphi(y) - \varphi(0) \leq -C|y|^2$ for y in (the image of) Γ. We then call Γ a *good contour* with respect to φ. If $u \in H_{\varphi,0}$, then only a neighborhood of $y = 0$ gives a contribution that is not exponentially small to the integral

$$I_\Gamma(h) = e^{-\varphi(0)/h}\int_\Gamma u(y;h)\,dy.$$

This integral is then well-defined modulo the sign (which depends on a choice of orientation of Γ) and modulo a term that is exponentially small as $h \to 0$. This last imprecision depends, on the one hand, on the fact that elements of $H_{\varphi,0}$ are only defined modulo "$\sim$" and, on the other hand, on the fact that the domain of definition of u does not necessarily contain the image of Γ, unless one restricts Γ to a sufficiently small neighborhood of 0. If Γ_1 is a second good contour, then by Stokes' formula, and with good choices of orientation, we have that $I_\Gamma(h) - I_{\Gamma_1}(h)$ is exponentially small as $h \to 0$. Indeed, the real

Morse's Lemma (see Hörmander [15]) allows us to find real coordinates (t,s), $t,s\in\mathbb{R}^k$, such that $\varphi(y)=\frac{1}{2}(t^2-s^2)$, so that Γ and Γ_1 are of the form $t=\gamma(s)$, $t=\gamma_1(s)$, with $|\gamma(s)|$, $|\gamma_1(s)|\le\theta|s|$, $\theta<1$, and hence we have an obvious deformation of good contours, applying Stokes' formula. Let us now add the parameters $x\in\mathbb{C}^n$, and we suppose that $\varphi(x,y)$ is real, C^∞, and defined in a neighborhood of $(0,0)\in\mathbb{C}^{n+k}$, and that $\varphi(0,y)$ satisfies the above hypotheses. We then let $\Phi(x)=\text{v.c.}_y\varphi(x,y)$. Then, if $u\in H_{\varphi,(0,0)}$, we put $U(x;h)=\int_{\Gamma_0}u(x,y;h)\,dy$, where Γ_0 is a good contour for $\varphi(0,y)$. Then $U(x;h)$ is holomorphic in x and defined in a neighborhood of 0. Applying the real Morse's Lemma and Stokes' formula, we see that, modulo an exponentially small error, we can replace the contour Γ_0 by a good contour Γ_x for $y\mapsto\varphi(x,y)$ (depending smoothly on x), and hence $U(x;h)\in H_{\Phi,0}$. We have thus defined, except for the sign, $\int u(x,y;h)\,dy\in H_{\Phi,0}$ for $u\in H_{\varphi,0}$.

Let us now return to the function φ, with φ^* as discussed earlier. Modulo the sign, we define for $u\in H_{\varphi,x_0}$

$$\mathcal{F}u(\xi;h)=\int_{\Gamma_\xi}e^{-ix\cdot\xi/h}u(x;h)\,dx\ \in H_{\varphi^*,\xi_0}$$

where Γ_ξ is a smooth family of good contours for $x\mapsto\varphi(x)+\operatorname{Im}x\cdot\xi$. Similarly, we define

$$\mathcal{G}v(x;h)=(2\pi h)^{-n}\int_{\Gamma_x^*}e^{ix\cdot\xi/h}v(\xi;h)\,d\xi\ \in H_{\varphi,x_0}$$

for $v\in H_{\varphi^*,\xi_0}$, where Γ_x^* is a smooth family of good contours for $\xi\mapsto\varphi^*(\xi)-\operatorname{Im}x\cdot\xi$. With good choices of orientation we have

Proposition 3.3. *For $u\in H_{\varphi,x_0}$, we have $u=\mathcal{G}\mathcal{F}u$ in H_{φ,x_0}.*

Proof. We write

$$\mathcal{G}\mathcal{F}u(x;h)=(2\pi h)^{-n}\iint_{\substack{\xi\in\Gamma^*(x)\\ y\in\Gamma(\xi)}}e^{i(x-y)\cdot\xi/h}u(y;h)\,dy\wedge d\xi. \tag{3.1}$$

The composed contour is automatically good for the function $(y,\xi)\mapsto\varphi(y)-\operatorname{Im}(x-y)\cdot\xi$, which has a saddle at $y=x$, $\xi=\frac{2}{i}\frac{\partial\varphi}{\partial x}(x)$. (We will prove this fact in greater generality in Section 4.) Another good contour is given by $\tilde\Gamma(x)$: $\xi=\frac{2}{i}\frac{\partial\varphi}{\partial x}(x)+iC(\overline{x-y})$, $|y-x|\le\frac{1}{C}$, if $C>0$ is sufficiently large. According to our previous remarks, (3.1) is still valid with the contour $\tilde\Gamma(x)$ in its place (modulo a negligible error). Thus we can easily apply Example 2.6 to conclude the proof. ☐

4. Pseudodifferential Operators and Fourier Integrals in the Complex Domain

Initially it will be convenient to have our operators act on the spaces of germs H_{φ,x_0}. It is clear, however, that one can control the sizes of the domains of definition, and indeed in Section 12 we will study the calculus of pseudodifferential operators with more precision.

Let $a(x,y,\theta;h)$ be an analytic symbol, defined in a neighborhood of $(x_0,x_0,\xi_0)\in\mathbb{C}^{3n}$, and let $\varphi(x)$ be a real C^∞ function (not necessarily pl.s.h.) with $\frac{2}{i}\frac{\partial\varphi}{\partial x}(x_0)=\xi_0$. For $u\in H_{\varphi,x_0}$ we then define $Au\in H_{\varphi,x_0}$ by

$$Au(x;h)=(2\pi h)^{-n}\iint_{\Gamma(x)} e^{i(x-y)\theta/h}a(x,y,\theta;h)u(y;h)\,dy\,d\theta \tag{4.1}$$

where $\Gamma(x)$ is a contour of the form

$$\theta=\frac{2}{i}\frac{\partial\varphi}{\partial x}(x)+iR(\overline{x-y}),\quad |x-y|\le r.$$

For $R>0$ sufficiently large and $r>0$ sufficiently small (and x sufficiently close to x_0), $\Gamma(x)$ is a good contour, because along $\Gamma(x)$ we have

$$\begin{aligned}e^{-\varphi(x)/h}\left|e^{i(x-y)\theta/h}\right|e^{\varphi(y)/h}&=\exp\Big(\frac{\varphi(y)-\varphi(x)}{h}-\frac{2}{h}\mathrm{Re}\left[\frac{\partial\varphi}{\partial x}(x)(y-x)\right]-\frac{R}{h}|x-y|^2\Big)\\&\le\exp\Big(-\frac{(R-C)}{h}|x-y|^2\Big)\end{aligned}$$

where $C>0$ depends on φ'' but not on R. The choice of r and of R does not affect Au on H_{φ,x_0}. When $a=1$ we have already seen that $Au=u$ on H_{φ,x_0} with a good choice of orientation for $\Gamma(x)$.

Let us clarify the meaning of the operator (4.1). Along $\Gamma(x)$ we have

$$dy\wedge d\theta=(-iR)^n dy\wedge d\overline{y}=(2R)^n\frac{dy\wedge d\overline{y}}{(2i)^n}=\underset{(-)}{+}(2R)^nL(dy),$$

hence

$$\begin{aligned}Au(x;h)&=\Big(\frac{R}{\pi h}\Big)^n\int_{|x-y|\le r}\exp\Big(\frac{2}{h}\frac{\partial\varphi}{\partial x}(x)(x-y)-\frac{R}{h}|x-y|^2\Big)\\&\qquad a\left(x,y,\frac{2}{i}\frac{\partial\varphi}{\partial x}(x)+iR(\overline{x-y});h\right)u(y;h)\,L(dy)\\&=\int k(x,y;h)u(y;h)\,L(dy).\end{aligned} \tag{4.2}$$

If a is a symbol of order 0, then

$$|k(x,y;h)|e^{-(\varphi(x)-\varphi(y))/h}\le C_{R,a}h^{-n}e^{-(R-C)|x-y|^2/h}.$$

We then have an obvious continuity result on the spaces L^2_φ.

For (x,ξ) near (x_0,ξ_0) we define the symbol of A, $\sigma_A(x,\xi;h)$, by

$$\sigma_A(x,\xi;h)=e^{-ix\xi/h}A(e^{i(\cdot)\xi/h}). \tag{4.3}$$

The method of stationary phase gives

$$\sigma_A(x,\xi;h)=\sum_{|\alpha|\le\frac{1}{Ch}}\frac{1}{\alpha!}\Big(\frac{h}{i}\Big)^{|\alpha|}(\partial_\xi^\alpha\partial_y^\alpha a)(x,x,\xi;h), \tag{4.4}$$

modulo an exponentially decreasing term, if $C>0$ is sufficiently large. We remark that $\sigma_A=a$ if $a=a(x,\theta;h)$ does not depend on y.

The mapping $a(x,y,\theta;h)\mapsto \sigma_A(x,\xi;h)$ is not injective, but we will show that the action of A on H_{φ,x_0} only depends on σ_A. By Stokes' formula, it in fact suffices to show:

Lemma 4.1. *If $\sigma_A=0$ in $H_{0,(x_0,\xi_0)}$, then there exists a symbol $b(x,y,\theta;h)\in H_{0,(x_0,x_0,\xi_0)}$, which takes values in the $(n-1)$-forms in θ, such that*

$$e^{i(x-y)\theta/h}a(x,y,\theta;h)\,d\theta = ih\,d_\theta(e^{i(x-y)\theta/h}b(x,y,\theta;h)) \tag{4.5}$$

in $H_{-Im\,(x-y)\theta,(x_0,x_0,\xi_0)}$.

Proof. We write

$$\begin{aligned}(2\pi h)^n\sigma_A(x,\eta;h) &= \iint e^{iy\theta/h}a(x,x-y,\theta;h)e^{-iy\eta/h}\,dy\,d\theta\\ &= \left(\mathcal{F}_{(y,\theta)\mapsto(\eta,\theta^*)}u\right)(\eta,0;h)\\ &= v(x,\eta,0;h),\end{aligned}$$

where $u(x,y,\theta;h)=a(x,x-y,\theta;h)e^{iy\theta/h}$, and where we consider x as a parameter. Here $u(x,.,..;h)$ is of class H_φ with $\varphi=-\mathrm{Im}\,(y\theta)$, which is pl.h., so the discussion of Section 3 applies. Hence $v(x,.,..;h)$ is of class H_{φ^*} where $\varphi^*=\mathrm{Im}\,\eta\cdot\theta^*$ is the pl.h. function dual to φ. Our hypothesis says that $v(x,\eta,0;h)$ is exponentially decreasing (with respect to $e^{\varphi^*(\eta,0)/h}=1$) in a neighborhood of $x=x_0$, $\eta=\xi_0$. After modification by a term $=0$ in $H_{\varphi^*,(x_0,\xi_0)}$ we can then suppose that $v=0$ on $\theta^*=0$. Then we multiply by $e^{i\eta\theta^*/h}$, apply Taylor's theorem to first order in θ^*, and then multiply by $e^{-i\eta\theta^*/h}$. Hence we find

$$v(x,\eta,\theta^*;h)=\sum_1^n \hat{v}_j(x,\eta,\theta^*;h)\theta_j^*\,,\quad \hat{v}_j\in H_{\varphi^*,(x_0,\xi_0)} \tag{4.6}$$

where the $\hat{v}_j$ depend holomorphically on x. By the Fourier inversion formula that we established in the previous section, we find

$$u(x,y,\theta;h)=\sum_1^n \frac{h}{i}\frac{\partial}{\partial\theta_j}v_j \qquad \text{in } H_{\varphi,(x_0,0,\xi_0)}$$

where

$$v_j=(2\pi h)^{-n}\iint e^{i(y\eta+\theta\theta^*)/h}\hat{v}_j(x,\eta,\theta^*;h)\,d\eta\,d\theta^* \ \in H_{\varphi,(x_0,0,\xi_0)}.$$

That is, $v_j=b_j(x,y,\theta;h)e^{iy\cdot\theta/h}$ where b_j is a symbol. Hence we obtain the lemma with

$$b=\sum_1^n(-1)^j b_j(x,x-y,\theta;h)\,d\theta_1\wedge\ldots\wedge\widehat{d\theta_j}\wedge\ldots\wedge d\theta_n.$$

□

To treat the composition, we will generalize our framework. Let $\varphi(z,y,\theta)$ be a real C^2 function defined near $(z_0,y_0,\theta_0)\in\mathbb{C}^{n_z+n_y+n_\theta}$, let $f(y)$ be a real C^2 function defined near y_0, and suppose that $(y,\theta)\mapsto\varphi(z,y,\theta)+f(y)$ has a saddle at (y_0,θ_0). We then let $g(z)=$

v.c.$(\varphi(z,y,\theta)+f(y))$ which will be pl.s.h. if φ and f are. If $a(z,y,\theta;h)\in H_{\varphi,(z_0,y_0,\theta_0)}$ we can then define (modulo the sign) an operator $A: H_{f,y_0}\to H_{g,z_0}$ by

$$Au(z;h)=\iint_{\Gamma_1(z)} a(z,y,\theta;h)u(y;h)\,dy\,d\theta$$

where $\Gamma_1(z)$ is a good contour for $(y,\theta)\mapsto\varphi(z,y,\theta)+f(y)$ which depends on z in a C^2 manner.

We now let $b(x,z,w;h)\in H_{\psi,(x_0,z_0,w_0)}$, $(x_0,z_0,w_0)\in\mathbb{C}^{n_x+n_z+n_w}$, and we make the same hypotheses on (b,ψ,g) as we did on (a,φ,f). Then we have an operator $B: H_{g,z_0}\to H_{\mathfrak{h},x_0}$ defined by

$$Bv(x;h)=\iint_{\Gamma_2(x)} b(x,z,w;h)v(z;h)\,dz\,dw$$

where $\mathfrak{h}(x)=\text{v.c.}(\psi(x,z,w)+g(z))$, and $B\circ A: H_{f,y_0}\to H_{\mathfrak{h},x_0}$ is given by

$$B\circ A\ u(x;h)=\iiiint_{\Gamma(x)} b(x,z,w;h)a(z,y,\theta;h)u(y;h)\,dy\,d\theta\,dz\,dw,$$

where $\Gamma(x)$ is the "composed contour": $(z,w)\in\Gamma_2(x)$, $(y,\theta)\in\Gamma_1(z)$.

Before continuing we make here a general remark. Let $F(x,y)$ be of class C^2 defined near 0 in $\mathbb{C}^{n+k}$ such that $F(0,y)$ has a saddle point at $y=0$. If $y=y(x)$ is the critical point of $y\mapsto F(x,y)$, we have already seen that $G(x)=F(x,y(x))$ has a saddle at 0 if and only if $F(x,y)$ has a saddle at $(0,0)$. Let $\Gamma(x)$ be a good contour for $y\mapsto F(x,y)$ and $\tilde\Gamma$ a good contour for G (while supposing that G has a saddle at 0). Then for (x,y) on the composed contour, $x\in\tilde\Gamma$, $y\in\Gamma(x)$, we have

$$F(x,y)\le G(x)-C|y-y(x)|^2\le G(0)-C|x|^2-C|y-y(x)|^2.$$

The composed contour is thus a good contour since $|x|^2+|y-y(x)|^2\sim|x|^2+|y|^2$.

Returning to the operator $B\circ A$ we note that $(z,w,y,\theta)\mapsto\psi(x,z,w)+\varphi(z,y,\theta)+f(y)$ has a saddle and that the composed contour $\Gamma(x)$ is a good contour. Then $B\circ A$ is a "Fourier integral operator" of the same type as A and B if one considers the (z,w,θ) as the "fiber variables."

We suppose now that $(z,w)\mapsto\psi(x_0,z,w)+\varphi(z,y_0,\theta_0)$ has a saddle at (z_0,w_0), and we let $\Gamma_3(x,y,\theta)$ be a good contour for $(z,w)\mapsto\psi(x,z,w)+\varphi(z,y,\theta)$ with $F(x,y,\theta)$ the critical value. Then $(y,\theta)\mapsto F(x,y,\theta)+f(y)$ has a saddle near (y_0,θ_0) and the critical value is $\mathfrak{h}(x)$. Let $\Gamma_4(x)$ be a good contour for $(y,\theta)\mapsto F(x,y,\theta)+f(y)$. Then the contour $\tilde\Gamma(x)$ given by $(y,\theta)\in\Gamma_4(x)$, $(z,w)\in\Gamma_3(x,y,\theta)$ is a good contour for $(z,w,y,\theta)\mapsto\psi(x,z,w)+\varphi(z,y,\theta)+f(y)$ and we obtain

$$\begin{aligned}B\circ A\,u(x;h)&=\iiiint_{\tilde\Gamma} bau\,dy\,d\theta\,dz\,dw\\&=\iint_{\Gamma_4(x)} c(x,y,\theta;h)u(y;h)\,dy\,d\theta\end{aligned}\tag{4.7}$$

where

$$c(x,y,\theta;h)=\iint_{\Gamma_3(x,y,\theta)} b(x,z,w;h)a(z,y,\theta;h)\,dz\,dw\in H_{F,(x_0,y_0,\theta_0)}. \tag{4.8}$$

The interpretation of our supplementary hypothesis (that

$$(z,w)\mapsto \psi(x_0,z,w)+\varphi(z,y_0,\theta_0)$$

has a saddle) and of (4.7) and (4.8) is that as soon as we can define $B(a(.,y,\theta;h))=c(x,y,\theta;h)$ then the equation $B\circ A$ is obtained by applying B under the integral sign of A. When B is a pseudodifferential operator, the supplementary hypothesis is always satisfied, and in particular for two pseudodifferential operators we obtain the following theorem:

Theorem 4.2. *Let $A,B:H_{\varphi,x_0}\to H_{\varphi,x_0}$ be two pseudodifferential operators. Then $B\circ A$ is a pseudodifferential operator with symbol*

$$\sigma_{B\circ A}(x,\xi;h)=\sum_{|\alpha|\le\frac{1}{Ch}}\frac{1}{\alpha!}\left(\frac{h}{i}\right)^{|\alpha|}\partial_\xi^\alpha\sigma_B(x,\xi;h)\partial_x^\alpha\sigma_A(x,\xi;h) \tag{4.9}$$

with $C>0$ sufficiently large.

Proof. Following Lemma 4.1 we can suppose that A, B are given by (4.1) with $a=\sigma_A(x,\theta;h)$, $b=\sigma_B(x,\theta;h)$ respectively. Following the preceding remarks we then have

$$B\circ A\,u(x;h)=(2\pi h)^{-n}\iint e^{i(x-y)\theta/h}c(x,\theta;h)u(y;h)\,dy\,d\theta$$

with

$$c(x,\theta;h)=e^{-ix\theta/h}B(\sigma_A(\cdot,\theta;h)e^{i(\cdot)\theta/h}).$$

This quantity is expanded by the method of stationary phase, which gives (4.9). ☐

Remark 4.3. We can also define pseudodifferential operators with nonstandard phases:

$$Au(x;h)=(2\pi h)^{-n}\iint e^{i\psi(x,y,\theta)/h}\tilde a(x,y,\theta;h)u(y;h)\,dy\,d\theta. \tag{4.10}$$

Here we assume that ψ is holomorphic, defined near $(x_0,x_0,\theta_0)\in\mathbb{C}^{3n}$, and is such that $\psi|_{x=y}=0$ and $\det\frac{\partial^2\psi}{\partial x\partial y}\neq 0$. Then by Kuranishi's trick we write

$$\psi(x,y,\theta)=(x-y)\xi(x,y,\theta)$$

where $\theta\mapsto\xi(x,y,\theta)$ is a local diffeomorphism and is holomorphic. If $\theta=\theta(x,y,\xi)$ is the inverse, we formally obtain

$$Au(x;h)=(2\pi h)^{-n}\iint e^{i(x-y)\xi/h}a(x,y,\xi;h)u(y;h)\,dy\,d\xi \tag{4.11}$$

with $a(x,y,\xi;h)=\tilde a(x,y,\theta;h)\det\left(\frac{\partial\theta}{\partial\xi}\right)$. A good contour for (4.11) gives a good contour for (4.10) and conversely. For two operators of type (4.10), not necessarily with the same phase, on the one hand Theorem 4.2 is valid, and on the other hand we obtain $B\circ A$ by formally applying B under the integral sign of A (or, just as well, by applying ${}^tA_y$ under the integral sign of B). The nonstandard phases appear after holomorphic changes of coordinates.

Remark 4.4. If one wants to study the action of a pseudodifferential operator simultaneously on several spaces H_{φ,x_0} where the φ vary slightly, one can use the *singular contours* of the form

$$\tilde{\Gamma}(x) : \theta = \frac{2}{i}\frac{\partial\varphi}{\partial x} + iR\frac{\overline{(x-y)}}{|x-y|} \quad , \quad |x-y| \leq r$$

in (4.1). Along $\tilde{\Gamma}(x)$ we have, modulo the terms in dy_k, $1 \leq k \leq n$:

$$d\theta_j \equiv \frac{R}{i}\overline{\partial_y}\left(\frac{\overline{y_j - x_j}}{|y-x|}\right) = \frac{R}{i|y-x|}\left(d\overline{y_j} - \frac{\overline{y_j - x_j}}{|y-x|}\,\overline{\partial_y}|y-x|\right).$$

Hence

$$\begin{aligned} dy \wedge d\theta = &\left(\frac{R}{i|y-x|}\right)^n dy_1 \wedge \ldots \wedge dy_n \wedge \left(d\overline{y_1} - \frac{\overline{y_1 - x_1}}{|y-x|}\overline{\partial_y}|y-x|\right) \wedge \\ &\ldots \wedge \left(d\overline{y_n} - \frac{\overline{y_n - x_n}}{|y-x|}\overline{\partial_y}|y-x|\right) \\ = &\left(\frac{R}{i|y-x|}\right)^n dy \\ &\wedge \left(d\overline{y} - \sum_1^n \frac{\overline{y_j - x_j}}{|y-x|} d\overline{y_1} \wedge \ldots \wedge \overline{\partial_y}|y-x| \wedge d\overline{y_{j+1}} \wedge \ldots \wedge d\overline{y_n}\right). \end{aligned}$$

Here we can replace $\overline{\partial_y}|y-x|$ by $\overline{\partial_{y_j}}|y-x| = \frac{1}{2}\frac{(y_j - x_j)}{|y-x|}\,d\overline{y_j}$, and along $\tilde{\Gamma}(x)$ we thus have

$$dy \wedge d\theta = \frac{1}{2}\left(\frac{R}{i|y-x|}\right)^n dy \wedge d\overline{y} = \frac{1}{2}\left(\frac{2R}{|y-x|}\right)^n L(dy).$$

Hence the integral (4.1) with $\Gamma(x)$ replaced by $\tilde{\Gamma}(x)$ becomes

$$\begin{aligned} \tilde{A}u(x;h) = \frac{1}{2}\left(\frac{R}{\pi h}\right)^n \int_{|x-y|\leq r} &\exp\left(\frac{2}{h}\frac{\partial\varphi}{\partial x}(x)(x-y) - \frac{R}{h}|x-y|\right)|x-y|^{-n} \\ &\times a\left(x, y, \frac{2}{i}\frac{\partial\varphi}{\partial x}(x) + iR\frac{\overline{x-y}}{|x-y|}; h\right) u(y;h)\; L(dy). \end{aligned} \tag{4.12}$$

The kernel is then locally integrable, and if a is a symbol of order zero we have an obvious continuity result on $L^2_{\tilde{\varphi}}$ if $\tilde{\varphi}$ is sufficiently close to φ in the C^1 (or even Lipschitz) norm.

Let us also verify that $\tilde{A}u = Au$ if in (4.1) we take the contour $\Gamma : \theta = \frac{2}{i}\frac{\partial\varphi}{\partial x}(x) + i\frac{R}{r}(\overline{x-y})$. For this we continuously deform the contour by introducing, for $t \geq 1$:

$$\Gamma_t(x) : \begin{cases} \theta = \frac{2}{i}\frac{\partial\varphi}{\partial x}(x) + it\frac{R}{r}(\overline{x-y}) & , \quad \text{if} \quad |x-y| \leq \frac{r}{t} \\ \theta = \frac{2}{i}\frac{\partial\varphi}{\partial x}(x) + iR\frac{(\overline{x-y})}{|x-y|} & , \quad \text{if} \quad \frac{r}{t} \leq |x-y| \leq r. \end{cases}$$

Then $\int_{\Gamma_t} \dots$ does not depend on t, and, on the other hand, $\lim_{t\to\infty}\int_{\Gamma_t} \dots dy\, d\theta = \int_{\tilde{\Gamma}} \dots dy\, d\theta$, because, for h, R, r fixed:

$$\iint_{\substack{\theta=\frac{2}{i}\frac{\partial\varphi}{\partial x}(x)+it\frac{R}{r}(\overline{x-y})\\ |x-y|\le\frac{r}{t}}} \dots dy\, d\theta = \mathcal{O}(1)t^{-2n}t^n \to 0, \qquad t\to+\infty.$$

To end this section, we slightly generalize the results of Section 3 concerning the Fourier transform. Let $\varphi(x,y)$ be a holomorphic function defined near $(x_0,y_0)\in\mathbb{C}^n\times\mathbb{C}^n$ such that

$$\det\frac{\partial^2\varphi}{\partial x\partial y}(x_0,y_0)\neq 0. \tag{4.13}$$

Let $f(y)$ be a C^∞ pl.s.h. function defined near y_0 such that

$$y\mapsto -\mathrm{Im}\varphi(x_0,y)+f(y) \qquad \text{has a saddle at } y_0. \tag{4.14}$$

Then $g(x)=\mathrm{v.c.}_y(-\mathrm{Im}\varphi(x,y)+f(y))$ is pl.s.h. Let $\psi(x,y)=-\mathrm{Im}\varphi(x,y)$. Then $\frac{2}{i}\frac{\partial\psi}{\partial x}=\frac{\partial\varphi}{\partial x}$ and (4.13) is equivalent to the condition

$$\left\{\left(x,\frac{2}{i}\frac{\partial\psi}{\partial x},y,-\frac{2}{i}\frac{\partial\psi}{\partial y}\right)\right\} \qquad \text{is the graph of a (symplectic) diffeomorphism} \tag{4.13'}$$

which we denote by κ. (4.14) implies that $\nabla_y^2(\psi(x,y)+f(y))$ is non-degenerate, which is equivalent to the fact that the map $\kappa(\Lambda_f)\ni(x,\xi)\mapsto x\in\mathbb{C}^n$ is a local diffeomorphism. Hence $\Lambda_g=\kappa(\Lambda_f)$. Since $\Lambda_f=\kappa^{-1}(\Lambda_g)$ and κ^{-1} is given by $-\psi$ with x and y permuted, it follows from (4.13), (4.14) that $\nabla_x^2(-\psi(x,y_0)+g(x))(x_0)$ is non-degenerate. Decreasing $f''(y_0)$ until it is a pl.h. form, we show as in Section 3 that $x\mapsto-\psi(x,y_0)+g(x)$ has a saddle at x_0. Also, $f(y)=\mathrm{v.c.}_x(-\psi(x,y)+g(x))+C$, and in putting $y=y_0$ we find that $C=0$. Hence, to summarize,

$$\begin{aligned} &x\mapsto-\psi(x,y)+g(x) \qquad \text{has a saddle near } x_0,\\ &\qquad\qquad \text{and the critical value is } f(y). \end{aligned} \tag{4.15}$$

We now let $a(x,y;h)$ be an elliptic classical analytic symbol of order zero defined near (x_0,y_0), and we let $T: H_{f,y_0}\to H_{g,x_0}$ be defined by

$$Tu(x;h)=\int_{\Gamma(x)} e^{i\varphi(x,y)/h}a(x,y;h)u(y;h)\,dy \tag{4.16}$$

where $\Gamma(x)$ is a good contour for $y\mapsto\psi(x,y)+f(y)$. We seek to invert T by an operator of the form

$$Sv(y;h)=h^{-n}\int_{\tilde{\Gamma}(y)} e^{-i\varphi(x,y)/h}b(x,y;h)v(x;h)\,dx, \tag{4.17}$$

where $\tilde{\Gamma}(y)$ is a good contour for $x\mapsto-\psi(x,y)+g(x)$ and where b is a classical analytic symbol of order zero.

Theorem 4.5. *There exists a classical analytic symbol b such that $S\circ T=I$ in H_{f,y_0} and $T\circ S=I$ in H_{g,x_0}, if S is given by (4.17).*

Proof. Let $\tilde{b}$ be a classical analytic symbol of order zero and let $\tilde{S}$ be the corresponding operator as in (4.17). Then

$$T \circ \tilde{S}u(x;h) = h^{-n} \iint_{\substack{y\in\Gamma(x)\\ z\in\tilde{\Gamma}(y)}} e^{i(\varphi(x,y)-\varphi(z,y))/h} a\,\tilde{b}\,u(z;h)\,dz\,dy.$$

By Kuranishi's trick, we recognize here an elliptic classical pseudodifferential operator. We then let R be a parametrix, that is, a pseudodifferential operator whose symbol is given by Theorem 1.5. We put $S = \tilde{S} \circ R$, and, as we saw earlier in this section, S is then of the form (4.17) with b given by

$${}^tR(x, \tilde{D}_x; h)(\tilde{b}(\cdot, y; h)e^{-i\varphi(\cdot,y)/h}) = b(x, y; h)e^{-i\varphi(x,y)/h}.$$

We have $T \circ S \equiv I$ in H_{g,x_0}. Similarly, we can construct an operator S_1 of the form (4.17) such that $S_1 \circ T \equiv I$ in H_{f,y_0}, and so we necessarily have $S \equiv S_1$. □

Example 4.6. We take $x_0 = y_0 = 0$, $\varphi = (x - y)^2$, and $f = g = 0$. If $u \in H_{0,0}$ is a classical analytic symbol, we easily see that Tu is of the same type, and we have a completely explicit stationary phase expansion for Tu. If v is a classical analytic symbol, to solve the equation $Tu = v$ we attempt to determine the successive terms of u by writing the expansion of Tu and identifying the terms. This proceeds very well on the level of formal classical symbols, but to bound u_k in $u = \sum_0^\infty h^k u_k$ we have difficulties, due to the fact that the k^{th} term in the expansion of Tu essentially uses $2k$ derivatives of u. It is clear that we cannot proceed only with estimates, since already the problem

$$u(x;h) + \epsilon h\partial_x^2 u(x;h) = v(x) \qquad \text{(one variable)}$$

in general does not have a classical analytic symbol as a solution, when v is a holomorphic function.

5. Pseudodifferential Operators in the Real Domain and Resolutions of the Identity

We essentially follow [30] and [31]. When one works in the real domain, one avoids writing pseudodifferential operators with the standard phase, because for this phase the contour $\mathbb{R}^n_y \times \mathbb{R}^n_\theta$ is not a good contour, and one easily falls into difficulties with cut-off functions. Let $(x_0, \xi_0) \in T^*\mathbb{R}^n$, and we write $\alpha = (\alpha_x, \alpha_\xi) \in \mathbb{C}^{2n}$. Let $\varphi(x, y, \alpha)$ be an analytic function defined near (x_0, x_0, x_0, ξ_0) satisfying:

(5.1) For $x = y = \alpha_x$ one has $\varphi = 0$ and $\dfrac{\partial\varphi}{\partial x} = -\dfrac{\partial\varphi}{\partial y} = \alpha_\xi$,

(5.2) On the real domain, $\operatorname{Im}\varphi \geq C(|x - \alpha_x|^2 + |y - \alpha_x|^2)$, where $C > 0$.

Example: $\varphi(x, y, \alpha) = (x - y)\alpha_\xi + \frac{i}{2}((x - \alpha_x)^2 + (y - \alpha_x)^2)$.

In general, if φ satisfies (5.1) and (5.2), then

$$\varphi(x,y,\alpha) = (x-y)\alpha_\xi + h(x,y,\alpha), \tag{5.3}$$

where

$$h(x,y,\alpha) = \mathcal{O}(|x-\alpha_x|^2 + |y-\alpha_x|^2), \tag{5.4}$$

$$\operatorname{Im} h(x,y,\alpha) \geq C(|x-\alpha_x|^2 + |y-\alpha_x|^2) \quad \text{on the real domain.} \tag{5.5}$$

In comparison with the standard phase, φ has n additional variables. Nevertheless, one can eliminate the variables α_x with the help of the stationary phase, for (5.4) and (5.5) show that for x near y and near the real domain, the function $\alpha_x \mapsto h(x,y,\alpha)$ has a non-degenerate critical point near $x \sim y$. If $g(x,y,\alpha_\xi)$ is the critical value, it is clear that

$$g(x,y,\alpha_\xi) = \mathcal{O}(|x-y|^2). \tag{5.6}$$

By applying Lemma 7.5 from Section 7 (or doubtless also by a direct analysis of the Hessian of g) one shows also that, for x, y, α_ξ real,

$$\operatorname{Im} g(x,y,\alpha_\xi) \geq C|x-y|^2 \tag{5.7}$$

where $C > 0$. Hence a Fourier integral operator expressed with the phase $\varphi(x,y,\alpha)$ brings itself back at least formally, with the assistance of the method of stationary phase, to a pseudodifferential operator expressed with the phase $(x-y)\alpha_\xi + g(x,y,\alpha_\xi)$. If $a(x,y,\alpha;h)$ is an analytic symbol defined near (x_0,x_0,x_0,ξ_0), and if $\psi(x)$ is a real-valued C^∞ function defined on a complex neighborhood of x_0 with $\frac{2}{i}\frac{\partial\psi}{\partial x}(x_0) = \xi_0$, then one has a pseudodifferential operator $A : H_{\psi,x_0} \to H_{\psi,x_0}$ defined by

$$Au(x;h) = h^{-3n/2} \iint_{\Gamma(x)} e^{i\varphi(x,y,\alpha)/h} a(x,y,\alpha;h) u(y;h)\, dy\, d\alpha, \tag{5.8}$$

where one finds the good contour $\Gamma(x)$ without difficulty, integrating first in α_x. The results and remarks of the preceding section remain true: one associates to A its symbol $\sigma_A(x,\xi;h)$ as before, and it is always true that if $\sigma_A = 0$ then there exists an analytic symbol $b(x,y,\alpha;h)$, with values in the $(2n-1)$-forms in α, such that

$$e^{i\varphi/h} a\, d\alpha = d_\alpha(e^{i\varphi/h} b)$$

near (x_0,x_0,x_0,ξ_0). In fact, as before one finds that

$$h^{3n/2}\sigma_A(x,\eta;h) = \left(\mathcal{F}_{(y,\alpha)\to(\eta,\alpha^*)} u_x\right)(\eta,0;h),$$

$$u_x(y,\alpha;h) = e^{i\varphi(x,x-y,\alpha)/h} a(x,x-y,\alpha;h).$$

We are not obligated to work only with germs; if $V \subset\subset \mathbb{C}^{2n}$ is open and φ and a are defined in a neighborhood of $\nabla\overline{V} = \{(x,y,\alpha) \in \overline{V};\ x = y = \alpha_x\}$, and $\sigma_A = 0$ as a formal symbol, then there exists a symbol $b(x,y,\alpha;h)$, defined in a neighborhood of $\nabla\overline{V}$ such that $a\, d\alpha = e^{-i\varphi/h} d_\alpha(e^{i\varphi/h} b)$ in the formal symbols. In addition, if a is a classical analytic symbol, one can choose b to be a classical analytic symbol.

Also as in Section 4, if A and B are two operators of the form (5.8), not necessarily with the same phase, then one obtains $A \circ B$, applying A under the integral sign of B, or applying ${}^tB$ under the integral sign of A. Moreover, the usual formula is true for $\sigma_{A\circ B}$.

Let now $V \subset\subset \mathbb{R}^n \times \mathbb{R}^n$ be an open neighborhood of (x_0, ξ_0), and let a and φ be as above and defined in complex neighborhoods of $\nabla\overline{V}$. Let $\chi(x, y, \alpha) \in C_0^\infty(\mathbb{R}^{4n})$ have support near $\nabla\overline{V}$, with $\nabla\overline{V} \cap \operatorname{supp}(\chi - 1) = \emptyset$. Then one has the "realization" of the operator A defined for $u \in \mathcal{D}'(\mathbb{R}^n)$ by

$$A^V u(x; h) = h^{-3n/2} \iint_{\alpha \in V} e^{i\varphi(x,y,\alpha)/h} a(x, y, \alpha; h)\chi(x, y, \alpha)u(y; h)\, dy\, d\alpha. \tag{5.9}$$

Thus $A^V : \mathcal{D}'(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n)$. Thanks to (5.2), A^V is independent of the choice of χ, modulo an operator whose distribution kernel is of exponential decrease on $C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ (that is, there exists a $C > 0$ such that $e^{C/h}k(x, y; h)$ is bounded on $C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, uniformly with respect to h). In the same way, if one replaces a by an equivalent symbol, then A^V is unchanged, modulo the same type of "negligible operator."

If $\sigma_A = 0$ as a formal symbol, then, as we have just noticed, $e^{i\varphi/h} a\, d\alpha \sim d_\alpha(e^{i\varphi/h} b)$, and, if the boundary of V is sufficiently regular, Stokes' formula shows that, modulo a negligible operator,

$$A^V u(x; h) \equiv h^{-3n/2} \iint_{\alpha \in \partial V} e^{i\varphi(x,y,\alpha)/h} \chi\, b\, u(y)\, dy. \tag{5.10}$$

When $\sigma_A = 1$, one calls

$$\Pi_{\alpha,h}(x, y) = e^{i\varphi(x,y,\alpha)/h} a(x, y, \alpha; h)$$

a resolution of the identity, and one writes formally

$$I = \int \Pi_{\alpha,h}\, d\alpha.$$

Lemma 5.1. *Let $\psi(x)$ be an analytic function defined near x_0 with $Im\, \psi(x_0) = 0$, $\psi'(x_0) = \xi_0$, and such that $Im\, \psi(x) \geq 0$ for x real. Let also $b(x; h)$ be an analytic symbol defined near x_0. Then for x in a real neighborhood of x_0, one has, modulo a negligible function (that is to say, of exponential decrease in C^∞):*

$$A^V(b(\cdot; h)e^{i\psi(\cdot)/h}) \equiv c(x; h)e^{i\psi(x)/h}$$

where $c(x; h)$ is an analytic symbol defined by

$$c\, e^{i\psi/h} = A(b\, e^{i\psi/h}) \quad \textit{in } H_{-\operatorname{Im}\psi, x_0}.$$

(It is understood that the support of χ is sufficiently small because $A^V(b\, e^{i\psi/h})$ is defined near x_0.)

Proof. One writes:

$$A^V(b\, e^{i\psi/h})(x) = h^{-3n/2} \iint_{\substack{\alpha \in V \\ y \in \mathbb{R}^n}} e^{i(\varphi(x,y,\alpha)+\psi(y))/h} \chi(x, y, \alpha)a(x, y, \alpha; h)b(y; h)\, dy\, d\alpha. \tag{5.11}$$

For $x = x_0$, the phase $\varphi + \psi$ has a non-degenerate critical point at $y = \alpha_x = x_0$, $\alpha_\xi = \xi_0$, and

$$\operatorname{Im}(\varphi + \psi) \geq C(|y - \alpha_x|^2 + |x - \alpha_x|^2).$$

However, this is not enough for $\alpha \in V$, $y \in \mathbb{R}^n$ to be a good contour, since it lacks the term $C|\alpha_\xi - \xi_0|^2$.

Let then $0 \le \tilde{\chi}(x, y, \alpha) \in C_0^\infty$ be a function which is $= 1$ near $\nabla \overline{V}$ and which has support in the region where $\chi = 1$. By Stokes' formula (in y), one can replace the real contour in (5.11) by the contour

$$\Gamma_\delta(x) : \begin{cases} \alpha \in V \\ \tilde{y} = y + i\delta\tilde{\chi}(x, y, \alpha)\overline{\partial_y(\varphi + \psi)}, \quad y \in \mathbb{R}^n. \end{cases}$$

For $(\alpha, \tilde{y}) \in \Gamma_\delta$ and $\delta > 0$ sufficiently small, one has with new constants > 0:

$$\begin{aligned} \operatorname{Im}(\varphi + \psi) &\ge C(|y - \alpha_x|^2 + |x - \alpha_x|^2 + \tilde{\chi}|\partial_y(\varphi + \psi)|^2) \\ &\ge C'(|y - \alpha_x|^2 + |x - \alpha_x|^2 + |\alpha_\xi - \partial_x\psi|^2). \end{aligned} \tag{5.12}$$

Hence $\Gamma_\delta(x_0)$ is a good contour along which the cut-off function only intervenes in the region where $\operatorname{Im}(\varphi + \psi) > 0$.

When x is a neighbor of x_0, one always has (5.12), but the contour does not necessarily pass through the critical point. Analyzing the situation in the coordinates given by Morse's Lemma, one notes nevertheless that after a supplementary deformation (of order $= \mathcal{O}(|x - x_0|)$) which does not change the contour in the cut-off region, one obtains a good contour, and one has reduced to the situation of Section 4. □

Remark 5.2. Let $\Pi_{\alpha,h} = a(x, y, \alpha; h)e^{i\varphi/h}$, $\tilde{\Pi}_{\alpha,h} = \tilde{a}\, e^{i\tilde{\varphi}/h}$ be resolutions of the identity that one realizes with the help of cut-off functions as before. Then $\Pi_{\alpha,h} \circ \Pi_{\beta,h}$ is negligible for $\alpha \ne \beta$. This is obvious for $\alpha_x \ne \beta_x$, and for $\alpha_x = \beta_x$, $\alpha_\xi \ne \beta_\xi$ it is shown by the same contour deformation as in the proof of Lemma 5.1.

Theorem 5.3. *Let A be as before, and let*

$$Bu(x; h) = \iint e^{i\psi(x,y,\alpha)/h} b(x, y, \alpha; h)u(y)\, dy\, d\alpha$$

be another operator of the same type. Let $c(x, y, \alpha; h)$ be the symbol in $H_{-Im\psi}$ given by

$$A(e^{i\psi(\cdot,y,\alpha)/h} b(\cdot, y, \alpha; h)) = e^{i\psi(x,y,\alpha)/h} c(x, y, \alpha; h).$$

Then if $U \subset\subset V \subset\subset \mathbb{R}^n \times \mathbb{R}^n$ are open sets (sufficiently small so that A^V and B^U are defined), one has, modulo a negligible operator,

$$A^V \circ B^U \equiv C^U$$

where C^U is a realization of

$$Cu(x; h) = \iint e^{i\psi/h} c\, u\, dy\, d\alpha.$$

6. The Analytic Wavefront Set, Essentially Following Bros-Iagolnitzer

In this section we introduce the analytic wavefront set (or "analytic singular spectrum") in the spirit of Bros-Iagolnitzer (see [8]). We recall that their definition followed the definitions of Sato [27] and Hörmander [13], and that Bony [1] has shown the equivalence of all these definitions, even within the framework of hyperfunctions, with regard to the definitions of Sato and Bros-Iagolnitzer. In the more simple framework of distributions, we show the equivalence between $WF_{a,Sato}$ and $WF_{a,Bros-Iagolnitzer}$. See also Lebeau [22] and the book of Hörmander [17].

Let $(x_0, \xi_0) \in T^*\mathbb{R}^n \backslash 0$ and let $\varphi(x, \alpha)$ be an analytic function defined in a neighborhood of (x_0, x_0, ξ_0) such that

$$\varphi = 0 \text{ and } \varphi'_x = \alpha_\xi \text{ for } x = \alpha_x \tag{6.1}$$

$$\operatorname{Im} \varphi \geq C|x - \alpha_x|^2 \quad \text{for real } x, \alpha. \tag{6.2}$$

Let $a(x, \alpha; h)$ be an elliptic classical analytic symbol defined near (x_0, x_0, ξ_0).

Definition 6.1. *Let $u \in \mathcal{D}'(X)$ where $X \subset \mathbb{R}^n$ is an open set containing x_0. One says that $u = 0$ (microlocally) at (x_0, ξ_0) if*

$$\int e^{i\varphi(x,\alpha)/h} a(x, \alpha; h)\chi(x)\overline{u(x)}\, dx$$

is exponentially decreasing when $h \to 0$, uniformly for α in a real neighborhood of (x_0, ξ_0). Here $\chi \in C_0^\infty(X)$ is equal to 1 near x_0.

This definition does not depend on the choice of χ.

Proposition 6.2. *Definition 6.1 does not depend on the choice of (a, φ).*

Proof. Suppose that $u = 0$ microlocally near (x_0, ξ_0) for the choice (a, φ), and let $(\tilde{a}, \tilde{\varphi})$ be another choice. The pseudodifferential operator in the complex domain

$$Au(x; h) = h^{-3n/2} \iint \exp\left\{\frac{i}{h}\left(\varphi(x, \alpha) - \overline{\varphi(\overline{y}, \overline{\alpha})}\right)\right\} a(x, \alpha; h)u(y; h)\, dy\, d\alpha$$

has an elliptic symbol σ_A and hence admits a parametrix in the spaces H_{ψ,x_0} with a suitable ψ. One can then find a classical analytic symbol $\tilde{b}(x, \alpha; h)$ defined near (x_0, x_0, ξ_0) such that

$$A(\tilde{b}(\cdot, \alpha; h)e^{i\tilde{\varphi}/h}) = \tilde{a}(x, \alpha; h)e^{i\tilde{\varphi}/h}.$$

By the results of Section 5, if $(x_0, \xi_0) \in V \subset\subset W \subset\subset \mathbb{R}^{2n}$, where V, W are sufficiently small open sets, one obtains for $\alpha \in V$ modulo a negligible function (that is to say exponentially decreasing in C_0^∞)

$$A^W(\tilde{b}(\cdot, \alpha; h)e^{i\tilde{\varphi}/h}) \equiv \tilde{a}(x, \alpha; h)e^{i\tilde{\varphi}(x,\alpha)/h}. \tag{6.3}$$

This relation expresses $\tilde{a}\, e^{i\tilde{\varphi}/h}$ as a superposition of functions $a\, e^{i\varphi/h}$:

$$\tilde{a}(x, \alpha; h)e^{i\tilde{\varphi}(x,\alpha)/h} \equiv \int_{\beta \in W} a(x, \beta; h)e^{i\varphi(x,\beta)/h} f(\alpha, \beta; h)\, d\beta \tag{6.4}$$

where

$$f(\alpha,\beta;h) = h^{-3n/2}\int \exp\left\{-\frac{i}{h}\overline{\varphi(\overline{x},\overline{\beta})}+\frac{i}{h}\tilde{\varphi}(x,\alpha)\right\}\tilde{b}(x,\alpha;h)\,dx$$

is of tempered increase in h. If one chooses W sufficiently small so that the integral in Definition 6.1 is uniformly exponentially decreasing for $\alpha\in W$, then for $\alpha\in V$ and modulo uniformly exponentially decreasing terms:

$$\begin{aligned}&\int e^{i\tilde{\varphi}(x,\alpha)/h}\tilde{a}(x,\alpha;h)\chi(x)\overline{u(x)}\,dx\\ &\qquad\equiv \int_{\beta\in W} f(\alpha,\beta;h)\int e^{i\varphi(x,\beta)/h}a(x,\beta;h)\chi(x)\overline{u(x)}\,dx\,d\beta\\ &\qquad\equiv 0.\end{aligned}\tag{6.5}$$

□

With the help of Proposition 6.2 one sees that the set of points $(x,\xi)\in T^*X\backslash 0$ where $u=0$ is a conic set (and is trivially open). One denotes by $WF_a(u)$ its complement in $T^*X\backslash 0$. This is the *analytic wavefront set* (or *analytic singular spectrum*) of u. Let $SS_a(u)$ denote the analytic singular support, that is to say the smallest closed subset of X outside of which u is analytic. If $\Pi : T^*X\backslash 0\to X$ is the natural projection, then one has the following:

Theorem 6.3. *For each $u\in\mathcal{D}'(X)$ one has $\Pi(WF_a(u)) = SS_a(u)$.*

Proof. First let $x_0\in X\backslash SS_a(u)$ and suppose without loss of generality that $u\in\mathcal{E}'(X)$. Writing $\alpha=(x,\xi)$ one can then study

$$\int \exp\left\{\frac{i}{h}\left((x-y)\xi+\frac{i}{2}(x-y)^2\right)\right\}u(y)\,dy \tag{6.6}$$

for x near x_0 and $\xi\neq 0$. Let $0\le\chi\in C_0^\infty(X)$ have support in a small neighborhood of x_0 where u is analytic and with $\chi(x_0)=1$. Then for $\epsilon>0$ sufficiently small one can replace $\mathbb{R}^n$ by the complex contour

$$\Gamma_\epsilon : y\mapsto y-i\epsilon\chi(y)\xi.$$

Then for x near x_0 and ξ in a compact subset of $\dot{\mathbb{R}}^n$ one notes that the integral (6.6) is uniformly exponentially decreasing when $h\to 0$. Hence $(x_0,\xi)\notin WF_a(u)$ for all $\xi\in\dot{\mathbb{R}}^n$, and one has shown that $\Pi(WF_a(u))\subset SS_a(u)$.

To show the opposite inclusion, one supposes that $(x_0,\xi)\notin WF_a(u)$ for all $\xi\in\dot{\mathbb{R}}^n$. We consider the identity

$$\delta(x) = \int e^{ix\xi}\frac{d\xi}{(2\pi)^n}, \tag{6.7}$$

where the integral is interpreted as an oscillatory integral, for example as the limit in $\mathcal{D}'(\mathbb{R}^n)$:

$$\lim_{\epsilon\to 0}\int e^{ix\xi-\epsilon\xi^2/2}d\xi. \tag{6.7'}$$

Following Lebeau [22], one replaces $\mathbb{R}^n_\xi$ by the complex contour $\xi \to \zeta = \xi + \frac{i}{2}|\xi|x$. Along this contour one has

$$e^{ix\zeta} = e^{ix\xi - |\xi|x^2/2}, \quad d\zeta_j = d\xi_j + \frac{i}{2}x_j\, d|\xi|.$$

Because $d|\xi| = \sum \frac{\xi_k}{|\xi|} d\xi_k$ one obtains

$$d\zeta_1 \wedge \cdots \wedge d\zeta_n = \left(1 + \frac{i}{2}\sum_1^n \frac{x_j\xi_j}{|\xi|}\right) d\xi_1 \wedge \cdots \wedge d\xi_n = a(x,\xi)\, d\xi_1 \wedge \cdots \wedge d\xi_n.$$

Passing through the integrals (6.7) one then easily shows that

$$\delta(x) = \int e^{ix\xi - |\xi|x^2/2} a(x,\xi)\, \frac{d\xi}{(2\pi)^n} \tag{6.8}$$

in the sense of oscillatory integrals. (In fact the argument with (6.7) does not work for $|x|$ small, but the integral (6.8) is evidently analytic outside of 0, hence is 0 everywhere in $\mathbb{R}^n\backslash 0$.)

Hence

$$u(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi - |\xi|(x-y)^2/2} a(x-y,\xi) u(y)\, dy\, d\xi \tag{6.9}$$

where the integral is in the sense of a limit in $\mathcal{D}'$ after the introduction of the convergence factor $e^{-\epsilon\xi^2/2}$.

Proposition 6.2 (and its proof) shows that

$$\int e^{i(x-y)\xi - |\xi|(x-y)^2/2} a(x-y,\xi) u(y)\, dy$$

decreases exponentially, uniformly when $|\xi| \to \infty$ for x in a *complex* neighborhood of x_0. Hence (6.9) shows that u is analytic at x_0. □

The following result is due to Sato [27] in the case of hyperfunctions and is also due to Hörmander [13] in the case of distributions.

Theorem 6.4. *Let $P(x,D)$ be a differential operator with analytic coefficients defined in a open set $X \subset \mathbb{R}^n$. Let $p(x,\xi)$ be its principal symbol, and suppose that $p(x_0,\xi_0) \neq 0$ where $(x_0,\xi_0) \in T^*X\backslash 0$. If $u \in \mathcal{D}'(X)$ then we have $u = 0$ at (x_0,ξ_0) if and only if $Pu = 0$ at (x_0,ξ_0). In particular, if P is elliptic at x_0 and Pu is analytic at x_0 then u is analytic at x_0.*

Proof. One can suppose that $u \in \mathcal{E}'(X)$. Then with a, φ as in Definition 6.1

$$\int e^{i\varphi(x,\alpha)/h} a(x,\alpha;h)\overline{Pu(x)}\, dx = \int e^{i\varphi(x,\alpha)/h} b(x,\alpha;h)\overline{u(x)}\, dx$$

where b is the elliptic symbol given by $b\, e^{i\varphi/h} = P^*(a\, e^{i\varphi/h})$. It then suffices to apply Proposition 6.2. □

We now establish the equivalence between the definitions of Bros-Iagolnitzer and of Sato. This is a particular case of a result of Bony [1] which has shown that all "reasonable" notions of $WF_a(u)$ satisfy certain coinciding functorial properties (and the same in the case of hyperfunctions).

Let $\Gamma \subset \dot{\mathbb{R}}^n$ be a connected open cone. Let $W \subset\subset \mathbb{C}^n$ be an open neighborhood of $x_0 \in \mathbb{R}^n$ and let u be a holomorphic function in $W \cap (\mathbb{R}^n \times i\Gamma)$. One says that u is of tempered increase if there exist constants $C > 0$ and $N > 0$ such that

$$|u(z)| \leq C|\mathrm{Im}\, z|^{-N}.$$

Then $\lim_{t\to+0} u(x + ity_0) \in \mathcal{D}'(W_\mathbb{R})$ exists for each $y_0 \in \Gamma$ and the limit is independent of y_0. For the proof of this fact, one can easily reduce to the case where Γ is of the form $|y'| < Cy_1$, $y = (y_1, y')$, and one shows next that in a neighborhood $V(x_1)$ of a point $x_1 \in W_\mathbb{R}$ one has

$$u(z) = \left(\frac{\partial}{\partial z_1}\right)^{N+1} v_{N+1}(z), \tag{6.10}$$

where v_{N+1} is continuous on $V(x_1) \cap (\mathbb{R}^n \times i\overline{\Gamma})$. We will omit the details. One writes

$$\lim_{t\to 0} u(x + ity_0) = b_\Gamma(u).$$

If $u \in \mathcal{D}'(X)$, then locally near a point $x_0 \in X$ one can write $u = \sum_1^N b_{\Gamma_j}(u_j)$ where the u_j are of tempered increase as above. (If $u \in \mathcal{E}'(\mathbb{R}^n)$ one can for example decompose $\hat{u} = \sum_1^N \hat{u}_j$ such that the $\hat{u}_j$ are supported in suitable cones.)

Theorem 6.5. *Let $u \in \mathcal{D}'(X)$, $(x_0, \xi_0) \in T^*X\backslash 0$. Then $(x_0, \xi_0) \notin WF_a(u)$ if and only if*

$$\begin{cases} \text{There exist connected open cones } \Gamma_1, \dots, \Gamma_N \subset \{y \in \mathbb{R}^n;\ y\xi_0 < 0\}, \\ \text{a complex neighborhood } W \text{ of } x_0, \text{ and holomorphic functions } u_j \\ \text{of tempered increase in } W \cap (\mathbb{R}^n \times i\Gamma_j),\ 1 \leq j \leq N, \\ \text{such that } u = \sum_1^N b_{\Gamma_j}(u_j) \text{ in } W_\mathbb{R}. \end{cases} \tag{6.11}$$

Proof. First suppose (6.11). Without loss of generality one can take $N = 1$ and after having replaced u_1 by an iterated primitive one can assume that u_1 is continuous.

Then by Stokes' formula, for (x, ξ) in a conic neighborhood of (x_0, ξ_0)

$$\int e^{i(x-y)\xi - |\xi|(x-y)^2/2} u(y)\, dy = \int_{\gamma_1} e^{i(x-y)\xi - |\xi|(x-y)^2/2} u_1(y)\, dy, \tag{6.12}$$

where to avoid cut-off functions one has also assumed that $u \in \mathcal{E}'(\mathbb{R}^n)$. Here γ_1 is a contour of the form $y \mapsto y + i\epsilon\chi(y)y_0$, where $\epsilon > 0$ is small, $y_0 \in \Gamma_1$, $|y_0| = 1$, $0 \leq \chi \in C_0^\infty(\mathbb{R}^n)$, $\chi(x_0) > 0$, and χ is supported in a small neighborhood of x_0. Along γ_1 one has

$$\mathrm{Im}\,(x-y)\xi - \frac{|\xi|}{2}(x-y)^2 = \frac{|\xi|}{2}((x - \mathrm{Re}\, y)^2 - \epsilon^2\chi(\mathrm{Re}\, y)^2) - \epsilon\chi(\mathrm{Re}\, y)(y_0\xi) > 0$$

when (x, ξ) is in a small conic neighborhood of (x_0, ξ_0) and $\epsilon > 0$ is sufficiently small. Then the integral (6.12) is exponentially decreasing, and $(x_0, \xi_0) \notin WF_a(u)$.

One supposes now that $(x_0, \xi_0) \notin WF_a(u)$. Then if W is a small complex neighborhood of x_0 and $V \subset \dot{\mathbb{R}}^n$ is a small conic neighborhood of ξ_0, we have by Proposition 6.2 that the integral

$$\int e^{i(x-y)\xi-|\xi|(x-y)^2/2}a(x-y,\xi)u(y)\,dy \tag{6.13}$$

is exponentially decreasing in $W \times V$. Here a is the same symbol as in (6.8).

For $\xi_1 \in \mathbb{R}^n\backslash V$ let Γ_{ξ_1} be a closed cone with nonempty interior such that

$$y\xi_1 > 0, \quad y\xi_0 < 0, \quad \forall y \in \Gamma_{\xi_1}.$$

Also let $\Gamma^*_{\xi_1}$ be a closed conic neighborhood of ξ_1 such that $y\xi > 0$ on $\Gamma_{\xi_1} \times \Gamma^*_{\xi_1}$. One can find $\xi_1, \ldots, \xi_N$ such that

$$\dot{\mathbb{R}}^n\backslash V = \bigcup_1^N (\Gamma^*_{\xi_j})^\circ.$$

Then as in the proof of Theorem 6.3 one obtains, modulo a function that is analytic near x_0,

$$\begin{aligned} u(x) &\equiv \sum_1^N \iint e^{i(x-y)\xi-|\xi|(x-y)^2/2}a(x-y,\xi)\chi_j\left(\frac{\xi}{|\xi|}\right)u(y)\,dy\,\frac{d\xi}{(2\pi)^n} \\ &= \sum_1^N u_j(x) \end{aligned} \tag{6.14}$$

where $\operatorname{supp}\chi_j \subset (\Gamma^*_{\xi_j})^\circ$ and $\sum_1^N \chi_j = 1$ on $\mathbb{R}^n\backslash V$. It is easy to see that $u_j(x)$ is the boundary value of

$$u_j(z) = \iint e^{i(z-y)\xi-|\xi|(z-y)^2/2}a(z-y,\xi)\chi_j\left(\frac{\xi}{|\xi|}\right)u(y)\,dy\,\frac{d\xi}{(2\pi)^n}$$

which is a holomorphic function of tempered increase in $\tilde{W} \cap (\mathbb{R}^n \times i\Gamma_{\xi_j})$, if $\tilde{W}$ is a small complex neighborhood of x_0. □

To conclude this section, we would have had to also establish the functorial properties of $WF_a(u)$: behavior of WF_a with respect to tensor products, images and inverse images, and the action of integral operators. This can be done along the same lines; see also Laubin [21].

7. The Fourier-Bros-Iagolnitzer (FBI) Transformation

These types of transformations have been introduced and studied by many mathematicians and physicists. To restrict oneself solely to works in partial differential equations, one can cite Kawai-Kashiwara [19], Boutet de Monvel [5], Lebeau [22], Trépreau [33], Hörmander [17], Córdoba-Fefferman [9], Unterberger [34], Boutet de Monvel-Sjöstrand [7], Melin-Sjöstrand [24], and Sjöstrand [32]. We present here yet another version.

Let $\varphi(x,y)$ be a holomorphic function, defined in a (complex) neighborhood of $(x_0,y_0)\in\mathbb{C}^n\times\mathbb{R}^n$, such that

$$\frac{\partial\varphi}{\partial y}(x_0,y_0)=:-\eta_0\in\dot{\mathbb{R}}^n, \tag{7.1}$$

$$\operatorname{Im}\frac{\partial^2\varphi}{\partial y^2}(x_0,y_0)>0, \tag{7.2}$$

$$\det\frac{\partial^2\varphi}{\partial x\partial y}(x_0,y_0)\neq 0. \tag{7.3}$$

We introduce $\varphi_1(x,y)=-\operatorname{Im}\varphi(x,y)$, which is then a pl.h. function with $\frac{2}{i}\partial\varphi_1=\partial\varphi$, and

$$\frac{2}{i}\frac{\partial\varphi_1}{\partial y}(x_0,y_0)=-\eta_0\in\dot{\mathbb{R}}^n, \tag{7.1$'$}$$

$$\nabla_y^2\varphi_1(x_0,y_0)|_{\mathbb{R}^n}<0, \tag{7.2$'$}$$

$$\det\nabla_x\nabla_y\varphi(x_0,y_0)\neq 0. \tag{7.3$'$}$$

Here ∇ denotes the gradient in the real direction.

For $x\in\mathbb{C}^n$ near x_0, the function $\varphi_1(x,\cdot)|_{\mathbb{R}^n_y}$ admits a strict maximum at a point $y=y(x)$ near y_0. This is the unique real point (near y_0) where $\frac{\partial\varphi}{\partial y}(x,y)$ is real, and we put

$$(y(x),\eta(x))=(y(x),-\frac{\partial\varphi}{\partial y}(x,y(x)))=(y(x),-\frac{2}{i}\frac{\partial\varphi_1}{\partial y}(x,y(x)))\in T^*\mathbb{R}^n\backslash 0.$$

Lemma 7.1. *The mapping $x\mapsto(y(x),\eta(x))$ is a diffeomorphism of a complex neighborhood of x_0 to a real neighborhood of (y_0,η_0).*

Proof. The mapping $(y,-\frac{\partial\varphi}{\partial y}(x,y))\mapsto x\in\mathbb{C}^n$ is well-defined on a complex neighborhood of (y_0,η_0). The restriction to $\mathbb{R}^{2n}$ then gives a left inverse for the mapping $x\mapsto(y(x),\eta(x))$. It suffices then to remark that $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ are of the same real dimension. $\square$

In what follows we work locally near x_0 or near y_0, but in order to not weigh down the formulas we present them as if they were global, and the reader will have to add in his head the suitable neighborhoods.

For $y\in\mathbb{R}^n$, let $\Gamma_y=\{x\in\mathbb{C}^n;\ y(x)=y\}$. Then the Γ_y form a fibration of a neighborhood of x_0. Each Γ_y is of real dimension n. Moreover, Γ_y is totally real; that is, $T_x(\Gamma_y)\cap iT_x(\Gamma_y)=\{0\}$, $x\in\Gamma_y$. In fact,

$$T_x(\Gamma_y)=\{t_x\in\mathbb{C}^n;\ \varphi''_{yx}t_x\in\mathbb{R}^n\}.$$

For x in a neighborhood of x_0 we put

$$\Phi(x)=\varphi_1(x,y(x))=\sup_{y\in\mathbb{R}^n}\varphi_1(x,y).$$

Φ is real-analytic and also pl.s.h. in that it is the supremum of a family of pluri(-sub)-harmonic functions. With $d(x, \Gamma_y)$ =distance from x to Γ_y, we even have

$$\Phi(x) \geq \varphi_1(x, y) + Cd(x, \Gamma_y)^2 , \quad C > 0,$$

which shows that Φ is in fact *strictly* plurisubharmonic.

We introduce the local complex (canonical) diffeomorphism

$$\kappa : (y, -\frac{2}{i}\frac{\partial \varphi_1}{\partial y}) \mapsto (x, \frac{2}{i}\frac{\partial \varphi_1}{\partial x})$$

of a complex neighborhood of (y_0, η_0) to a complex neighborhood of (x_0, ξ_0), where $\xi_0 = \frac{2}{i}\frac{\partial \varphi_1}{\partial x}(x_0, y_0) = \frac{2}{i}\frac{\partial \Phi}{\partial x}(x_0)$. We also let $\Lambda_\Phi = \{(x, \frac{2}{i}\frac{\partial \Phi}{\partial x}); x \in \mathbb{C}^n\}$. Then $\Lambda_\Phi = \kappa(T^*\mathbb{R}^n)$.

We let $a(x, y; h)$ be an elliptic classical analytic symbol of order 0, defined in a neighborhood of (x_0, y_0). Let $Y \subset \mathbb{R}^n$ be a small neighborhood of y_0, and let $\chi \in C_0^\infty(Y)$ be equal to 1 near y_0. Then, if $X \subset \mathbb{C}^n$ is a small neighborhood of x_0, we put

$$Tu(x; h) = \int e^{i\varphi(x,y)/h} a(x, y; h)\chi(y)u(y)\, dy \tag{7.4}$$

for $x \in X$ and $u \in \mathcal{D}'(Y)$. (We choose X small enough so that $\kappa^{-1}(\Pi^{-1}(X) \cap \Lambda_\Phi) \subset T^*\mathbb{R}^n\backslash 0$ projects onto an open set where $\chi = 1$.) It is clear that $T : \mathcal{D}'(Y) \to H_\Phi^{\mathrm{loc}}(X)$. If one writes $\alpha = (y(x), \eta(x))$ and $\psi(y, \alpha) = \varphi(x, y) + i\Phi(x)$, it becomes clear that T permits a characterization of $WF_a(u)$.

Proposition 7.2. *Let $u \in \mathcal{D}'(Y)$ and $x_1 \in X$. Then $(y(x_1), \eta(x_1)) \notin WF_a(u)$ if and only if $Tu(x; h) = 0$ in H_{Φ,x_1}.*

We now investigate the link between $\tilde{T}u$ and Tu, if $\tilde{T}$ is another FBI transform given by (7.4) with (φ, a) replaced by $(\tilde{\varphi}, \tilde{a})$, where $\tilde{\varphi}$ and $\tilde{a}$ are defined near $(\tilde{x}_0, y_0) \in \mathbb{C}^n \times \mathbb{R}^n$ and $\frac{\partial \tilde{\varphi}}{\partial y}(\tilde{x}_0, y_0) = -\eta_0$. The natural idea is to give a meaning to the relation $\tilde{T}u = (\tilde{T}T^{-1})Tu$. As in Section 4, we can formally invert T by an operator of the form

$$Sv(y; h) = h^{-n} \int e^{-i\varphi(x,y)/h} b(x, y; h)v(x; h)\, dx. \tag{7.5}$$

For $u \in \mathcal{D}'(Y)$ one cannot hope in general to have $u \equiv STu$ modulo an analytic function, since Tu only describes u microlocally near $(y_0, h^{-1}\xi_0)$. Nevertheless, we look for a contour in (7.5) that is as good as possible, that is to say a contour on which

$$-\varphi_1(x, y) + \Phi(x) = -\varphi_1(x, y) + \varphi_1(x, y(x)) \sim |y - y(x)|^2$$

is as small as possible. The natural choice is to take Γ_y on which $-\varphi_1(x, y) + \Phi(x) = 0$. The contour Γ_y is not even completely good when $v \in H_\Phi$, but, if $v \in H_\Psi$, where $\Psi \leq \Phi$, $\Psi(x_0) = \Phi(x_0)$, and $\nabla^2\Psi(x_0)|_{\Gamma_{y_0}} < \nabla^2\Phi(x_0)|_{\Gamma_{y_0}}$, then Γ_{y_0} is a good contour, and hence Sv is well-defined; and, following Section 4, we have $TSv \equiv v$ in a neighborhood of x_0, modulo equivalence in H_Ψ.

We show that, despite this difficulty, one can define $\tilde{T}S$ as an operator $H_\Phi \to H_{\tilde{\Phi}}$.

Lemma 7.3. *For x in a neighborhood of $\tilde{x}_0$, the function $(y,z)\mapsto\tilde{\varphi}_1(x,y)-\varphi_1(z,y)+\Phi(z)$ has a saddle near (y_0,x_0), given by $y=\tilde{y}(x)=y(z)$ and $\tilde{\eta}(x)=\eta(z)$. The critical value is $\tilde{\Phi}(x)$.*

Proof. The indicated point is clearly a critical point, and the critical value is clearly $\tilde{\Phi}(x)$, so the point is to show that the Hessian is non-degenerate and of signature 0. This can be done directly. One first shows without too much difficulty that the Hessian is non-degenerate. The signature is 0 because on the contour $y\in\mathbb{R}^n$, $z\in\Gamma_y$, one has $\tilde{\varphi}_1(x,y)-\varphi_1(z,y)+\Phi(z)\le\tilde{\Phi}(x)$, and a standard deformation of this contour gives a good contour. A still simpler method is to use the general results of Section 11 (Exercise). □

As in Section 4, we can now define the operator $\tilde{T}S : H_{\Phi}^{\mathrm{loc}}(X)\to H_{\tilde{\Phi}}^{\mathrm{loc}}(\tilde{X})$ by

$$(\tilde{T}S)u(x;h)=h^{-n}\iint_{V_x}e^{i(\tilde{\varphi}(x,y)-\varphi(z,y))/h}\tilde{a}(x,y;h)b(z,y;h)u(z;h)\,dz\,dy, \tag{7.6}$$

where V_x denotes a good contour, and X and $\tilde{X}$ are small neighborhoods of x_0 and $\tilde{x}_0$, respectively. In particular, $(TS)u\sim u$ in $H_{\Phi}^{\mathrm{loc}}(X)$.

Proposition 7.4. *For $u\in\mathcal{D}'(Y)$ one has $\tilde{T}u\sim(\tilde{T}S)Tu$ in $H_{\tilde{\Phi}}^{loc}(\tilde{X})$, if $\tilde{X}$ is a small neighborhood of $\tilde{x}_0$.*

Proof. If $\psi(y)$ is a pl.s.h. C^∞ function defined near y_0 and is such that $\psi(y_0)=0$, $\frac{2}{i}\frac{\partial\psi}{\partial y}(y_0)=\eta_0$, and $\psi(y)\le 0$ for y real, and if $u\in H_{\psi}^{\mathrm{loc}}(\tilde{Y})$ (where $\tilde{Y}\subset\mathbb{C}^n$ is an open set with $\tilde{Y}\cap\mathbb{R}^n=Y$), then $Tu\in H_{\psi^*}^{\mathrm{loc}}(X)$, where $\psi^*(x)$ is the critical value of $y\mapsto\varphi_1(x,y)+\psi(y)$. To estimate $\psi^*(x)$ we apply the following lemma of Melin-Sjöstrand [23]:

Lemma 7.5. *Let $f(z,w)$ be a real C^∞ function defined near $(0,0)$ in $\mathbb{C}_z^n\times\mathbb{C}_w^k$. One supposes that $f(x,w)\le 0$ for $x\in\mathbb{R}^n$, and that $f(0,0)=0$, $\frac{\partial f}{\partial z}(0,0)=0$, and that $z=0$ is a saddle for $f(z,0)$. If $z(w)$ denotes the critical point near 0 of $z\mapsto f(z,w)$, then for w in a neighborhood of 0 one has $f(z(w),w)\le -C|Im\,z(w)|^2$, where $C>0$.*

Proof. We choose the real C^∞ coordinates

$$\tilde{y}(z,w)=(\tilde{y}_1,\dots,\tilde{y}_n),\qquad\tilde{x}(z,w)=(\tilde{x}_1,\dots,\tilde{x}_n)$$

as in Morse's Lemma:

$$f(z,w)-f(z(w),w)=\tilde{y}^2-\tilde{x}^2.$$

As we have seen in Section 2, we find $\mathbb{R}_z^n$ as a graph $\tilde{y}=H(\tilde{x},w)$, where H is a real C^∞ function with $H(0,0)=0$. We note, on the one hand, that $|\mathrm{Im}\,z(w)|\sim|H(0,w)|$, and, on the other hand, that since $f\le 0$ for $\tilde{x}=0$ and $\tilde{y}=H(0,w)$, we have $f(z(w),w)\le -|H(0,w)|^2$. □

We now apply the lemma to $y\mapsto\varphi_1(x,y)+\psi(y)-\Phi(x)$. We then find that $\psi^*(x)\le\Phi(x)-C|\mathrm{Im}\,\tilde{y}(x)|^2$, where $\tilde{y}(x)$ is the complex critical point of $y\mapsto\varphi_1(x,y)+\psi(y)$. Also,

$\psi^*(x_0) = \Phi(x_0)$. If we restrict x to Γ_{y_0}, we know that $\mathbb{R}^n \ni y \mapsto \varphi_1(x,y) + \psi(y) - \Phi(x)$ attains its strict maximum at $y = y_0$, where the value is 0. Then, for $y \in \mathbb{R}^n$,

$$|\nabla_{\operatorname{Re} y, \operatorname{Im} y}(\varphi_1(x,y) + \psi(y) - \Phi(x))| \sim |x - x_0| + |y - y_0|.$$

Hence $|\operatorname{Im} \tilde{y}(x)| \sim |x - x_0|$, and, using Lemma 7.5,

$$\psi^*(x) \leq \Phi(x) - C|x - x_0|^2 \qquad \text{for } x \in \Gamma_{y_0}. \tag{7.7}$$

It is then clear that $T : H_\psi^{\mathrm{loc}} \to H_{\psi^*}^{\mathrm{loc}}$ admits as its inverse $S : H_{\psi^*}^{\mathrm{loc}} \to H_\psi^{\mathrm{loc}}$ (defined with the contour of integration Γ_y as in (7.5)). Also, for $u \in H_\psi^{\mathrm{loc}}$ we have $\tilde{T}u \sim (\tilde{T}S)Tu$ in $H_{\tilde{\psi}^*}^{\mathrm{loc}}$, if $\tilde{\psi}^*$ is defined in the same manner as ψ^*.

We let V be a small real neighborhood of (y_0, η_0) in $\mathbb{R}^{2n}$ and I^V a resolution of the identity as in Section 5. We then have, for $u \in \mathcal{D}'$: $\tilde{T}u \sim \tilde{T}I^V u$ in $H_{\tilde{\Phi}}^{\mathrm{loc}}$ and $Tu \sim TI^V u$ in H_Φ^{loc}. On the other hand, $\tilde{T}I^V u \sim (\tilde{T}S)TI^V u$ in $H_{\tilde{\Phi}}^{\mathrm{loc}}$, since $I^V u = \int_{\alpha \in V} u_\alpha(x;h)\, d\alpha$, where $u_\alpha \in H_{\psi_\alpha}$ and ψ_α is essentially a function as above. This completes the proof of Proposition 7.4. □

In our discussion, we have used the fact that $\tilde{T}$ has an elliptic symbol. We can then consider the particular case $\tilde{T} = h^m T \circ P$, where P is a differential operator of order m with analytic coefficients. Then, by Kuranishi's trick,

$$\tilde{T}S = h^m TPS = \tilde{P}$$

is a classical pseudodifferential operator of order 0. We see immediately that the principal symbols of P and of $\tilde{P}$ are related by the classical relation $\tilde{p} \circ \kappa = p$, where we recall that $\kappa = \kappa_T$ is the canonical transformation given by $\kappa : (y, -\frac{\partial \varphi}{\partial y}) \mapsto (x, \frac{\partial \varphi}{\partial x})$.

Example 7.6. We consider $\varphi(x,y) = \frac{i}{2}(x-y)^2$ and $\varphi_1(x,y) = -\operatorname{Re} \frac{1}{2}(x-y)^2$. For y real, $\varphi_1(x,y) = -\frac{1}{2}(\operatorname{Re} x - y)^2 + \frac{1}{2}(\operatorname{Im} x)^2$, hence $\Phi(x) = \frac{1}{2}(\operatorname{Im} x)^2$ and $\Lambda_\Phi : \xi = -\operatorname{Im} x$. If we take $a = 1$ in (7.4), then T and P commute when P is an operator with constant coefficients.

In the rest of this section, we will show how to transform a principal-type operator (not necessarily having a real principal symbol) into $\tilde{D}_{x_n}$ by a suitable FBI transform. This result, in another formalism, is due to Trépreau [33], who developed an idea of Kashiwara and Kawai.

Let $p(y, \eta)$ be an analytic function defined near a point $(y_0, \eta_0) \in T^*\mathbb{R}^n \backslash 0$ satisfying $p(y_0, \eta_0) = 0$ and $dp(y_0, \eta_0) \neq 0$.

Lemma 7.7. *There exists a holomorphic function defined near (x_0, y_0) (where $x_0 \in \mathbb{C}^n$ is a suitable point) which satisfies (7.1)–(7.3) and*

$$\frac{\partial \varphi}{\partial x^n} = p\left(y, -\frac{\partial \varphi}{\partial y}\right). \tag{7.8}$$

Proof. One first puts

$$\varphi(x',0,y)=\frac{i}{2}(x'-y')^2-\eta_0^n y^n+iC(y^n-y_0^n)^2 \tag{7.9}$$

where C must satisfy $\operatorname{Re} C>0$. (One uses the notation $x=(x^1,\dots,x^n)=(x',x^n)$.) Let $x_0\in\mathbb{C}^n$ be given by $\operatorname{Re} x_0'=y_0'$, $\operatorname{Im} x_0'=-\eta_0'$, $x_0^n=0$. The equation (7.8) then gives φ in a neighborhood of (x_0,y_0), and (7.2) is satisfied because $\operatorname{Re} C>0$. One also has (7.1), and to have (7.3) one can first suppose (after a real change of coordinates in y) that $\frac{\partial p}{\partial \eta^n}(y_0,\eta_0)\neq 0$ or $\left[\frac{\partial p}{\partial \eta}(y_0,\eta_0)=0 \text{ and } \frac{\partial p}{\partial y^n}(y_0,\eta_0)\neq 0\right]$. Then one can find C with $\operatorname{Re} C>0$ such that

$$\frac{\partial}{\partial y^n}p(y,-\frac{\partial\varphi}{\partial y})\neq 0 \quad \text{at } (x_0,y_0). \tag{7.10}$$

Now one has the following equivalences:

(7.3)
$\Longleftrightarrow$
The differential of $y\mapsto \frac{\partial\varphi}{\partial x}$ is bijective at (x_0,y_0)
$\Longleftrightarrow$
The differential of $y\mapsto \left(\frac{\partial\varphi}{\partial x'},p(y,-\frac{\partial\varphi}{\partial y})\right)$ is bijective at (x_0,y_0)
$\Longleftrightarrow$
(7.10)

The last equivalence results from the fact that $\frac{\partial^2\varphi}{\partial x'\partial y^n}=0$ and $\det\frac{\partial^2\varphi}{\partial x'\partial y'}\neq 0$ at (x_0,y_0). ☐

Now let P be a formal classical pseudodifferential operator of order 0 with principal symbol p. It is then well-known (Mizohata [25], Hörmander [13], Sato-Kawai-Kashiwara [28]) that one can find a formal analytic symbol of order 0 that is elliptic in a neighborhood of (x_0,y_0) such that formally

$$(\tilde{D}_{x^n}-{}^tP(y,\tilde{D}_y;h))(a(x,y;h)e^{i\varphi(x,y)/h})=0. \tag{7.11}$$

(We will give a proof of this fact in Section 9.) Here φ is the function given in Lemma 7.7. (a is determined by solving the classical transport equations in each degree of homogeneity.) We have then shown:

Theorem 7.8. *Let P be a formal classical analytic pseudodifferential operator of order 0 defined near (y_0,η_0) whose principal symbol p satisfies $dp(y_0,\eta_0)\neq 0$ and $p(y_0,\eta_0)=0$. One can then find an FBI transformation T such that formally $\tilde{D}_{x^n}T=TP$.*

Now let P be a differential operator whose principal symbol satisfies the hypotheses of Theorem 7.8. One identifies P with a pseudodifferential operator by putting $P(x,\tilde{D}_x;h)=h^mP(x,D_x)$ where m is the order of P. Let u be a distribution defined near y_0 and suppose that

$$(y_0,\eta_0)\notin WF_a(Pu). \tag{7.12}$$

We say that P is analytic hypoelliptic on germs if (7.12) implies that $(y_0,\eta_0)\notin WF_a(u)$.

Let $U = Tu \in H_{\Phi,x_0}$. Then by Theorem 7.8 the property (7.12) is equivalent to

$$\tilde{D}_{x^n} U = 0 \quad \text{in } H_{\Phi,x_0}. \tag{7.13}$$

More explicitly, under the hypothesis (7.12) there exists a neighborhood V of x_0 and a number $\epsilon_0 > 0$ such that

$$\tilde{D}_{x^n} U(x; h) = \mathcal{O}(e^{(\Phi(x)-\epsilon_0)/h}), \quad x \in V. \tag{7.14}$$

One can suppose that $V = V'_{x'} \times V^n_{x^n}$, where V^n is a disc centered at $x^n_0 = 0$ and V' is a neighborhood of x'_0. Possibly decreasing V with ϵ_0 fixed, one can suppose that $\Phi(x_0) - \frac{\epsilon_0}{4} \le \Phi(x) \le \Phi(x_0) + \frac{\epsilon_0}{4}$ on V. Then one obtains from (7.14) that

$$|U(x; h) - U(x', 0; h)| = \mathcal{O}(e^{(\Phi(x_0) - \frac{3}{4}\epsilon_0)/h}) \tag{7.15}$$

and that for every $\epsilon > 0$:

$$|U(x', 0; h)| \le C_\epsilon e^{(\Psi_V(x') + \epsilon)/h}, \quad x' \in V', \tag{7.16}$$

where $\Psi_V(x') = \inf_{x^n \in V^n} \Phi(x)$ (which is Lipschitz). If $\Psi_V(x'_0) < \Phi(x_0)$, one deduces that $U = 0$ in H_{Φ,x_0}, hence that $(y_0, \eta_0) \notin WF_a(u)$. More generally, let $\tilde{\Psi}_V(x') = \sup f(x')$, where $f(x')$ is plurisubharmonic and $f \le \Psi_V(x')$ on V'. Since $U(x', 0; h)$ is holomorphic, one knows that $\log |U(x', 0; h)|$ is plurisubharmonic. Hence by (7.16):

$$|U(x', 0; h)| \le C_\epsilon e^{(\tilde{\Psi}_V(x') + \epsilon)/h}, \quad x' \in V', \tag{7.17}$$

for each $\epsilon > 0$. When one decreases V then Ψ_V and $\tilde{\Psi}_V$ increase. Our discussion immediately gives:

Theorem 7.9. *Let P be a differential operator with analytic coefficients, whose principal symbol satisfies the hypotheses of Theorem 7.8. One supposes in addition that for each $V = V' \times V^n$ one has $\tilde{\Psi}_V < \Phi(x_0)$ in a neighborhood of x'_0. Then P is analytic hypoelliptic on germs at (y_0, η_0).*

In using the initial version of this method, Trépreau [33] has demonstrated that the subelliptic operators (in the sense of Egorov) are analytic hypoelliptic.

8. Holmgren's Uniqueness Theorem and Extensions

We first recall the classical theorem of Holmgren:

Theorem 8.1. *Let $\Omega \subset \mathbb{R}^n$ be an open neighborhood of 0 and let P be a differential operator of order m on Ω with analytic coefficients, of the form*

$$P = D^m_{x^n} + \sum_{j=1}^{m} A_j(x, D_{x'}) D^{m-j}_{x^n}.$$

Then if $u \in \mathcal{D}'(\Omega)$, $Pu = 0$, and $u|_{x^n<0} = 0$, one has $u = 0$ in a neighborhood of 0.

If $u \in \mathcal{D}'(\Omega)$, $\operatorname{supp} u \subset \{x^n \geq 0\}$, and $0 \in \operatorname{supp} u$, then u cannot be analytic in a neighborhood of 0 and hence there exists $\xi \in \dot{\mathbb{R}}^n$ such that $(0,\xi) \in WF_a(u)$. We have the following result which can be found as a remark of Kashiwara in Sato-Kawai-Kashiwara [28] and which has also been proven in a slightly weaker form by Hörmander [13].

Theorem 8.2. *Let $u \in \mathcal{D}'(\Omega)$ be such that $\operatorname{supp} u \subset \{x^n \geq 0\}$ and $0 \in \operatorname{supp} u$. Then $(0,(0,\dots,0,1))$ and $(0,(0,\dots,-1))$ belong to $WF_a(u)$.*

For $n = 1$ the proof of this theorem is simple enough if one uses the definition of WF_a in terms of boundary values. In higher dimensions one can reduce to the case $n = 1$ by using the standard convexification in the proof of Holmgren's Theorem. As Hörmander [13] and Sato-Kawai-Kashiwara [28] have noticed, Theorem 8.2 gives Theorem 8.1. In fact, for P and u as in Theorem 8.1 one has $p_m(0,(0,\dots,\pm 1)) \neq 0$ and hence $(0,(0,\dots,\pm 1)) \notin WF_a(u)$.

Theorem 8.2 follows easily from the following result due to Kashiwara.

Theorem 8.3. ("the Watermelon") *Let $u \in \mathcal{D}'(\Omega)$ be such that $\operatorname{supp} u \subset \{x^n \geq 0\}$. Then if $(0,\eta_0) \notin WF_a(u)$ for a $\eta_0 \neq 0$, one also has $(0,(\eta_0',t)) \notin WF_a(u)$ for all $t \in \mathbb{R}$.*

There is an even finer result due to Kashiwara, linked to the second microlocalization. This result and its generalizations will be discussed in Section 16. Under the hypotheses of Theorem 8.2 one knows that there exists a point $(0,\xi_0) \in WF_a(u)$. Then Theorem 8.3 shows that $(0,(\xi_0',t)) \in WF_a(u)$, and in taking $t \to \pm\infty$ one obtains $(0,(0,\dots,\pm 1)) \in WF_a(u)$. Hence Theorem 8.3 implies Theorem 8.2.

We have learned the essential idea of the following proof from A. Grigis and P. Schapira during the Colloque de Marathéa in September 1980. Also, Hörmander has recently given a similar proof.

Proof. One could work with an arbitrary FBI transformation, which would better show the invariant aspects, but in order to not be too complicated, one works with the transformation of Example 7.6. We can suppose that $u \in \mathcal{E}'(\mathbb{R}^n)$, supp $u \subset \{x^n \geq 0\}$, and we also put, for $x \in \mathbb{C}^n$,

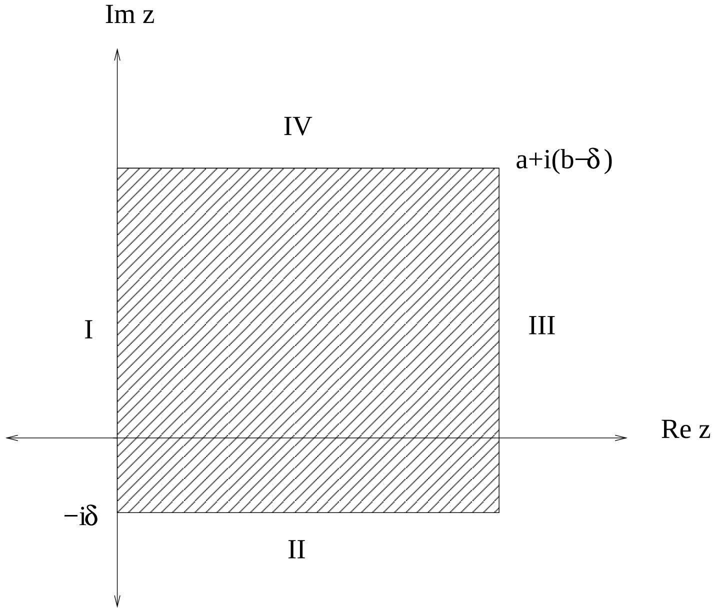
$$U(x;h) = \int e^{-(x-y)^2/(2h)} u(y)\, dy.$$

Then $U \in H_\Phi^{\mathrm{loc}}(\mathbb{C}^n)$ where

$$\Phi(x) = \begin{cases} \frac{1}{2}(\operatorname{Im} x)^2 & \text{if } \operatorname{Re} x^n \geq 0 \\ \frac{1}{2}\left[(\operatorname{Im} x)^2 - (\operatorname{Re} x^n)^2\right] & \text{if } \operatorname{Re} x^n \leq 0. \end{cases} \tag{8.1}$$

The crucial observation is now that $x^n \mapsto \Phi(x)$ is harmonic in the region $\operatorname{Re} x^n \leq 0$.

Lemma 8.4. *Let $\varphi(z) = 0$ in the upper half-plane and $= \frac{1}{2}(Im\, z)^2$ in the lower half-plane. If $a,b,c > 0$ then there exists a continuous function $\psi(z) \leq \varphi(z)$ in the rectangle*



$R: Re\, z \in [0,a]$, $|Im\, z| \le b$ with $\psi(x) < \varphi(x)$ for $x \in [0,a)$ such that $|u(z)| \le e^{\psi(z)/h}$ in R if $h > 0$ and u is a holomorphic function, continuous in R and satisfying $|u(z)| \le e^{\varphi(z)/h}$ in R and $|u(z)| \le e^{(\varphi(z)-c)/h}$ for $z \in i[-b,b]$.

Proof. (*of Lemma 8.4.*) For $0 < \delta < b$ one considers the closed rectangle Ω, whose boundary is $I \cup II \cup III \cup IV$ as in the figure. Then $h \log |u| - \frac{\delta^2}{2} \le 0$ on $II \cup III \cup IV$ and $h \log |u| - \frac{\delta^2}{2} \le -c \sin \frac{\pi}{b}(y+\delta)$ on I. By the maximum principle one then has $h \log |u| - \frac{\delta^2}{2} \le -cw(z)$ on Ω, where $w(z) = f(x) \sin(\frac{\pi}{b}(y+\delta))$ is the harmonic function given by

$$f(x) = (e^{\pi a/b} - e^{-\pi a/b})^{-1}(e^{-\pi(x-a)/b} - e^{\pi(x-a)/b}).$$

Here $f(x) > 0$ for $0 \le x < a$ and hence $-cw(x) + \frac{\delta^2}{2} < 0$ on each compact interval $\subset [0,a)$ if $\delta > 0$ is sufficiently small. This then gives the lemma. □

The lemma is still true if one replaces the real axis by the imaginary axis, R by a rectangle $\tilde{R}$ symmetric with respect to the imaginary axis, and φ by the function

$$\tilde{\varphi}(z) = \begin{cases} \frac{1}{2}(\operatorname{Re} z)^2 & \text{if } \operatorname{Re} z \ge 0 \\ 0 & \text{if } \operatorname{Re} z \le 0. \end{cases}$$

If $u \in \mathcal{E}'(\mathbb{R}^n)$, supp $u \subset \{x^n \ge 0\}$, $(0, \eta_0) \notin WF_a(u)$, then $U \in H^{\mathrm{loc}}_{\tilde{\Phi}}$ where $\tilde{\Phi} \le \Phi$ (where Φ is given by (8.1)) with strict inequality in a neighborhood of $-i\eta_0$. It suffices then to apply the modified form of Lemma 8.4 with functions $x^n \mapsto \exp\{\frac{1}{2h}(x^n)^2 - \frac{1}{2h}(\operatorname{Im} x')^2\}U(x;h)$ for x' near $-i\eta_0'$, and one deduces that $U \in H^{\mathrm{loc}}_{\tilde{\Phi}}$ where $\tilde{\Phi} \le \Phi$ with strict inequality near each point $i(-\eta_0', t)$, $t \in \mathbb{R}$. This gives the theorem. □

One can note as a curiosity that if η_0 is of the form $(0,\dots,0,\pm 1)$ then $U=0$ in $H_{\Phi,0}$. Then for x real and in a neighborhood of 0 one has

$$u(x)=\lim_{h\to 0}(2\pi h)^{-n/2}\int e^{-(x-y)^2/(2h)}u(y)\,dy=0.$$

This gives a more direct proof of Theorem 8.2 where one does not use the fact that $\partial\text{supp }(u)\subset\text{singsupp}_a(u)$.

We go now to a discussion of extensions of Holmgren's Theorem in the case of a hypersurface which may be characteristic. They are the purely geometric extensions of the classical Holmgren's Theorem, and the "WF_a" does not intervene. The most remarkable result in this direction is due to Bony [2]. We present here an improvement of this result which contains also the results of Hörmander [16]. We will not discuss here the much more difficult case of C^∞ coefficients, nor certain results of Baouendi-Zachmanoglou concerning the uniqueness from a submanifold of codimension ≥ 2.

Let $X\subset\mathbb{R}^n$ be an open set, and let $K\subset X$ be a closed subset. Following Bony, one defines the conormal fiber of K:

Definition 8.5. *Let N^*K be the set of $(x_0,\xi_0)\in T^*X\backslash 0$ with $x_0\in K$ and such that there exists $h\in C^\infty(X;\mathbb{R})$ with $\xi_0=\pm h'(x_0)$ and with $h|_K$ admitting a local maximum at x_0.*

One can describe N^*K as the points (x_0,ξ_0) where ξ_0 is normal to a hypersurface H which touches K at x_0 and is such that K is "locally on one side of H." If K is a smooth submanifold, then N^*K is the usual conormal fiber. The reader is encouraged to calculate N^*K when $K\subset\mathbb{R}^n$ is a closed quadrant or the complement of an open quadrant. It will show that N^*K is not necessarily closed. Theorem 8.2 takes the following more invariant form:

Theorem 8.6. *If $u\in\mathcal{D}'(X)$, then $N^*supp\ u\subset WF_a(u)$.*

We will now establish an entirely geometric result on N^*K which will involve the different extensions of Holmgren's Theorem. If $f(x,\xi)$ is a real C^∞ function, defined near $(x_0,\xi_0)\in T^*\mathbb{R}^n$, one defines its Hamiltonian field $H_f=\sum\frac{\partial f}{\partial\xi^j}\frac{\partial}{\partial x^j}-\frac{\partial f}{\partial x^j}\frac{\partial}{\partial\xi^j}$, and one remarks that f is constant along the integral curves of H_f, since $H_f(f)=0$. Let $h(x)$ be a real-valued function defined near x_0 with $h'(x_0)=\xi_0$. Then the Hamilton-Jacobi theory tells us that the problem

$$\begin{cases}\frac{\partial\varphi}{\partial t}+f(x,\varphi'_x)=0\\ \varphi(0,x)=h(x)\end{cases}\tag{8.2}$$

admits a unique C^∞ solution in a suitable neighborhood of $(0,x_0)\in\mathbb{R}\times\mathbb{R}^n$. If one introduces

$$\Lambda_{\varphi_t}=\{(x,\varphi'_x(t,x))\},\quad \Lambda_h=\{(x,h'_x(x))\}$$

(locally near (x_0, ξ_0)), then $\varphi(t, \cdot)$ is determined up to a constant C_t, by the (well-known) fact that

$$\Lambda_{\varphi_t} = \exp(tH_f)(\Lambda_h). \tag{8.3}$$

The constant C_t is determined by

$$\varphi(t, x(t)) - \varphi(0, x_0) = \int_0^t \left[f'_\xi(x(s), \xi(s)) \cdot \xi(s) - f(x(s), \xi(s)) \right] \, ds \ , \tag{8.4}$$

where $(x(s), \xi(s)) = \exp(sH_f)(x_0, \xi_0)$.

Theorem 8.7. *Let $K \subset X$ be a closed set, let $(x_0, \xi_0) \in N^*K$, and let $f(x, \xi)$ be a real C^∞ function, defined near (x_0, ξ_0) and such that $f|_{N^*K} = 0$. Then for $t_0 > 0$ sufficiently small, $\exp tH_f(x_0, \xi_0) \in N^*K$ when $|t| \le t_0$.*

Proof. Let $h(x)$ be a real C^∞ function such that $\xi_0 = h'(x_0)$ and such that $h|_K$ admits an isolated local maximum (or minimum) at x_0. Let φ be the solution of (8.2). Then, if W is a small neighborhood of x_0 and $|t|$ is sufficiently small, we know that $\varphi(t, \cdot)|_{K \cap W}$ attains its maximum in the interior of W on a set $A_t \subset \partial K$ which is uniformly compact in W. Let

$$\Phi(t) = \max_{K \cap W} \varphi(t, \cdot).$$

If one writes $\Phi(t) = \varphi(t, x_t)$, $\Phi(s) = \varphi(s, x_s)$, $x_t \in A_t$, $x_s \in A_s$, one obtains

$$|\Phi(t) - \Phi(s)| \le \begin{cases} \varphi(t, x_t) - \varphi(s, x_t) & \text{if } \Phi(t) \ge \Phi(s) \\ \varphi(s, x_s) - \varphi(t, x_s) & \text{if } \Phi(t) \le \Phi(s) \end{cases} \ ;$$

that is,

$$|\Phi(t) - \Phi(s)| \le \max\left(|\varphi(t, x_t) - \varphi(s, x_t)|, \, |\varphi(t, x_s) - \varphi(s, x_s)| \right).$$

However,

$$\frac{\partial \varphi}{\partial t}(t, x_t) = -f(x_t, \varphi'_x(t, x_t)) = 0$$
$$\frac{\partial \varphi}{\partial s}(s, x_s) = -f(x_s, \varphi'_x(s, x_s)) = 0,$$

since $f|_{N^*K} = 0$. Hence $|\Phi(t) - \Phi(s)| \le C(t-s)^2$; that is, $\Phi'(t) = 0$, and so Φ is independent of t.

Now one puts $g(x) = h(x) - (x - x_0)^2$, so that g has the same properties as h, and one denotes by $\psi(t, x)$ the solution of (8.2) with h replaced by g. Then, for $|t|$ sufficiently small,

$$\max_{K \cap W} \varphi(t, \cdot) = \max_{K \cap W} \psi(t, \cdot) = h(x_0) = g(x_0). \tag{8.5}$$

On the other hand, with $(x(t), \xi(t)) = \exp(tH_f)(x_0, \xi_0)$, one obtains from (8.3), (8.4):

$$\begin{cases} \varphi'_x(t, x(t)) = \psi'_x(t, x(t)) = \xi(t) \\ \varphi(t, x(t)) = \psi(t, x(t)). \end{cases} \tag{8.6}$$

The second derivatives do not change much if $|t|$ is small, and hence

$$\psi(t, x) \le \varphi(t, x) - \frac{1}{2}(x - x(t))^2, \quad x \in W. \tag{8.7}$$

Let $\tilde{x}_t \in K$ be a point where $\psi(t,\cdot)|_{K\cap W}$ attains its maximum. Then (8.5) and (8.7) show that $\tilde{x}_t = x(t)$, since otherwise one would have $\Phi(t) > \Psi(t)$. Hence by (8.6) one has $(x(t),\xi(t)) \in N^*K$. □

Theorem 8.7 generalizes a result of Bony [2]:

Corollary 8.8. *Let $K \subset X$ be a closed subset, and let $(x_0,\xi_0) \in N^*K$. Let $f(x,\xi)$, $g(x,\xi)$ be two C^∞ functions defined near (x_0,ξ_0), such that $f|_{N^*K} = g|_{N^*K} = 0$. Then the Poisson bracket, defined by*

$$\{f,g\} := \sum \frac{\partial f}{\partial \xi^j}\frac{\partial g}{\partial x^j} - \frac{\partial f}{\partial x^j}\frac{\partial g}{\partial \xi^j},$$

vanishes on N^*K.

Proof. One may suppose that f is real. Then $\{f,g\} = H_f(g) = 0$ since the integral curve of H_f issuing from a point of N^*K remains in N^*K for sufficiently small times. □

We return now to Holmgren's Theorem. Let $h(x)$ be a real C^∞ function defined in an open set $X \subset \mathbb{R}^n$ with $dh \neq 0$ and with $h(x_0) = 0$ for some $x_0 \in X$. One puts

$$X_\pm = \{x \in X;\, \pm h(x) > 0\}, \quad \xi_0 = h'(x_0).$$

Let $P(x,D_x)$ be a differential operator on X whose principal symbol $p(x,\xi)$ has analytic coefficients. Let $I(p,x_0,\xi_0)$ be the smallest set of germs of C^∞ functions at (x_0,ξ_0) that contains each function which vanishes on $\{(x,\xi);\, p(x,\xi) = 0\}$ and which is closed under taking Poisson brackets: $f,g \in I \Rightarrow \{f,g\} \in I$. Then I is an ideal, since

$$a\{f,g\} = \{af,g\} - \{f,ag\}, \quad a \in C^\infty, \quad f,g \in I.$$

In the following theorem, part (A) is due to Bony [2], and part (B) generalizes a result of Hörmander [16].

Theorem 8.9. *If there exists $u \in \mathcal{D}'(X)$ with $Pu = 0$, $x_0 \in$ supp $u \subset \overline{X}_+$, then*

(A) $f(x_0,\xi_0) = 0$ for each $f \in I(p,x_0,\xi_0)$.

(B) If $f \in I(p,x_0,\xi_0)$ is real-valued, then for $|t|$ sufficiently small one has $\exp tH_f(x_0,\xi_0) \in N^*(\mathrm{supp}\, u)$.

Proof. Let $K = \mathrm{supp}\, u$. Then $N^*K \subset p^{-1}(0)$ by Theorem 8.6 (or directly from the classical Holmgren's Theorem). Hence if f and g are germs of C^∞ functions which vanish on $p^{-1}(0)$, then they also vanish on N^*K, and hence $\{f,g\}|_{N^*K} = 0$ by Corollary 8.8. By iteration, one finds that $f|_{N^*K} = 0$ for each $f \in I(p,x_0,\xi_0)$, which gives (A), and (B) follows from Theorem 8.7. □

To illustrate the result, the reader may reflect on the trivial case where P is given by a vector field. There are also less obvious applications: Let $iP = Q_1 + iQ_2$ or $-P = \sum_1^d Q_j^2$, where $Q_1,\dots,Q_d$ are real analytic vector fields.

We first recall a theorem of Nagano [26]:

Theorem 8.10. *Let $Q_1, \dots, Q_d$ be real analytic vector fields defined near $x_0 \in \mathbb{R}^n$, and let $\mathcal{L}$ be the Lie algebra generated by $Q_1, \dots, Q_d$ and their commutators. Let $k = \dim \mathcal{L}(x_0)$, where $\mathcal{L}(x_0) := \{\nu(x_0); \nu \in \mathcal{L}\}$. Then there exists a real analytic manifold $\Gamma \subset \mathbb{R}^n$ of dimension k such that $x_0 \in \Gamma$ and $\mathcal{L}(x) = T_x(\Gamma)$ for each $x \in \Gamma$. (Moreover, Γ will be unique in "suitable" neighborhoods of x_0 if one requires additionally that Γ is connected and closed.)*

From Theorems 8.9 and 8.10 one deduces the following result of Zachmanoglou [35].

Theorem 8.11. *Let $iP = Q_1 + iQ_2$ or $-P = \sum_1^d Q_j^2$, where the Q_j are real analytic vector fields on X. If there exists $u \in \mathcal{D}'(X)$ such that $Pu = 0$, $x_0 \in supp\ u \subset \overline{X}_+$, then*

(A) Each $Q \in \mathcal{L}(Q_1, \dots, Q_d)$ is tangent to $\{h = 0\}$ at x_0.

(B) If $\Gamma = \Gamma(Q_1, \dots, Q_d)$ is the connected and closed foliation of Nagano in a small neighborhood of x_0, then one has $\Gamma(Q_1, \dots, Q_d) \subset \partial supp\ u \subset \overline{X}_+$.

Proof. (*of Nagano's Theorem* (Derridj [10])). One proves the theorem in the more general case where one has a countable set of generators defined in a complex neighborhood Ω of $x_0 = 0$. The cases $k = 0$, $k = n$, and $n = 1$ are trivial. One supposes by induction that the theorem is true in dimension $n - 1$. Let $0 < k < n$ be the dimension of $\mathcal{L}(Q_1, Q_2, \dots)(0)$. One may assume that $Q_1(0) \neq 0$, and, after a change of coordinates, that $Q_1 = \frac{\partial}{\partial x^1}$. For $j \geq 2$ one can replace Q_j by $Q_j - f_j(x)\frac{\partial}{\partial x_1}$, with $f_j(x)$ holomorphic in Ω, without changing $\mathcal{L}(x)$, $x \in \Omega$. One chooses $f_j(x)$ as the coefficient of $\frac{\partial}{\partial x_1}$ in the expression for Q_j, and one is then reduced to the case where $Q_j = Q_j(x, \frac{\partial}{\partial x'})$ for $j \geq 2$ and $Q_1 = \frac{\partial}{\partial x^1}$, $x = (x^1, x')$. We have $Q_j = \sum_{\nu=0}^{\infty} x_1^{\nu} Q_{j,\nu}(x', \frac{\partial}{\partial x'})$, $j \geq 2$, and one remarks that

$$(\text{Ad } Q_1)^{\mu} Q_j = \mu! Q_{j,\mu} + x^1 R_{j,\mu}(x, \frac{\partial}{\partial x'}),$$

which shows that $\mathcal{L}'(x') \subset \mathcal{L}(0, x')$, if $\mathcal{L}'$ denotes the Lie algebra generated by the $Q_{j,\nu}$. In particular, $\dim \mathcal{L}'(0) = k - 1$, and if Γ' is an integral foliation of $\mathcal{L}'$ (which exists by the induction hypothesis), then one can take $\Gamma = (-\epsilon, \epsilon) \times \Gamma'$. ☐

9. A Study of Operators of Principal Type Using the Method of Geometric Optics

In this section we will prove two results on the propagation of analytic singularities for operators of principal type. The first is due to N. Hanges [11], and both are particular cases of a conjecture of Hanges which has been proven by Hanges and the author in [12], and both are even particular cases of the results of Section 15 (obtained in collaboration with A. Grigis and P. Schapira). We nonetheless treat these two results themselves to show how the method of geometric optics is simply integrated with the point of view of Bros-Iagolnitzer. This also gives us an additional pretext to treat the transport equations in a direct manner.

Let $X \subset \mathbb{R}^n$ be an open set, $(x_0, \xi_0) \in T^*X\backslash 0$, and P a differential operator with analytic coefficients with principal symbol p. We suppose that $p(x_0, \xi_0) = 0$.

Theorem 9.1. (Hanges [11].) *We suppose that H_p admits a real integral curve $\gamma : [-a, a] \to T^*X\backslash 0$ with $\gamma(0) = (x_0, \xi_0)$. If $u \in \mathcal{D}'(X)$ and $WF_a(Pu) \cap \gamma([-a, a]) = \emptyset$, then either $\gamma([-a, a]) \subset WF_a(u)$ or $\gamma([-a, a]) \cap WF_a(u) = \emptyset$.*

When p is real this is a classical result of Sato-Kawai-Kashiwara [28] and Hörmander [13]. When p is not real the result is in general false for singularities mod C^∞.

Proof. For $\alpha = (\alpha_x, \alpha_\xi)$, let

$$v(x, \alpha; h) = e^{i\psi(x,\alpha)/h}, \quad \psi(x, \alpha) = (x - \alpha_x)\alpha_\xi + \frac{i}{2}(x - \alpha_x)^2.$$

Let $Q(x, \tilde{D}_x; h) = h^m P^*(x, D_x)$, where m is the order of P. For α near (x_0, ξ_0), x near x_0, $t \in [-a, a]$, and $a > 0$ sufficiently small, we will approximately construct $w(t, x, \alpha; h) = \exp(-itQ/h)(v)$. (By a covering argument it suffices to prove the theorem for $a > 0$ sufficiently small.) More explicitly, one wants to approximately solve

$$\begin{cases} (\tilde{D}_t + Q)w = 0. \\ w(0, x, \alpha; h) = v(x, \alpha; h). \end{cases} \tag{9.1}$$

One proceeds by the method of geometric optics. First let $\varphi(t, x, \alpha)$ be the (local) solution to the eikonal equation:

$$\frac{\partial \varphi}{\partial t} + q(x, \varphi'_x) = 0, \quad \varphi(0, x, \alpha) = \psi(x, \alpha). \tag{9.2}$$

Since γ is also a bicharacteristic for $q = \overline{p}$ one easily verifies that with $(x_t, \xi_t) = \gamma(t)$ and for $\alpha = (x_0, \xi_0)$:

$$\varphi(t, x_t, \alpha) = 0, \quad \varphi'_x(t, x_t, \alpha) = \xi_t, \quad \operatorname{Im} \varphi''_{xx}(t, x_t, \alpha) > 0. \tag{9.3}$$

When α is near (x_0, ξ_0) then $\varphi(t, \cdot, \alpha)$ is a small perturbation of $\varphi(t, \cdot, \alpha_0)$, $\alpha_0 = (x_0, \xi_0)$, and hence $\inf \operatorname{Im} \varphi(t, \cdot, \alpha)$ is near 0 and is equal to 0 at a point $\tilde{x}_t(\alpha)$ near x_t. Moreover, $\tilde{\xi}_t(\alpha) = \frac{\partial \varphi}{\partial x}(t, \tilde{x}_t, \alpha)$ is near ξ_t.

We now look for w of the form

$$e^{i\varphi(t,x,\alpha)/h} a(t, x, \alpha; h),$$

where $a = \sum_0^\infty a_k(t, x, \alpha)h^k$ is a formal classical analytic symbol. Substitution in (9.1) then gives the following transport equations:

$$\begin{aligned} &La_0 = 0 &&, \quad a_0|_{t=0} = 1, \\ &La_1 + f_1(a_0) = 0 &&, \quad a_1|_{t=0} = 0, \\ &\vdots \\ &La_k + f_k(a_0, \dots, a_{k-1}) = 0 &&, \quad a_k|_{t=0} = 0, \\ &\vdots \end{aligned} \tag{9.4}$$

where

$$L = \frac{\partial}{\partial t} + \sum_1^n q^{(j)}(x, \varphi'_x)\frac{\partial}{\partial x_j} + s(x, \alpha)$$

with $s(x, \alpha)$ analytic and where $f_k(a_0, \dots, a_{k-1})$ is a linear expression with analytic coefficients of the derivatives of $a_0, \dots, a_{k-1}$. It is obvious that one can successively solve the equations in a complex domain independent of k. It is less clear that a becomes an analytic symbol, but this (well-known) fact will be established later on.

We also denote a representative by $a(t, x, \alpha; h)$. Then $(\tilde{D}_t + Q)w$ is of uniform exponential decrease and $w(0, x, \alpha; h) = v(x, \alpha; h)$ if $w = e^{i\varphi/h}a$. Without loss of generality one can suppose that u has support in a small neighborhood of x_0. Then for α in a small neighborhood of α_0 and $t \in [-a, a]$ we have, modulo terms of uniform exponential decrease:

$$(9.5) \qquad \begin{aligned} -\frac{\partial}{\partial t}(w(t, \cdot, \alpha; h), u)_{L^2} &\equiv h^{m-1}(P^* w, u)_{L^2} \\ &\equiv h^{m-1}(w(t, \cdot, \alpha; h), Pu)_{L^2} \\ &\equiv 0, \end{aligned}$$

where the last equivalence follows from the fact that $\gamma([-a, a]) \cap WF_a(Pu) = \emptyset$. If $\gamma(t_0) \notin WF_a(u)$ for $t_0 \in [-a, a]$ then $(w(t_0, \cdot, \alpha; h), u)_{L^2} \equiv 0$ for α in a neighborhood of α_0, and integrating (9.5) from 0 to t_0 one obtains $(v(\cdot, \alpha; h), u)_{L^2} \equiv 0$ and hence that $\gamma(0) \notin WF_a(u)$. Slightly modifying our argument, one obtains also $\gamma(t_1) \notin WF_a(u)$ for each $t_1 \in [-a, a]$. □

Theorem 9.2. *Let $\Gamma \subset p^{-1}(0)$ be a connected real-analytic submanifold of dimension 2 such that the tangent space at each point is generated by H_{Rep} and H_{Imp}. If $u \in \mathcal{D}'(X)$ and $\Gamma \cap WF_a(Pu) = \emptyset$, then either $\Gamma \subset WF_a(u)$ or $\Gamma \cap WF_a(u) = \emptyset$.*

Proof. One proceeds as in the proof of Theorem 9.1 with the difference that the real variable t is now replaced by a complex variable z and that "$\frac{\partial}{\partial t}$" is replaced by "$\frac{\partial}{\partial \overline{z}}$". Let q be the holomorphic extension of $\overline{p}$ and consider the complexification $\tilde{\Gamma}$ of Γ in a neighborhood of the point $\alpha_0 = (x_0, \xi_0) \in \Gamma$. Then H_p and H_q (viewed as real vector fields in the complex domain) are tangent to $\tilde{\Gamma}$, and if $\rho \in \tilde{\Gamma}$ is a point near α_0 then $\{\exp zH_q(\rho);\, z \in \mathbb{C} \cap \text{neighborhood of } 0\}$ cuts Γ transversally at a point near α_0.

Hence for $z \in \mathbb{C} \cap$ neighborhood of 0 we have

$$(9.6) \qquad \gamma(z) := \exp(f(z, \overline{z})H_q) \circ \exp(zH_p)(\alpha_0) \in \Gamma$$

where $f(z, \overline{z})$ is a real-analytic function. Moreover one easily sees that γ is a local diffeomorphism which allows a local parametrization of Γ by $z \in \mathbb{C}$.

Let $v(x, \alpha; h)$ be as before. By a geometric optics construction in two stages one can then construct as in the proof of Theorem 9.1:

$$(9.7) \qquad \begin{aligned} w(z, x, \alpha; h) &= \exp(h^{-1} f(z, \overline{z})P^*) \circ \exp(h^{-1} zP)v \\ &= e^{i\varphi(z, x, \alpha)/h} a(z, x, \alpha; h) \end{aligned}$$

for (z, x, α) in a neighborhood of $(0, x_0, \alpha_0)$. If one denotes $\gamma(z) = (x_z, \xi_z)$ then for $\alpha = (x_0, \xi_0)$ one always has (9.3) with "t" replaced by "z".

Now let $r_0 > 0$ be sufficiently small but independent of $u \in \mathcal{D}'(X)$, and suppose that $WF_a(Pu) \cap \{\gamma(z) \in \Gamma;\ |z| \le r_0\} = \emptyset$. Without loss of generality one can also suppose that the support of u is in a small neighborhood of x_0. Let then

$$U(z; h) = (w(z, \cdot, \alpha; h), u)_{L^2}.$$

Then for $|z| \le r_0$ and α in a small neighborhood of α_0 one obtains, modulo terms of uniform exponential decrease:

$$\begin{aligned}\frac{\partial}{\partial \overline{z}} U &\equiv h^{-1} \frac{\partial f}{\partial \overline{z}} (P^* \exp h^{-1} f(z, \overline{z}) P^* \circ \exp h^{-1} z P v, u)_{L^2} \\ &\equiv h^{-1} \frac{\partial f}{\partial \overline{z}} (w, Pu)_{L^2} \equiv 0.\end{aligned}$$

After a modification by a term $\equiv 0$ one can suppose that U is holomorphic.

Now suppose that $\gamma(z_0) \notin WF_a(u)$ for some z with $|z| \le r_0$. Then for (z, α) in a neighborhood of (z_0, α_0) we have $U \equiv 0$. On the other hand, for each $\epsilon > 0$ we have $U(z, \alpha; h) = \mathcal{O}(e^{\epsilon/h})$ for $|z| \le r_0$ if α is sufficiently close to α_0. Applying a Möbius transformation and Hadamard's Three-Circle Theorem one finds that $U(0, \alpha; h) \equiv 0$ for α in a neighborhood of α_0 and hence that $\alpha_0 \notin WF_a(u)$. The theorem then follows from a simple geometric argument. ☐

To conclude this section, we treat the transport equations. Let $P(x, \tilde{D}_x; h)$ be a formal classical analytic pseudodifferential operator of order 0 defined near $(x_0, \xi_0) \in \mathbb{C}^{2n}$ such that $p(x_0, \xi_0) = 0$, $\frac{\partial p}{\partial \xi}(x_0, \xi_0) \ne 0$ for the principal symbol p. Let $\varphi(x)$ be a holomorphic function defined near x_0 which satisfies

$$p(x, \varphi'(x)) = 0, \quad \varphi'(x_0) = \xi_0. \tag{9.8}$$

Let $H \subset \mathbb{C}^n$ be a complex hypersurface passing through x_0 such that $\frac{\partial p}{\partial \xi}(x_0, \xi_0) \frac{\partial}{\partial x}$ is transversal to H at x_0. After a change of variables one reduces to the case $x_0 = 0$, $H : x_n = 0$.

Theorem 9.3. *Let $v(x; h)$, $w(x'; h)$ be formal classical analytic symbols of order 0 defined near 0 in $\mathbb{C}^n$ and $\mathbb{C}^{n-1}$ respectively. Then there exists a symbol u in the same class such that*

$$h^{-1} e^{-i\varphi/h} P e^{i\varphi/h} u = v, \quad u|_H = w. \tag{9.9}$$

Proof. By Kuranishi's trick etc. we already know that $e^{-i\varphi/h} P e^{i\varphi/h}$ is a pseudodifferential operator, and the problem is reduced to the case $\xi_0 = 0$, $\varphi = 0$. After a change of variables which does not modify near H one can also suppose that $\frac{\partial p}{\partial \xi'}(x, 0) = 0$ and $\frac{\partial p}{\partial \xi^n}(x, 0) = i$; that is, $p(x, \xi) = i\xi_n + \mathcal{O}(|\xi|^2)$.

If one writes $P = \sum_0^\infty p_k(x, \xi) h^k$, $p_0 = p$, then the first part of (9.9) becomes

$$\frac{\partial u}{\partial x_n} + p_1(x, 0) u(x; h) + h^{-1} A u = v \tag{9.10}$$

where

$$A = \sum_{k+|\alpha|\geq 2} \frac{h^k}{\alpha!} p_k^{(\alpha)}(x,0)\tilde{D}_x^\alpha, \tag{9.11}$$

and, as we have already remarked, if one regroups the terms with the same homogeneity in (9.10) then one finds a sequence of transport equations which uniquely determine u in a suitable neighborhood of 0. One remarks also that $A = \sum_2^\infty h^k A_k$ is a differential operator of order infinity of the same type as in Section 1.

Let

$$\Omega = \left\{ x \in \mathbb{C}^n;\ \frac{|x'|}{R} + \frac{|x_n|}{r} \leq 1 \right\}$$

where $R, r > 0$ are sufficiently small so that $\overline{\Omega}$ is contained in the domain of definition of v. One supposes without loss of generality that $w = 0$. After conjugation with $\exp \int^{x_n} p_1\, dx_n$ in (9.10) one can also suppose that $p_1(x,0) = 0$. For $0 \leq t \leq r$ one defines $\Omega_t \subset \mathbb{C}^n$ by $\frac{|x'|}{R-\frac{Rt}{r}} + \frac{|x_n|}{r-t} \leq 1$. Let $a(x)$ be holomorphic on $\Omega = \Omega_0$ and such that for some $k > 1$:

$$\sup_{\Omega_t} |a| \leq C(a,k)t^{-k}, \quad 0 < t \leq r.$$

Let

$$(\partial_{x_n})^{-1}a = \int_0^{x_n} a(x', y_n)\, dy_n.$$

Then

$$\sup_{\Omega_t} |(\partial_{x_n})^{-1}a| \leq C(a,k)\int_t^\infty s^{-k}\, ds = \frac{C(a,k)}{(k-1)t^{k-1}}.$$

Now let $a = \sum_2^\infty a_k(x)h^k$ be a symbol of order -2 on Ω such that

$$||a_k||_{\Omega_t} = \sup_{\Omega_t} |a_k| \leq \frac{f(a,k)k^k}{t^k}, \quad 0 < t \leq r, \tag{9.12}$$

where the sequence of (best) constants $f(a,k)$ is at most of exponential increase. Then $b = (h\partial_{x_n})^{-1}a = \sum_1^\infty b_k h^k$ where $b_k = (\partial_{x_n})^{-1}a_{k+1}$, and hence

$$||b_k||_{\Omega_t} \leq \frac{f(a,k+1)}{k\, t^k}(k+1)^{k+1} \leq 2e\, \frac{f(a,k+1)k^k}{t^k}. \tag{9.13}$$

Hence $f(b,k) \leq 2e\, f(a,k+1)$ if one defines $f(b,k)$ as in (9.12). Let

$$||a||_\mu = \sum_2^\infty f(a,k)\mu^k, \quad ||b||_\mu = \sum_1^\infty f(b,k)\mu^k.$$

Then

$$||b||_\mu \leq 2e \sum_1^\infty f(a,k+1)\mu^k = \frac{2e}{\mu}||v||_\mu. \tag{9.14}$$

The equation (9.10) with initial data 0 can be written

$$u + (h\partial_{x_n})^{-1}Au = \tilde{v}, \quad \text{where } \tilde{v} = h(h\partial_{x_n})^{-1}v. \tag{9.15}$$

Here $\tilde{v}$ is of order 0. If one defines $||A||_\mu$ as in Section 1, then $||A||_\mu = \mathcal{O}(\mu^2)$. One also sees, as in Section 1, that $||Au||_\mu \leq ||A||_\mu ||u||_\mu$. Hence with (9.14) one obtains

$$||(h\partial_{x_n})^{-1}Au|| < C\mu||u||_\mu$$

where $C > 0$ does not depend on μ. It is then clear that $||u||_\mu < \infty$ for μ sufficiently small. Hence u is an analytic symbol. $\square$

See also Remark 12.13.

10. The Method of Non-characteristic Lagrangians

It is a matter of old ideas that are well known for example in the context of Holmgren's Theorem. They are equally well known in the microlocal context, by the experts of the theory of hyperfunctions. Classically one seeks to deform the hypersurfaces or domains to obtain the extension results. We give here a sufficiently direct version with the help of the spaces H_φ. We will prove also some results of propagation of singularities which unfortunately are all contained in the results of Section 15 (obtained later). It is nevertheless not at all clear that the method elaborated further in Sections 12, 13, and 14 (and which contain implicitly anyway the non-characteristic deformations) is in any case the best.

Let $\Omega \subset\subset \mathbb{C}^n$ be an open set, and let $\Phi \in C^{1,1}(\overline{\Omega})$ be a real-valued function. Here $C^{1,1}(\overline{\Omega})$ denotes the space of $C^1(\overline{\Omega})$ functions whose first derivatives are Lipschitz on $\overline{\Omega}$. Let $P(x,\xi;h)$ be a classical analytic symbol of order 0, defined in a neighborhood of $\Lambda_\Phi = \{(x, \frac{2}{i}\frac{\partial\Phi}{\partial x}(x));\, x \in \overline{\Omega}\}$. For $u \in H_\Phi^{\mathrm{loc}}(\Omega)$ one then puts

$$Pu(x;h) = (2\pi h)^{-n} \iint_{\Gamma(x)} e^{\frac{i}{h}(x-y)\theta} P(x,\theta;h)\chi(x,y)u(y)\, dy\, d\theta \tag{10.1}$$

where $\Gamma(x)$ is the singular contour: $\theta = \frac{2}{i}\frac{\partial\Phi}{\partial x}(x) + \frac{i}{C_0}\frac{\overline{(x-y)}}{|x-y|}$ and where $\chi \in C^\infty(\Omega\times\Omega)$ satisfies:

$$\chi(x,y) = 1 \quad \text{for } |x-y| \le \frac{1}{2C_1} d(x, \complement\Omega),$$
$$\chi(x,y) = 0 \quad \text{for } |x-y| \ge \frac{1}{C_1} d(x, \complement\Omega).$$

(The existence of such a function will be established in Section 12.) Here the constants $C_0 > 0$ and $C_1 > 1$ have to be sufficiently large so that (x,θ) remains well inside the domain of definition of $P(x,\theta;h)$ for $(y,\theta)\in\Gamma(x)$, $(x,y)\in \mathrm{supp}\chi$. One also requires that

$$\left|\frac{2}{i}\frac{\partial\Phi}{\partial x}(x) - \frac{2}{i}\frac{\partial\Phi}{\partial x}(y)\right| \le \frac{1}{4C_0} \quad \text{for } |x-y| \le \frac{1}{C_1} d(x, \complement\Omega). \tag{10.2}$$

Then on $\Gamma(x)\cap \mathrm{supp}\chi$ one has

$$|e^{\frac{i}{h}(x-y)\theta - \frac{1}{h}(\Phi(x)-\Phi(y))}| \le e^{-\frac{3}{4hC_0}|x-y|}.$$

For $u \in H_\Phi^{\mathrm{loc}}(\Omega)$ one has

$$\overline{\partial} Pu \in L^{2,\mathrm{loc}}_{\Phi_1}, \qquad \Phi_1(x) = \Phi(x) - \frac{3}{8C_0C_1} d(x, \complement\Omega). \tag{10.3}$$

For $\Phi \in C^2$ this results from the formula for $\overline{\partial}Pu$ which we will establish in Section 12, and for $\Phi \in C^{1,1}(\overline{\Omega})$ one then obtains (10.3) by a simple regularization argument.

More generally, if $\varphi \in C^1(\Omega) \cap C(\overline{\Omega})$ and $\left|\frac{2}{i}\frac{\partial\varphi}{\partial x} - \frac{2}{i}\frac{\partial\Phi}{\partial x}\right| \leq \frac{1}{4C_0}$ then for $u \in H^{\mathrm{loc}}_{\varphi}(\Omega)$ one has $Pu \in L^{2,\mathrm{loc}}_{\varphi}(\Omega)$, $\overline{\partial}Pu \in L^{2,\mathrm{loc}}_{\varphi_1}(\Omega)$, where $\varphi_1(x) = \varphi(x) - \frac{1}{4C_0C_1}d(x, \complement\Omega)$. In fact, for $|x-y| \leq \frac{1}{C_1}d(x, \complement\Omega)$, $(y,\theta) \in \Gamma(x)$, one has

$$\left|e^{\frac{i}{h}(x-y)\theta - \frac{1}{h}(\varphi(x)-\varphi(y))}\right| = \left|e^{\frac{i}{h}(x-y)\theta - \frac{1}{h}(\Phi(x)-\Phi(y)) - \frac{1}{h}((\varphi-\Phi)(x)-(\varphi-\Phi)(y))}\right|$$
$$\leq e^{-\frac{1}{2hC_0}|x-y|}.$$

If, near a point $x_0 \in \Omega$, φ is also of class $C^{1,1}$ and $P_\varphi u$ denotes an operator provided with a regular contour adapted to φ, then for $u \in H^{\mathrm{loc}}_{\varphi}(\Omega)$ one has $Pu - P_\varphi u = 0$ in H_{φ,x_0}. This is proven by Stokes' formula.

Proposition 10.1. *Let P, Ω, Φ be as above, satisfying (10.2), and let $\varphi \in C(\overline{\Omega}) \cap C^{1,1}(\Omega)$, such that*

$$\varphi|_{\partial\Omega} \geq \tilde{\Phi}|_{\partial\Omega}, \qquad \left|\frac{2}{i}\frac{\partial\varphi}{\partial x} - \frac{2}{i}\frac{\partial\Phi}{\partial x}\right| \leq \frac{1}{4C_0}, \tag{10.4}$$

$$\varphi(x) < \tilde{\Phi}(x) \Rightarrow p\left(x, \frac{2}{i}\frac{\partial\varphi}{\partial x}\right) \neq 0. \tag{10.5}$$

Here $\tilde{\Phi} \leq \Phi$ is in $C(\overline{\Omega})$. If $u \in H^{loc}_{\tilde{\Phi}}(\Omega)$, $Pu \in L^{2,loc}_{\varphi}$, then $u \in H^{loc}_{\varphi}(\Omega)$.

Proof. Let $\Omega_1 \subset\subset \Omega$. Then if P_φ denotes the operator P provided with a suitable regular contour $\Sigma(x)$, adapted to φ, one finds with the aid of Stokes' formula that for each function u holomorphic in Ω,

$$||Pu - P_\varphi u||_{L^2_\varphi(\Omega_1)} \leq Ce^{-\frac{\epsilon}{h}}||u||_{L^2_\varphi(\Omega_2)},$$

where $\Omega_1 \subset\subset \Omega$ and $C, \epsilon > 0$. Here $||\cdot||_{L^2_\varphi(\Omega_1)}$ denotes the L^2 norm on Ω_1 with the weight $e^{-\frac{2\varphi(x)}{h}}L(dx)$. If Qu denotes the operator, obtained by replacing $P(x,\theta)$ by $q(x) = p\left(x, \frac{2}{i}\frac{\partial\varphi}{\partial x}\right)$ in the integral formula for $P_\varphi u(x)$, one sees that

$$||Qu - P_\varphi u||_{L^2_\varphi(\Omega_1)} \leq Ch^{\frac{1}{2}}||u||_{L^2_\varphi(\Omega_2)}.$$

With a suitable choice of $\Gamma(x)$, one finds, with the help of the stationary phase method as in Example 2.6, that

$$||Qu - q.u||_{L^2_\varphi(\Omega_1)} \leq Ce^{-\frac{\epsilon}{h}}||u||_{L^2_\varphi(\Omega_2)}.$$

Hence for each function u holomorphic in Ω,

$$||Pu - q.u||_{L^2_\varphi(\Omega_1)} \leq Ch^{\frac{1}{2}}||u||_{L^2_\varphi(\Omega_2)}.$$

For a given $\epsilon > 0$, let us now take

$$\Omega_1 = \{x \in \Omega;\ \varphi(x) < \tilde{\Phi}(x) - \epsilon\}.$$

Then $\frac{1}{|q|}$ is bounded on Ω_1, and one finds that

$$||u||_{L^2_\varphi(\Omega_1)} \leq C.||q.u||_{L^2_\varphi(\Omega_1)} \leq C_1\left(||Pu||_{L^2_\varphi(\Omega_1)} + h^{\frac{1}{2}}||u||_{L^2_\varphi(\Omega_2)}\right).$$

Hence, for h sufficiently small:

$$||u||_{L^2_\varphi(\Omega_1)} \leq 2C_1\left(||Pu||_{L^2_\varphi(\Omega_1)} + h^{\frac{1}{2}}||u||_{L^2_\varphi(\Omega_2\setminus\Omega_1)}\right).$$

We have $\varphi \geq \tilde{\Phi} - \epsilon$ on $\Omega_2 \backslash \Omega_1$, and if u satisfies the hypotheses of the proposition then one deduces that:

$$||u||_{L^2_\varphi(\Omega_1)} \leq C(\epsilon) e^{\frac{2\epsilon}{h}}.$$

When $\epsilon \to 0$, Ω_1 tends to $\{x \in \Omega;\, \varphi(x) < \tilde{\Phi}(x)\}$ and one finds that $u \in H^{\mathrm{loc}}_\varphi(\Omega)$. $\square$

In applications one has $u \in H^{\mathrm{loc}}_\Phi(W)$, where $W \subset\subset \mathbb{C}^n$ is an open set and $\Omega \subset\subset W$, $\Phi \in C^{1,1}(\overline{W})$ and $P_\Phi u|_\Omega \in L^{2,\mathrm{loc}}_{\Phi-2\epsilon}(\Omega)$ for an $\epsilon > 0$. Here $P_\Phi u$ is defined with a regular contour adapted to Φ. Free to modify $\epsilon > 0$ one has, more explicitly,

$$P_\Phi u(x;h) = (2\pi h)^{-n} \iint_{\Gamma(x)} e^{\frac{i}{h}(x-y)\theta} P(x,\theta;h) u(y)dy\,d\theta \tag{10.6}$$

where $\Gamma(x)$ is the contour $\theta = \frac{2}{i}\frac{\partial \Phi}{\partial x}(x) + \frac{i}{C_0 d_0}(\overline{x-y})$, $|x-y| \leq d_0$, where $d_0 < d(\Omega, \complement W)$ is sufficiently small so that

$$\left| \frac{2}{i}\left(\frac{\partial \Phi}{\partial x}(x) - \frac{\partial \Phi}{\partial x}(y) \right) \right| \leq \frac{1}{4C_0} \qquad \text{for } x \in \Omega,\ |x-y| \leq d_0.$$

As before one can replace $\Gamma(x)$ in (10.6) by the singular contour

$$\theta = \frac{2}{i}\frac{\partial \Phi}{\partial x}(x) + \frac{i}{C_0}\frac{(\overline{x-y})}{|x-y|},$$

and with χ and P as before (and with $C_1 > 0$ such that $\frac{d(x,\complement\Omega)}{C_1} \leq d_0$), one obtains

$$Pu - P_\Phi u \in L^{2,\mathrm{loc}}_{\Phi_1}(\Omega). \tag{10.7}$$

Here Φ_1 is defined in (10.3).

We now additionally suppose that $u \in H^{\mathrm{loc}}_{\tilde{\Phi}}(\Omega)$, where $\tilde{\Phi} \in \mathrm{Lip}(\overline{\Omega})$ and $\left| \frac{2}{i}\left(\frac{\partial \tilde{\Phi}}{\partial x} - \frac{\partial \Phi}{\partial x} \right) \right| \leq \frac{1}{4C_0}$. Then one obtains $Pu \in L^{2,\mathrm{loc}}_{\min(\tilde{\Phi},\max(\Phi-2\epsilon))}(\Omega)$. Since $\min(a,\max(b,c)) \leq \max\left(\frac{a+b}{2}, \frac{a+c}{2}\right)$, one then has $Pu \in L^{2,\mathrm{loc}}_{\max(\frac{1}{2}(\Phi+\tilde{\Phi}) - \frac{3}{16C_0C_1}d(x,\complement\Omega), \frac{\Phi+\tilde{\Phi}}{2}-\epsilon)}$. One may always assume that $\tilde{\Phi} \leq \Phi$, perhaps after replacing $\tilde{\Phi}$ by $\min(\Phi, \tilde{\Phi})$, and then

$$Pu \in L^{2,\mathrm{loc}}_{\max(\frac{1}{2}(\Phi+\tilde{\Phi}) - \frac{3}{16C_0C_1}d(x,\complement\Omega), \Phi-\epsilon)}(\Omega).$$

Corollary 10.2. *Let Ω, C_0, C_1 as before, $\Omega \subset\subset W$ where W is an open set, and $\Phi \in C^{1,1}(\overline{W})$. One supposes that $u \in H^{loc}_\Phi(W)$, $P_\Phi u|_\Omega \in L^{2,loc}_{\Phi-2\epsilon}(\Omega)$ where $\epsilon > 0$ and P_Φ is given by (10.6) with d_0 satisfying the supplementary conditions given after (10.6). Let $\tilde{\Phi} \leq \Phi$ be a Lipschitz function on $\overline{\Omega}$ such that $\left| \frac{2}{i}\left(\frac{\partial \Phi}{\partial x} - \frac{\partial \tilde{\Phi}}{\partial x} \right) \right| \leq \frac{1}{4C_0}$, and suppose that $u \in H^{loc}_{\tilde{\Phi}}(\Omega)$. Let $\varphi \in C(\overline{\Omega}) \cap C^{1,1}(\Omega)$ be such that $\varphi \geq \tilde{\Phi}$ on $\partial\Omega$, $\left| \frac{2}{i}\left(\frac{\partial \Phi}{\partial x} - \frac{\partial \tilde{\Phi}}{\partial x} \right) \right| \leq \frac{1}{4C_0}$, and $p\left(x, \frac{2}{i}\frac{\partial \varphi}{\partial x}\right) \neq 0$ on Ω. If in addition $\varphi \geq \max\left(\frac{1}{2}(\Phi + \tilde{\Phi}) - \frac{3}{16C_0C_1}d(x, \complement\Omega), \Phi - \epsilon \right)$, then $u \in H^{loc}_\varphi(\Omega)$.*

Here one can shrink Ω without changing ϵ, C_0, C_1, and Corollary 10.2 is still valid (but in general it is necessary to modify the choice of φ).

As a first application, we will prove an important result of Kawai-Kashiwara [19] on the propagation of analytic singularities for microhyperbolic operators. First let $p(x)$ be real-analytic defined near $x_0 \in \mathbb{R}^n$. If $y_0 \in \mathbb{R}^n$, one says that p is microhyperbolic at x_0 in the direction y_0 if there exists a real neighborhood V_0 of x_0 and $\epsilon_0 > 0$ such that $p(x + ity_0) \neq 0$ for $x \in V_0$, $0 < t \leq \epsilon_0$. Following a microlocal version of Bochner's tube theorem, due to Kashiwara [18] (see also Komatsu [20]), there exist real neighborhoods V of x_0 and W of y_0 such that $p(x + ity) \neq 0$ for $x \in V$, $y \in W$, $0 < t \leq \epsilon_0$. This "stability" in the definition of microhyperbolicity is very important; the first consequence is that the definition extends to the case where $\mathbb{R}^n$ is replaced by an analytic manifold and y_0 by a tangent vector. If one replaces x by (x, ξ) and if $p = p(x, \xi)$ is the principal symbol of a differential operator P, then one has an obvious definition of microhyperbolicity for P at a point $(x_0, \xi_0) \in T^*\mathbb{R}^n\backslash 0$ in a real direction $(y_0, \eta_0) \in T_{(x_0,\xi_0)}(T^*\mathbb{R}^n\backslash 0)$.

Theorem 10.3. *(Kashiwara–Kawai* [19]*): Let P be a differential operator with analytic coefficients defined in an open set $X \subset \mathbb{R}^n$. Let $(x_0, \xi_0) \in T^*X\backslash 0$ and let $\psi(x, \xi) \in C^2(V)$ be a real function, zero at (x_0, ξ_0), such that P is microhyperbolic at (x_0, ξ_0) in the direction H_ψ. Here $V \subset T^*X\backslash 0$ is a neighborhood of (x_0, ξ_0). If $u \in \mathcal{D}'(X)$, $(x_0, \xi_0) \notin WF_a(Pu)$, and $\{(x, \xi) \in V;\ \psi(x, \xi) < 0\} \cap WF_a(u) = \emptyset$, then $(x_0, \xi_0) \notin WF_a(u)$.*

Proof. Without loss of generality, one may suppose that ψ is analytic. For $\delta > 0$ sufficiently small, one puts

$$\tilde{\psi} = \psi - \frac{\delta^3}{2} + \delta((x - x_0)^2 + (\xi - \xi_0)^2).$$

One may also replace V by the ball $(x - x_0)^2 + (\xi - \xi_0)^2 \leq \delta^2$. Then P is microhyperbolic at each point of $\overline{V}$ in the direction $H_{\tilde{\psi}}$, and $\tilde{\psi}(x_0, \xi_0) < \psi(x_0, \xi_0) = 0$, while $\tilde{\psi} > \psi$ on ∂V. Hence

$$WF_a(u) \cap \partial V \cap \{\tilde{\psi} \leq 0\} = \emptyset \tag{10.8}$$

and it suffices to show that

$$WF_a(u) \cap V \cap \{\tilde{\psi} < 0\} = \emptyset. \tag{10.9}$$

One now takes an FBI transform T which transforms $T^*\mathbb{R}^n$ into Λ_{Φ_0}, and by an abuse of notation one denotes again by $P, p, \tilde{\psi}$ the transformed quantities. Then one knows that $p|_{\Lambda_{\Phi_t}} \neq 0$ if Φ_t is such that $\Lambda_{\Phi_t} = \exp itH_{\tilde{\psi}}(\Lambda_{\Phi_0})$, $0 < t \leq t_0$.

In general, if $q(x, \xi)$ is holomorphic and $\varphi(x)$ is a real C^2 function satisfying $q\left(x, \frac{2}{i}\frac{\partial \varphi}{\partial x}\right) = 0$, then the real vector fields $\widehat{H_q}$ and $\widehat{H_{iq}}$ associated to H_q and H_{iq} are tangent to $\Lambda_\varphi = \{\xi = \frac{2}{i}\frac{\partial \varphi}{\partial x}\}$. (Also, as we will see in Section 11, $\mathrm{Re}\, q$ and $\mathrm{Im}\, q$ are in involution with respect to the real symplectic form $\mathrm{Im}\, \sigma = \mathrm{Im} \sum_1^n d\xi_j \wedge dx_j$, and Λ_φ is a Lagrangian manifold for this form.) Applying these observations to $\tau + i\tilde{\psi}(x, \xi)$, one sees that one can take

$\Phi_t = \Phi(t, x)$ to be the solution of the Hamilton-Jacobi system (in involution for $\operatorname{Im}\sigma$)

$$\begin{cases} \frac{2}{i}\frac{\partial \Phi}{\partial t} + i\tilde{\psi}(x, \frac{2}{i}\frac{\partial \Phi}{\partial x}) = 0 \\ \Phi(0, x) = \Phi_0(x). \end{cases} \tag{10.10}$$

Here one restricts to $0 < t \le \epsilon_0$, and one observes that

$$\Phi(t, x) = \Phi_0(x) + \frac{t}{2}\tilde{\psi}\left(x, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}\right) + \mathcal{O}(t^2).$$

Let $\tilde{V} \subset \mathbb{C}^n_x$ be the set corresponding to V and W a small neighborhood of $\overline{\tilde{V}}$. The hypotheses on $WF_a(u)$ imply that $U = Tu \in H^{\rm loc}_{\tilde{\Phi}}(W)$, where $\tilde{\Phi} \le \Phi_0$ and $\tilde{\Phi} < \Phi$ on $\partial\tilde{V} \cap \{\tilde{\psi} \le 0\}$. Here $\tilde{\Phi}$ is a small perturbation of Φ_0 in the C^2 norm, and for $\epsilon > 0$ sufficiently small we have $PU \in H^{\rm loc}_{\Phi_0 - 2\epsilon}(\tilde{V})$. Then for $t_0 > 0$ sufficiently small one has $\Phi(t_0, x) \ge \tilde{\Phi}$ on $\partial\tilde{V}$, and

$$\Phi(t_0, x) \ge \max\left(\frac{1}{2}(\Phi_0 + \tilde{\Phi}) - \frac{3}{16C_0C_1}d(x, \complement\tilde{V}),\ \Phi_0 - \epsilon\right) \qquad \text{in } \tilde{V}.$$

In fact, this inequality is trivial when $\tilde{\psi} \ge Ct_0$ because then $\Phi(t_0, x) \ge \Phi_0$, and in the region $\tilde{\psi} \le Ct_0$ and for $t_0 > 0$ sufficiently small one has

$$\frac{1}{2}(\Phi + \tilde{\Phi}) - \frac{3}{16C_0C_1}d(x, \complement\tilde{V}) \le \Phi - a,$$

where $a > 0$ does not depend on t_0.

One can then apply Corollary 10.2 with $\varphi = \Phi(t_0, x)$, and one finds that $U \in H^{\rm loc}_{\Phi_{t_0}}(\tilde{V})$ for $0 < t_0 \le \epsilon_0$ if ϵ_0 is sufficiently small. Hence $U \equiv 0$ in H_{Φ_0, x_1} for each $x_1 \in \tilde{V}$ with $\tilde{\psi}(x_1) < 0$, and by the inverse FBI transform one obtains (10.9). ☐

As our second application we will consider certain "perturbations" of operators studied by Bony-Shapira [3]. Let $V \subset T^*\mathbb{R}^n$ be the involutive manifold given by $\xi'' = 0$, where one writes $x = (x', x'')$, $\xi = (\xi', \xi'')$, $\xi'' \in \mathbb{R}^d$. Let $\Gamma_0 \subset V$ be the bicharacteristic leaf given by $x' = 0$, $\xi' = (1, 0, \dots, 0)$, $\xi'' = 0$. One considers a differential operator P with analytic coefficients, defined in a neighborhood X of 0, such that the principal symbol p satisfies in a neighborhood W of $(0, \eta_0) = (0, (1, 0, \dots, 0))$ the following conditions:

$$p = \mathcal{O}(d^2_{\Gamma_0}), \quad \text{where } d_{\Gamma_0} = d_{\Gamma_0}(x, \xi) \text{ denotes the distance from } (x, \xi) \text{ to } \Gamma_0, \tag{10.11}$$

$$p|_V = \mathcal{O}(d^4_{\Gamma_0}),\ \nabla p|_V = \mathcal{O}(d^2_{\Gamma_0}), \tag{10.12}$$

$$\begin{gathered} \text{On } \Gamma_0 \text{ one has } \langle \nu, p''\nu\rangle \ne 0 \\ \text{for each real tangent vector } \nu = (t_x, t_\xi) \text{ with } t_{\xi''} \ne 0. \end{gathered} \tag{10.13}$$

The hypothesis (10.12) implies that $T_{\Gamma_0}V$ is in the kernel of p'', and (10.13) is a condition of transversal ellipticity (with respect to V). With the help of (10.12) and (10.13), one obtains in a small complex neighborhood of $(0,\eta_0)$:

$$p(x,\xi)\neq 0 \quad \text{for } |\mathrm{Im}\,\xi''|+|x'|^2+|\xi'-\eta_0'|^2<\frac{1}{C_0}|\mathrm{Re}\,\xi''|, \tag{10.14}$$

if the constant C_0 is sufficiently large. In fact, it suffices to write the Taylor expansion:

$$p(x,\xi)=\sum_{|\alpha|=2}a_\alpha(x,\xi)(\xi'')^\alpha+\mathcal{O}((|x'|^2+|\xi'-\eta_0'|^2)|\xi''|+|x'|^4+|\xi'-\eta_0'|^4). \tag{10.15}$$

Theorem 10.4. *Suppose additionally that $\Gamma_0\cap W$ is connected. If $u\in\mathcal{D}'(X)$ and $WF_a(Pu)\cap W\cap\Gamma_0=\emptyset$, then either $\Gamma_0\cap W\cap WF_a(u)=\emptyset$ or $\Gamma_0\cap W\subset WF_a(u)$.*

This result is due to Bony-Schapira in the case where (10.11) and (10.12) are replaced by the stronger condition: $p=\mathcal{O}(d_V^2)$. The result will be further generalized in Section 15.

Proof. (of Theorem 10.4): One first of all applies the FBI transform

$$Tu(x;h)=\int e^{\frac{i}{h}(i(x-y)^2/2-y\eta_0)}u(y)\,dy \tag{10.16}$$

for which $\Phi=\frac{1}{2}(\mathrm{Im}\,x)^2$ and $\Lambda_\Phi:\xi=-\mathrm{Im}\,x$. The point $(0,\eta_0)$ is transformed to $(0,0)$, the complexification $V^{\mathbb{C}}$ of V is preserved, while the complex leaf $\Gamma_0^{\mathbb{C}}$ becomes $x'=\xi'=0$, $\xi''=0$. If $\tilde p$ is the principal symbol of the transformed operator $\tilde P$, then (10.14) becomes

$$\tilde p(x,\xi)\neq 0 \quad \text{for } |\mathrm{Im}\,\xi''|+|x'|^2+|\xi'|^2<\frac{1}{C_0}|\mathrm{Re}\,\xi''| \quad \text{in a neighborhood of } (0,0). \tag{10.14$'$}$$

In terms of weight functions, this is written as

$$\begin{aligned}\tilde p\left(x,\frac{2}{i}\frac{\partial\psi}{\partial x}(x)\right)\neq 0 \quad &\text{for } \left|\frac{\partial\psi}{\partial\mathrm{Re}\,x''}\right|+|x'|^2+\left|\frac{2}{i}\frac{\partial\psi}{\partial x'}\right|^2<\frac{1}{C_0}\left|\frac{\partial\psi}{\partial\mathrm{Im}\,x''}\right|\\ &\text{and } \left(x,\frac{2}{i}\frac{\partial\psi}{\partial x}\right) \text{ in a neighborhood of } (0,0).\end{aligned} \tag{10.14$''$}$$

We need the following lemma:

Lemma 10.5. *For any $r_1,r_2\in(0,1]$, $C>0$, there exists a constant $\epsilon>0$ and a function*

$$\varphi_0(x'')\in C^{1,1}(\{|Re\,x''|\leq r_1,\ |Im\,x''|\leq r_2\})$$

such that $0\leq\varphi_0\leq\frac{1}{2}(Im\,x'')^2$, φ_0 is radial in $Im\,x''$, and $\left|\frac{\partial\varphi_0}{\partial Re\,x''}\right|\leq\frac{1}{C}\frac{\partial\varphi_0}{\partial|Im\,x''|}$, $\varphi_0=\frac{1}{2}(Im\,x'')^2$ when $|Re\,x''|=r_1$, where $|Im\,x''|=r_2$, $\varphi_0=0$ for $|Im\,x''|\leq\epsilon(r_1^2-|Re\,x''|^2)$. Finally, one may choose ϵ, φ_0 such that $||\varphi_0||_{C^{1,1}}\leq\tilde C$, where $\tilde C$ is a constant which does not depend on r_1, r_2, or C.

Proof. (of Lemma 10.5): Let

$$\chi(x'') = \frac{(|\mathrm{Im}\, x''| - \epsilon(r_1^2 - |\mathrm{Re}\, x''|^2))_+}{(r_2 - \epsilon(r_1^2 - |\mathrm{Re}\, x''|^2))},$$

where $a_+ = \max(a, 0)$. Then $\chi(x'')$ is Lipschitz and is C^∞ at places where $\chi \neq 0$. In the region $\chi \neq 0$, one obtains the following estimates, uniformly in r_1, r_2, and $\epsilon \leq \frac{r_2}{2r_1^2}$:

$$\frac{\partial \chi}{\partial |\mathrm{Im}\, x''|} \sim \frac{1}{r_2}, \quad \frac{\partial \chi}{\partial \mathrm{Re}\, x''} = \mathcal{O}\left(\frac{\epsilon}{r_2}\right), \quad \nabla^2_{x''}\chi = \mathcal{O}\left(\frac{1}{r_2 |\mathrm{Im}\, x''|}\right).$$

One then puts $\varphi_0(x'') = \frac{1}{2} r_2^2 \chi^2$ and one verifies the desired properties (with $\epsilon > 0$ sufficiently small). □

One now places oneself in a domain:

$$\Omega: \ |\mathrm{Re}\, x''| < r_1, \quad |\mathrm{Im}\, x''| < r_2, \quad |x'| < |\mathrm{Im}\, x''|,$$

with r_1, r_2 sufficiently small. With $C = C_0$, where C_0 is given in (10.14″), one chooses $\varphi_0(x'')$ as in the lemma, and with a constant $C_1 > 0$ sufficiently large one considers

$$\psi_t(x) = |x'|^2 + C_1 |\mathrm{Im}\, x''||x'|^2 + \varphi_0(x'') + t(\mathrm{Im}\, x'')^2$$

for $0 \leq t \leq 1$. One takes $r_2 < \frac{1}{C_1}$. Then ψ_t is continuous on $\overline{\Omega}$ and is of class $C^{1,1}$ on Ω. The $C^{1,1}$ norm of ψ_t can by majorized by a constant which does not depend on r_0, r_1, C_0, or C_1 (free to choose "ϵ" sufficiently small in Lemma 10.5). Thus ψ_t will also be as near to $\frac{1}{2}(\mathrm{Im}\, x'')^2$ as one wants in the C^1 norm, free to choose $r_2 > 0$ sufficiently small.

We have

$$\psi_t|_{\partial\Omega} \geq |x'|^2 + \varphi_0(x'')|_{\partial\Omega} \geq \frac{1}{2} |\mathrm{Im}\, x''|^2|_{\partial\Omega}.$$

If C_1 is sufficiently large, $0 < t \leq 1$, then one also has on Ω

$$\left|\frac{\partial \psi_t}{\partial \mathrm{Re}\, x''}\right| + |x'|^2 + \left|\frac{\partial \psi_t}{\partial x'}\right|^2 < \frac{1}{C_0}\left|\frac{\partial \psi_t}{\partial \mathrm{Im}\, x''}\right|.$$

If r_2 is sufficiently small, depending also on $U = Tu$, one can then apply Corollary 10.2 with $\varphi = \psi_t$, $0 < t \leq 1$. Letting t go to 0 one obtains:

$$Tu \in H^{\mathrm{loc}}_{\psi_0}(\Omega). \tag{10.17}$$

The only condition on r_1 is that it should be sufficiently small so that one does not meet the region corresponding to $WF_a(Pu)$. Let $\rho_1 < r_1$. Then for $\epsilon_0 > 0$ sufficiently small one has $\psi_0(x) = |x'|^2 + C_1 |\mathrm{Im}\, x''||x'|^2 \leq 2|x'|^2$ for $|x'| \leq |\mathrm{Im}\, x''| \leq \epsilon_0$, $|\mathrm{Re}\, x''| \leq \rho_1$. Since $\frac{1}{2}|\mathrm{Im}\, x|^2 \leq 2|x'|^2$ for $|\mathrm{Im}\, x''| \leq |x'|$, one then has

$$Tu \in H^{\mathrm{loc}}_{2|x'|^2} \tag{10.18}$$

in the region $|\mathrm{Re}\, x''| \leq \rho_1$, $|\mathrm{Im}\, x''| \leq \epsilon_0$, $|x'| \leq \epsilon_0$. Here $2|x'|^2$ is trivially pluriharmonic in x'', and one can apply the maximum principle to $x'' \mapsto e^{-\frac{2|x'|^2}{h}} Tu$ to conclude that if $Tu = 0$ in $H_{\Phi,0}$ then $Tu = 0$ in $H_{\Phi,(x_0',0)}$ for each real x_0' with $|x_0'| < \rho_1$. We then have shown that if $(0, \eta_0) \notin WF_a(u)$ then $(x_0', 0, \eta_0) \notin WF_a(u)$ for $|x_0'| < \rho_1$. The same argument works if one replaces $(0, \eta_0)$ by another point on $\Gamma_0 \cap W$, and so one obtains Theorem 10.4. □

The invariant version of Theorem 10.4 is the following: Let $\Gamma \subset T^*X\backslash 0$ be a connected real-analytic submanifold that is isotropic with respect to the symplectic 2-form. One then supposes that the principal symbol p satisfies the following conditions in a neighborhood of Γ:

$$p = \mathcal{O}(d_\Gamma^2), \tag{10.19}$$

(10.20) If V is an involutive submanifold on which Γ is a bicharacteristic leaf, then

$$\nabla p|_V = \mathcal{O}(d_\Gamma^2), \quad p|_V = \mathcal{O}(d_\Gamma^4),$$

$$\langle p''\nu, \nu\rangle \neq 0 \quad \text{for all } 0 \neq \nu \in N_\Gamma(V). \tag{10.21}$$

It is easy to see that (10.20) and (10.21) do not depend on the choice of V, since for $\rho \in \Gamma$ one has $T_\rho(V) = T_\rho(\Gamma)^\perp$, the orthogonal subspace of $T_\rho(\Gamma)$ with respect to the symplectic form.

Let κ be a canonical transformation which sees V locally as $\xi'' = 0$. Then, modulo a change of variables in y, it is well known that there exists a real analytic function $\varphi(x, \eta)$ of $2n$ variables such that κ is given by

$$\left(\frac{\partial \varphi}{\partial \eta}(x, \eta), \eta\right) \mapsto \left(x, \frac{\partial \varphi}{\partial x}\right),$$

and of course $\det \frac{\partial^2 \varphi}{\partial x \partial \eta} \neq 0$. One may then consider the formal Fourier integral operator

$$Ru(x; h) = \iint e^{\frac{i}{h}(\varphi(x,\eta) - y\eta)} u(y) dy\, d\eta,$$

and one may take the formal composition $T \circ R$ where T is as before. To calculate this composition, one applies ${}^tR_y$ to the kernel $K(x, y; h)$ of T and one then finds a new FBI transform $\tilde{T}$ which transforms P into an operator $\tilde{P}$ of the same type as in the proof of Theorem 10.4. One then obtains

Theorem 10.6. *Under the more general hypotheses (10.19)–(10.21), where $\Gamma \subset T^*X\backslash 0$ is a connected isotropic real-analytic manifold, if $u \in \mathcal{D}'(X)$ and $\Gamma \cap WF_a(Pu) = \emptyset$, then either $\Gamma \cap WF_a(u) = \emptyset$ or $\Gamma \subset WF_a(u)$.*

This theorem applies to the Kohn Laplacian on a real hypersurface of $\mathbb{C}^2$ which contains a complex curve; see Section 15.

11. Toward a General Theory

In the geometric aspects we were inspired by Schapira [29] who observed the usefulness of distinguishing between $\operatorname{Re}\sigma$ and $\operatorname{Im}\sigma$ when one works in the complex domain. Here $\sigma = \sum d\zeta_j \wedge dz_j$, $z_j = x_j + iy_j$, $\zeta_j = \xi_j + i\eta_j$,

$$\operatorname{Re}\sigma = \frac{1}{2}(\sigma + \overline{\sigma}) = \sum d\xi_j \wedge dx_j - d\eta_j \wedge dy_j$$

$$\operatorname{Im}\sigma = \frac{1}{2i}(\sigma - \overline{\sigma}) = \sum d\xi_j \wedge dy_j + d\eta_j \wedge dx_j.$$

We verify that $\operatorname{Re}\sigma$ and $\operatorname{Im}\sigma$ are real symplectic forms, that is, closed non-degenerate 2-forms. For example, $\operatorname{Im}\sigma$ becomes the standard symplectic form on $\mathbb{R}^{2n}_X \times \mathbb{R}^{2n}_\Xi$ if one puts $X = (y, x)$ and $\Xi = (\xi, \eta)$.

We thus have a complex symplectic geometry and two real symplectic geometries given by σ, $\operatorname{Re}\sigma$, and $\operatorname{Im}\sigma$. We study first the relation between the Hamiltonian fields. Let $r = p + iq$ be a holomorphic function (defined on an open set in $\mathbb{C}^{2n}$). The Cauchy-Riemann equations $\overline{\partial}p + i\overline{\partial}q = 0$, $\partial p - i\partial q = 0$ give $\partial p = i\partial q$, $\partial r = 2i\partial q$. Let $H_r \in T_{1,0}(\mathbb{C}^{2n}) \subset \mathbb{C} \otimes T(\mathbb{C}^{2n})$ be the (holomorphic) Hamiltonian vector defined at each point by $H_r \lrcorner \sigma = -dr$ (where we observe that σ is a $(2,0)$-form and dr is a $(1,0)$-form). Explicitly, $H_r = \sum \frac{\partial r}{\partial \zeta_j}\frac{\partial}{\partial z_j} - \frac{\partial r}{\partial z_j}\frac{\partial}{\partial \zeta_j}$. By definition we thus have (at each point) $\langle \sigma, t \wedge H_r\rangle = -\langle dr, t\rangle$ for each $t \in T_{1,0}(\mathbb{C}^{2n})$. This relation is trivially again true for $t \in T_{0,1}(\mathbb{C}^{2n})$, hence for $t \in \mathbb{C} \otimes T(\mathbb{C}^{2n})$ and in particular for $t \in T(\mathbb{C}^{2n})$. Let $\widehat{H_r}$ be the real vector field associated to H_r; that is, $\widehat{H_r} \in T(\mathbb{C}^{2n})$ is defined by $H_r - \widehat{H_r} \in T_{0,1}$. Then $\langle \sigma, t \wedge (\widehat{H_r} - H_r)\rangle = 0$ for all $t \in \mathbb{C} \otimes T(\mathbb{C}^{2n})$. Hence $\langle \sigma, t \wedge \widehat{H_r}\rangle = -\langle dr, t\rangle$, and so we deduce

$$\widehat{H_r} \lrcorner \sigma = -dr. \tag{11.1}$$

Separating the real and imaginary parts we obtain

$$\widehat{H_r} \lrcorner \operatorname{Re}\sigma = -dp, \qquad \widehat{H_r} \lrcorner \operatorname{Im}\sigma = -dq. \tag{11.2}$$

On the other hand, let $H_p^{\operatorname{Im}\sigma}$ and $H_p^{\operatorname{Re}\sigma}$ be the Hamiltonian fields for $\operatorname{Im}\sigma$ and $\operatorname{Re}\sigma$, respectively. Then, by definition,

$$H_p^{\operatorname{Im}\sigma} \lrcorner \operatorname{Im}\sigma = H_p^{\operatorname{Re}\sigma} \lrcorner \operatorname{Re}\sigma = -dp \tag{11.3}$$

$$H_q^{\operatorname{Im}\sigma} \lrcorner \operatorname{Im}\sigma = H_q^{\operatorname{Re}\sigma} \lrcorner \operatorname{Re}\sigma = -dq. \tag{11.4}$$

Comparing with (11.2) we find

$$\widehat{H_r} = H_p^{\operatorname{Re}\sigma} = H_q^{\operatorname{Im}\sigma}. \tag{11.5}$$

The same discussion applies also to $ir = -q + ip$:

$$\widehat{H_{ir}} = -H_q^{\operatorname{Re}\sigma} = H_p^{\operatorname{Im}\sigma}. \tag{11.6}$$

One immediate consequence is that p and q are in involution for $\operatorname{Im}\sigma$:

$$\{p, q\}^{\operatorname{Im}\sigma} = H_p^{\operatorname{Im}\sigma}(q) = -H_q^{\operatorname{Re}\sigma}(q) = 0.$$

In general, the Hamiltonian fields preserve the symplectic form:

$$\mathcal{L}_{H_p^\omega}(\omega) = 0,$$

where ω is a real symplectic form, p is a real C^2 function, and $\mathcal{L}$ denotes the Lie derivative.

Proposition 11.1. *Let $p(z,\zeta)$ be a real C^2 function on an open set in $\mathbb{C}^{2n}$. Then*

$$\mathcal{L}_{H_p^{Im\,\sigma}}(Re\ \sigma) = \frac{2}{i}\overline{\partial}\partial p.$$

Proof. We have

$$\mathcal{L}_{H_p^{\mathrm{Im}\,\sigma}}(\mathrm{Re}\ \sigma) = H_p^{\mathrm{Im}\ \sigma} \lrcorner\, d\,\mathrm{Re}\ \sigma + d(H_p^{\mathrm{Im}\,\sigma} \lrcorner\, \mathrm{Re}\ \sigma) = d(H_p^{\mathrm{Im}\ \sigma} \lrcorner\, \mathrm{Re}\ \sigma) \tag{11.7}$$

because $d\,\mathrm{Re}\ \sigma = 0$. To calculate $H_p^{\mathrm{Im}\,\sigma} \lrcorner \mathrm{Re}\,\sigma$ at a given point, we can suppose that p is linear, hence pl.h.; say $r = p + iq$, $\overline{\partial} r = 0$. Then, by (11.6):

$$H_p^{\mathrm{Im}\,\sigma} \lrcorner \mathrm{Re}\,\sigma = -H_q^{\mathrm{Re}\,\sigma} \lrcorner \mathrm{Re}\,\sigma = dq = \partial q + \overline{\partial} q = \frac{1}{i}\partial p - \frac{1}{i}\overline{\partial} p.$$

Substituting in (11.7) gives

$$\mathcal{L}_{H_p^{\mathrm{Im}\,\sigma}}(\mathrm{Re}\,\sigma) = (\partial + \overline{\partial})(\frac{1}{i}\partial p - \frac{1}{i}\overline{\partial} p) = \frac{2}{i}\overline{\partial}\partial p.$$

□

Following the terminology of Schapira [29], one says that a manifold Λ in $\mathbb{C}^{2n}$ is I-Lagrangian (resp. $\mathbb{R}$-Lagrangian) if Λ is Lagrangian for $\mathrm{Im}\,\sigma$ ($\mathrm{Re}\,\sigma$). If Λ is a C^∞ I-Lagrangian manifold such that $\Lambda \ni (z,\zeta) \mapsto z \in \mathbb{C}^n$ is a local diffeomorphism, then, since $d(\mathrm{Im}\,(\zeta dz)|_\Lambda) = \mathrm{Im}\,\sigma|_\Lambda = 0$, there exists locally a *real* C^∞ function φ on Λ, such that

$$\mathrm{Im}\,(\zeta dz) = -d\varphi.$$

That is,

$$\frac{1}{2i}\zeta dz - \frac{1}{2i}\overline{\zeta} d\overline{z} = -\partial\varphi - \overline{\partial}\varphi.$$

Considering φ as a function of z, we then find the following local expression for Λ:

$$\zeta = \frac{2}{i}\frac{\partial\varphi}{\partial z}.$$

Conversely, if Λ is of this form, then Λ is I-Lagrangian. (One writes $\Lambda = \Lambda_\varphi$.)

The manifolds that are both I- and $\mathbb{R}$-Lagrangian are precisely the complex Lagrangian manifolds. One such manifold Λ is written in suitable canonical holomorphic coordinates: $\zeta = \frac{\partial r}{\partial z}$, where r is holomorphic. Then $\Lambda = \Lambda_\varphi$, where $\varphi = -\mathrm{Im}\,r$.

One says that Λ is $\mathbb{R}$-symplectic if $\mathrm{Re}\,\sigma|_\Lambda$ is non-degenerate. As an example we observe that $T^*\mathbb{R}^n$ is I-Lagrangian and $\mathbb{R}$-symplectic. An I-Lagrangian manifold of the form $\Lambda = \Lambda_\varphi$ is $\mathbb{R}$-symplectic if and only if the Levi form $\partial\overline{\partial}\varphi$ is non-degenerate. In fact, since $\zeta dz = \frac{2}{i}\partial\varphi$ on Λ and $\sigma = d(\zeta dz)$, we obtain

$$\mathrm{Re}\,\sigma|_\Lambda = (\partial + \overline{\partial})\zeta dz|_\Lambda = \frac{2}{i}\overline{\partial}\partial\varphi. \tag{11.8}$$

To decide if φ is strictly pl.s.h. it is also necessary to know something about the choice of coordinates (z,ζ) and more particularly about the choice of a $\mathbb{C}$-Lagrangian manifold F (that is, a complex Lagrangian manifold) which reduces to $z = 0$.

To analyze this problem we begin with a study of the linear situation. We let $\Lambda \subset \mathbb{C}^{2n}$ be a (linear) I-Lagrangian subspace, and we let F be a $\mathbb{C}$-Lagrangian subspace such that $F \pitchfork \Lambda$, where "$\pitchfork$" denotes transversal intersection; that is, $F + \Lambda = \mathbb{C}^{2n}$. Then after a (linear) $\mathbb{C}$-canonical transformation we can reduce F to $z = 0$ and Λ to Λ_φ, where φ is a real quadratic form on $\mathbb{C}^n$.

Definition 11.2. *One says that Λ is F-pseudoconvex if φ is pl.s.h.*

This definition does not depend on the choice of the $\mathbb{C}$-canonical transformation which reduces F to $x = 0$. In fact, if $\mathcal{H}$ is $\mathbb{C}$-canonical with $\mathcal{H}(F) = F$ (supposing that F is already given by $x = 0$) then $\mathcal{H} = \mathcal{H}_2 \circ \mathcal{H}_1$ where $\mathcal{H}_1(y, \eta) = (y, \eta + Ay)$, with A a symmetric matrix and $\mathcal{H}_2(y, \eta) = (B^{-1}y, {}^tB\eta)$ where B is invertible. It is obvious that $\mathcal{H}_1$ and $\mathcal{H}_2$ preserve the plurisubharmonicity of the generating functions.

Let $L^k(\mathbb{C}^n)$ be the set of I-Lagrangian planes with $\operatorname{Re}\sigma|_\Lambda$ of rank $2k$. $L^0(\mathbb{C}^{2n})$ is then the set of $\mathbb{C}$-Lagrangian planes and $L^n(\mathbb{C}^{2n})$ is the set of planes that are I-Lagrangian and $\mathbb{R}$-symplectic. Let $(L^0 \times L^k)^t = \{(F, \Lambda) \in L^0 \times L^k;\ F \pitchfork \Lambda\}$.

Proposition 11.3. *Let $(F, \Lambda) \in (L^0 \times L^k)^t$ be such that Λ is F-ps.c. Then for $(\tilde{F}, \tilde{\Lambda}) \in (L^0 \times L^k)^t$ the following two properties are equivalent:*

(i) $\tilde{\Lambda}$ is $\tilde{F}$-ps.c.

(ii) $(\tilde{F}, \tilde{\Lambda})$ and (F, Λ) belong to the same connected component of $(L^0 \times L^k)^t$.

Proof. $(i) \Rightarrow (ii)$: After an initial deformation by a continuous family of $\mathbb{C}$-canonical transformations, we can assume that $F = \tilde{F} = \{(x, \xi) \in \mathbb{C}^{2n};\ x = 0\}$. Then $\Lambda = \Lambda_\varphi$ and $\tilde{\Lambda} = \Lambda_{\tilde{\varphi}}$, where the Levi matrices of φ and $\tilde{\varphi}$ have the same rank k. We can then find a continuous deformation $[0, 1] \ni t \mapsto \varphi_t$ of real quadratic forms, such that $\varphi_0 = \varphi$, $\varphi_1 = \tilde{\varphi}$, and such that the rank of $\mathcal{L}_{\varphi_t}$ is constant, where $\mathcal{L}_{\varphi_t}$ denotes the Levi matrix. Then $t \mapsto (F, \Lambda_{\varphi_t})$ gives the desired deformation.

$(ii) \Rightarrow (i)$: Let (F_t, Λ_t), $0 \le t \le 1$, be a curve in $(L^0 \times L^k)^t$ with $(F_0, \Lambda_0) = (F, \Lambda)$ and $(F_1, \Lambda_1) = (\tilde{F}, \tilde{\Lambda})$. We can then find a continuous family of $\mathbb{C}$-canonical transformations κ_t with $\kappa_t(F_t) = F$. Hence by a canonical transformation (depending continuously on t) we reduce F_t to $x = 0$ and reduce Λ to Λ_{φ_t}, where φ_t varies continuously with t and the rank of $\mathcal{L}_{\varphi_t}$ is constant. Then the plurisubharmonicity of $\varphi = \varphi_0$ is preserved by the deformation, and in particular φ_1 is pl.s.h. Hence $\tilde{\Lambda}$ is $\tilde{F}$-pseudoconvex. □

When Λ is I-Lagrangian and $\mathbb{R}$-symplectic, we have that Λ is a totally real subspace, and we can define the mapping $\mathbb{C}^{2n} \ni u \mapsto \overline{u} \in \mathbb{C}^{2n}$ as the unique anti-linear mapping that is equal to the identity on Λ. (One can find $\mathbb{C}$-canonical coordinates such that Λ is identified with $\mathbb{R}^{2n}$.) If F is a $\mathbb{C}$-Lagrangian subspace, then $F \pitchfork \Lambda$ if and only if the Hermitian form $\frac{1}{i}\sigma(u, \overline{u})$ on $F \times F$ is non-degenerate. We say that F *is strictly negative with respect to* Λ if this form is negative definite. This property is evidently stable under continuous deformations in $(L^0 \times L^n)^t$, and, conversely, if (F, Λ) and $(\tilde{F}, \tilde{\Lambda})$ both have this property, after a continuous deformation we may assume that $\Lambda = \tilde{\Lambda} = \mathbb{R}^{2n}$. Then one easily verifies that F and $\tilde{F}$ are of the form $\xi = \frac{2}{i}\frac{\partial f}{\partial x}$, $\xi = \frac{2}{i}\frac{\partial \tilde{f}}{\partial x}$, where f and $\tilde{f}$ are pluriharmonic and > 0 on $\dot{\mathbb{R}}^n$. There is then an obvious deformation of F into $\tilde{F}$.

Proposition 11.4. *Let $(F, \Lambda) \in (L^0 \times L^n)^t$. Then Λ is F-pseudoconvex if and only if F is strictly negative with respect to Λ.*

Proof. Following Proposition 11.3 and the preceding remarks, it suffices to find a single pair $(F, \Lambda) \in (L^0 \times L^n)^t$ which has the two properties. We take $F : x = 0$ and $\Lambda = \Lambda_\varphi$ with $\varphi = \frac{1}{2}|x|^2$. Clearly Λ is F-pseudoconvex. We have $\Lambda : \xi = \frac{1}{i}\overline{x}$, and $u \mapsto \overline{u}$ becomes $(y, \eta) \mapsto (\frac{1}{i}\overline{\eta}, \frac{1}{i}\overline{y})$. Hence for $u = (0, \eta) \in F$:

$$\frac{1}{i}\sigma(u, \overline{u}) = \frac{1}{i}\sigma((0, \eta), (\frac{1}{i}\overline{\eta}, 0)) = -|\eta|^2$$

which shows that F is strictly negative with respect to Λ. □

We still let $\Lambda \subset \mathbb{C}^{2n}$ be an I-Lagrangian subspace. If $\Lambda^{\mathrm{Re}\,\sigma}$ (Λ^σ) denotes the orthogonal subspace with respect to $\mathrm{Re}\,\sigma$ (σ), we put $L = \Lambda^{\mathrm{Re}\,\sigma} \cap \Lambda = \Lambda^\sigma \cap \Lambda$. We note that L is $\mathbb{C}$-isotropic, and, if $V = L^\sigma$ is the corresponding $\mathbb{C}$-involutive subspace, then $\Lambda \subset V$. If $d = \dim_{\mathbb{C}} L$, then $\dim_{\mathbb{C}} V = 2n - d$ and $V/V^\sigma = V/L$ is a $\mathbb{C}$-symplectic subspace of dimension $2(n-d)$. If Λ' is the image of Λ in V/L, then Λ' is I-Lagrangian and $\mathbb{R}$-symplectic (of real dimension $2(n-d)$). In suitable $\mathbb{C}$-symplectic coordinates $(x, \xi) = (x', x'', \xi', \xi'')$, we have

$$\Lambda = \{(x, \xi) \in \mathbb{C}^{2n};\ \xi'' = 0,\ (x', \xi') \in \mathbb{R}^{2(n-d)}\}. \tag{11.9}$$

We now let $F \subset \mathbb{C}^{2n}$ be a $\mathbb{C}$-Lagrangian subspace transversal to Λ. Then F is transversal to V, and $V \cap F$ is of dimension $n - d$ and is transversal to L in V. Hence the projection $F' \subset V/L$ of $V \cap F$ is of dimension $n - d$, is clearly $\mathbb{C}$-Lagrangian, and is transversal to Λ'. We can now choose $\mathbb{C}$-symplectic coordinates such that $F : x = 0$ and $V : \xi'' = 0$. We parametrize V/L by (x', ξ'), and hence F' is given by $x' = 0$, while $\Lambda' : \xi' = \frac{2}{i}\frac{\partial \varphi}{\partial x'}$. Hence $\Lambda : \xi = \frac{2}{i}\frac{\partial \varphi}{\partial x'}(x')$, that is, $\xi'' = 0$, $(x', \xi') \in \Lambda'$, and we deduce that Λ is F-ps.c. if and only if Λ' is F'-ps.c.

Proposition 11.5. *Let (F_0, Λ), $(F_1, \Lambda) \in (L^0 \times L^{n-d})^t$ where Λ is both F_0- and F_1-pseudoconvex. Then there exists a continuous deformation $[0, 1] \ni t \mapsto F_t$ of $\mathbb{C}$-Lagrangian planes transversal to Λ, which links F_0 to F_1.*

Proof. We first reduce Λ to the form (11.9). According to the discussion surrounding Proposition 11.4, we know that F_0' and F_1' are of the form

$$-x' = \frac{2}{i}\frac{\partial \psi_j}{\partial \xi'}(\xi'), \quad j = 0, 1$$

where the ψ_j are pl.h. and < 0 on $\dot{\mathbb{R}}^{n-d}$. Recalling also that V is of the form $\xi'' = 0$, we note that the projections $F_j \ni (x, \xi) \mapsto \xi \in \mathbb{C}^n$ are bijective and hence that F_j is of the form

$$-x = \frac{2}{i}\frac{\partial \varphi_j}{\partial \xi}, \quad j = 0, 1$$

where the φ_j are pl.h. and $\varphi_j(\xi', 0) = \psi_j(\xi')$. To conclude, it suffices to take $F_t : -x = \frac{2}{i}\frac{\partial \varphi_t}{\partial \xi}$ where $\varphi_t = (1 - t)\varphi_0 + t\varphi_1$. □

Definition 11.6. *Let Λ_1, $\Lambda_2 \subset \mathbb{C}^{2n}$ be two I-Lagrangian subspaces, and let $F \subset \mathbb{C}^{2n}$ be a $\mathbb{C}$-Lagrangian subspace. One says that $\Lambda_1 \le \Lambda_2$ relative to F if $\kappa(\Lambda_j) = \Lambda_{\varphi_j}$ with $\varphi_1 \le \varphi_2$ plurisubharmonic, if κ is a $\mathbb{C}$-canonical transformation such that $\kappa(F) : x = 0$.*

This definition does not depend on the choice of κ. It is also independent of the choice of F, provided that Λ_2 is F-pseudoconvex:

Proposition 11.7. *Let $\Lambda_1 \le \Lambda_2$ relative to F, and let $\tilde{F}$ be a $\mathbb{C}$-Lagrangian subspace such that Λ_2 is $\tilde{F}$-pseudoconvex. Then $\Lambda_1 \le \Lambda_2$ relative to $\tilde{F}$.*

We can then define an obvious relation "$\Lambda_1 \le \Lambda_2$", and we remark that, if $\Lambda_1 \le \Lambda_2$ and $\Lambda_2 \le \Lambda_3$, then $\Lambda_1 \le \Lambda_3$. If $\Lambda_1 \le \Lambda_2$ and $\Lambda_2 \in L^0$, then $\Lambda_1 = \Lambda_2$.

Proof. (of Proposition 11.7): Following Proposition 11.5, we can find a continuous family $[0,1] \ni t \mapsto F_t \in L^0$ such that $F_0 = F$, $F_1 = \tilde{F}$ and such that Λ_2 is F_t-pseudoconvex for all t. Let κ_t be a continuous family of $\mathbb{C}$-canonical transformations with $\kappa_t(F_t) = \{x = 0\}$. Then $\kappa_t(\Lambda_2) = \Lambda_{\varphi_{2,t}}$, where $\varphi_{2,t}$ is pl.s.h. and where the Levi matrix $\mathcal{L}_{\varphi_{2,t}}$ is of constant rank k.

For $t \in [0, a)$ with $a > 0$ sufficiently small, we know that $F_t \pitchfork \Lambda_1$, and hence that $\kappa_t(\Lambda_1) = \Lambda_{\varphi_{1,t}}$, where $\varphi_{1,t}$ is pl.s.h., since the Levi matrix is of constant rank. Moreover the rank of $\varphi_{2,t} - \varphi_{1,t}$ is constant (measuring the dimension of $\Lambda_2 \cap \Lambda_1$), and $\varphi_{2,0} - \varphi_{1,0} \ge 0$ by hypothesis. Hence $\varphi_{2,t} - \varphi_{1,t} \ge 0$ for $t \in [0, a)$.

Here we remark that if $f \le g$ are pl.s.h. quadratic forms, that we decompose as $f = f_\ell + f_{\mathfrak{h}}$ and $g = g_\ell + g_{\mathfrak{h}}$ as in the beginning of Section 3, then with $\Im f(x) = f(ix)$ we deduce from $f_{\mathfrak{h}} \le g$ that $-f_{\mathfrak{h}} \le \Im g$, that is, $f_{\mathfrak{h}} \ge -\Im g$. Hence $-\Im g \le f \le g$, and, in our situation,

$$-\Im\varphi_{2,t} \le \varphi_{1,t} \le \varphi_{2,t}.$$

Hence $F_a \pitchfork \Lambda_1$ and it is clear that in fact $F_t \pitchfork \Lambda_1$ for all $t \in [0, 1]$ and that $\varphi_{1,t}$ is pl.s.h., $\varphi_{1,t} \le \varphi_{2,t}$. □

Still in the linear case, we now discuss the phase functions.

Definition 11.8. *A real quadratic form $\varphi(x, \theta)$ on $\mathbb{C}^n_x \times \mathbb{C}^N_\theta$ is called an "admissible phase" if*

- *1^o φ is pl.s.h.*
- *2^o There exists a pl.s.h. quadratic form $\psi(x, w)$, $w \in \mathbb{C}^M$, such that $\varphi(x, \theta) + \psi(x, w)$ is non-degenerate of signature 0.*

Remark 11.9. In 2^o we can always replace ψ by a pl.h. form bounded by ψ. We can hence suppose that ψ is pl.h., and, after a slight perturbation, that $\psi(0, w)$ is non-degenerate. If $\tilde{\psi}(x) = \text{v.c.}_w\psi(x, w)$ then $\tilde{\psi}(x) + \varphi(x, \theta)$ is non-degenerate of signature 0. In 2^o we can then take $\psi = \psi(x)$ pl.h.

Remark 11.10. If $\varphi(x,\theta)$ is admissible and $\tilde{\varphi}(x,\theta) \le \varphi$ is a pl.s.h. quadratic form, then $\tilde{\varphi}$ is admissible. We remark also that the admissible phase $\varphi(x,\theta)$ is always non-degenerate in the sense of Hörmander [15]; that is, the mapping $(x,\theta) \mapsto \nabla_\theta \varphi(x,\theta)$ is surjective. This results from the fact that $(x,\theta) \mapsto (\nabla_x(\varphi(x,\theta)+\psi(x)), \nabla_\theta\varphi(x,\theta))$ is bijective if ψ is as in Definition 11.8. Hence, if φ is an admissible phase, we know that each pl.s.h. form $\tilde{\varphi} \le \varphi$ is non-degenerate in the sense of Hörmander.

Example 11.11. Let $N = n$ and $\varphi(x,\theta) = |\overline{x}-\theta|^2$. This is a strictly pl.s.h. phase that is non-degenerate in the sense of Hörmander, with critical manifold

$$C_\varphi := \{(x,\theta) \in \mathbb{C}^{n+N};\ \nabla_\theta\varphi = 0\} = \{(x,\theta);\ \theta = \overline{x}\}.$$

φ is not admissible, since $\varphi \ge 0$ and the function 0 is pl.h. but not admissible. On the other hand, the functions $\varphi_t(x,\theta) = |\overline{x}-\theta|^2 - t|\overline{x}+\theta|^2$, $0 < t \le 1$, are pl.s.h. and non-degenerate of signature 0, hence are admissible, and φ_1 is pl.h.

Let $\varphi(x,\theta)$ and $\psi(x)$ be as in Definition 11.8, with $\psi(x)$ pl.h. We put

$$\Lambda_\psi = \left\{ \left(x, \frac{2}{i}\frac{\partial\psi}{\partial x}(x) \right);\ x \in \mathbb{C}^n \right\}$$

$$\Lambda_\varphi = \left\{ \left(x, \frac{2}{i}\frac{\partial\varphi}{\partial x}(x,\theta) \right);\ \frac{\partial\varphi}{\partial\theta}(x,\theta) = 0 \right\}.$$

We have already seen that Λ_ψ is $\mathbb{C}$-Lagrangian, and we show that Λ_φ is I-Lagrangian: in fact, the mapping $C_\varphi \ni (x,\theta) \mapsto \left(x, \frac{2}{i}\frac{\partial\varphi}{\partial x}\right) \in \mathbb{C}^{2n}$ is injective, hence Λ_φ is of real dimension $2n$, and $\operatorname{Im}\sigma|_{\Lambda_\varphi} = \operatorname{Im} d(\xi dx)|_{\Lambda_\varphi} \simeq \operatorname{Im} d\left(\frac{2}{i}\frac{\partial\varphi}{\partial x}dx\right)|_{C_\varphi} = \operatorname{Im} d\left(\frac{2}{i}\frac{\partial\varphi}{\partial x}dx + \frac{2}{i}\frac{\partial\varphi}{\partial\theta}d\theta\right)|_{C_\varphi} = \operatorname{Im} d\frac{2}{i}\partial\varphi|_{C_\varphi} = \operatorname{Im}\frac{2}{i}\overline{\partial}\partial\varphi|_{C_\varphi} = 0$ because $\frac{2}{i}\overline{\partial}\partial\varphi$ is real.

The non-degeneracy of $\psi(x)+\varphi(x,\theta)$ is equivalent to the transversality of Λ_φ and of $\Lambda_{-\psi}$.

Proposition 11.12. *Let $\psi(x)$ be pl.h. and let $\varphi(x,\theta)$ be an admissible phase. Then $\psi(x)+\varphi(x,\theta)$ is non-degenerate of signature 0 if and only if Λ_φ and $\Lambda_{-\psi}$ are pseudoconvex.*

Proof. To replace (φ,ψ) by $(\varphi+\psi,0)$ is equivalent to the $\mathbb{C}$-canonical transformation $(x,\xi) \mapsto \left(x, \xi + \frac{2}{i}\frac{\partial\psi}{\partial x}(x)\right)$. We can then suppose that $\psi = 0$, that is, that $\Lambda_{-\psi}$ is given by $\xi = 0$. If φ is of signature 0, then $H(\xi) = \text{v.c.}_{(x,\theta)}(\varphi(x,\theta) + \operatorname{Im} x\xi)$ is pl.s.h., and Λ_φ is given by $-x = \frac{2}{i}\frac{\partial H}{\partial\xi}$, which shows that Λ_φ is $\Lambda_{-\psi}$–pseudoconvex.

Conversely, let Λ_φ be $\Lambda_{-\psi}$–pseudoconvex and let $\psi_0(x)$ be pl.h. such that $\varphi+\psi_0$ is non-degenerate of signature 0. Then Λ_φ is $\Lambda_{-\psi_0}$–pseudoconvex, and, following Proposition 11.5, we can find a homotopy $[0,1] \ni t \mapsto F_t \in L^0$ with $F_0 = \Lambda_{-\psi_0}$, $F_1 = \Lambda_{-\psi}$, and with F_t transversal to Λ_φ. After a perturbation we may assume moreover that the F_t are transversal to $x = 0$ and hence are of the form $\Lambda_{-\psi_t}$ with ψ_t pl.h. and $\psi_1 = \psi$. Then $\psi_t + \varphi$ is non-degenerate and hence is of constant signature $= 0$. □

The same perturbation argument as in Remark 11.9 shows that the Proposition remains true when $\psi = \psi(x,w)$ is pl.h. on $\mathbb{C}^{n+M}$. We now let $\varphi_1(x,\theta)$ and $\varphi_2(x,\tilde{\theta})$ be two admissible phases; then to say that $\varphi_1(x,\theta)+\varphi_2(x,\tilde{\theta})$ is non-degenerate of signature 0 is equivalent to the condition that $\varphi_1(x,\theta)+\varphi_2(y,\tilde{\theta})-\mathrm{Im}\,(x-y)w$ is non-degenerate of signature 0 on $\mathbb{C}^{3n+N+\tilde{N}}_{x,y,w,\theta,\tilde{\theta}}$. Hence with $\psi=-\mathrm{Im}\,(x-y)w$ and $\varphi=\varphi_1(x,\theta)+\varphi_2(y,\tilde{\theta})$ we obtain:

Proposition 11.13. *Let $\varphi_1(x,\theta)$ and $\varphi_2(x,\tilde{\theta})$ be two admissible phases. Then $\varphi_1(x,\theta)+\varphi_2(x,\tilde{\theta})$ is non-degenerate of signature* 0 *if and only if $\Lambda_{\varphi_1}\times\Lambda_{\varphi_2}$ is F-pseudoconvex. Here $F\subset\mathbb{C}^{2n}\times\mathbb{C}^{2n}$ is the fiber conormal to the diagonal,*

$$F=\{(x,\xi,x,-\xi)\in\mathbb{C}^{2n}\times\mathbb{C}^{2n};\,(x,\xi)\in\mathbb{C}^{2n}\}=N^*(\{x=y\}).$$

Remark 11.14. If $\varphi(x,\theta)$ is admissible and $\psi(x,\theta)\le\varphi(x,\theta)$ is pl.s.h., then $\Lambda_\psi\le\Lambda_\varphi$. In fact, it suffices to repeat the first part of the proof of Proposition 11.12 while using again the Fundamental Lemma. We remark also that, for $F\in L^0$, $\Lambda\in\mathrm{L}^k$ is F-pseudoconvex if and only if $\tilde{\Lambda}\pitchfork F$ for each $\tilde{\Lambda}\le\Lambda$. One direction results from the definition of "$\le$" and its invariance. For the other direction, we suppose that $\tilde{\Lambda}\pitchfork F$ for all $\tilde{\Lambda}\le\Lambda$, and $F_0\in L^0$ such that Λ is F_0-pseudoconvex. (The existence of F_0 results from the discussion preceding Proposition 11.5.) After a small perturbation of F_0 we may assume that $F_0\pitchfork F$, and so after a canonical transformation we may assume that $F_0=\{x=0\}$ and $F=\{\xi=0\}$. Hence $\Lambda=\Lambda_\varphi$ where $\varphi(x)$ is pl.s.h. and non-degenerate of signature 0. Proposition 11.12 then shows that Λ is F-pseudoconvex.

We now discuss the non-linear situation. In what follows we only consider the (micro-)local theory, and the different phase functions and manifolds are only germs, defined near certain fixed points. (Nevertheless, the theorem below should be able to serve as a point of departure for a global theory.) All of the phases and manifolds below are supposed to be of class C^∞. (However it seems possible to relax this regularity condition to $C^{1,1}$ phases and to Lipschitz I-Lagrangian manifolds.)

We let $\Lambda\subset\mathbb{C}^{2n}$ be a C^∞ I-Lagrangian manifold defined near $(x_0,\xi_0)\in\mathbb{C}^{2n}$ (containing this point), and we let $F\subset\mathbb{C}^{2n}$ be a $\mathbb{C}$-Lagrangian manifold defined near (x_0,ξ_0) which intersects Λ transversally at (x_0,ξ_0). After a $\mathbb{C}$-canonical transformation κ, we can reduce to the case $(x_0,\xi_0)=(0,0)$, $F:\{x=0\}$; Λ is then of the form Λ_φ, where $\varphi(x)$ is C^∞, real, and defined in a neighborhood of 0.

Definition 11.15. *One says that Λ is F-pseudoconvex at (x_0,ξ_0) if φ is pl.s.h. in a neighborhood of* 0.

The plurisubharmonicity of φ near 0 is equivalent to saying that $T_{(y,\eta)}(\Lambda_\varphi)$ is $T_{(y,\eta)}(\{x=y\})$–pseudoconvex for $\eta=\frac{2}{i}\frac{\partial\varphi}{\partial x}(y)$ and for each y near 0, and so our results on the F-pseudoconvexity in the linear case show that Definition 11.15 does not depend on the choice of the $\mathbb{C}$-canonical transformation, which reduces F to $\{x=0\}$ and (x_0,ξ_0) to $(0,0)$. Moreover, if Λ is F-pseudoconvex at (x_0,ξ_0), and if $\tilde{F}$ is another $\mathbb{C}$-Lagrangian manifold passing through (x_0,ξ_0), then Λ is $\tilde{F}$-pseudoconvex at (x_0,ξ_0) if and only if $T_{(x_0,\xi_0)}(\Lambda)$ is $T_{(x_0,\xi_0)}(\tilde{F})$–pseudoconvex.

As before, we have a partial order relation. If Λ_1 and Λ_2 are germs of I-Lagrangian manifolds at (x_0, ξ_0), and if F is a germ of a $\mathbb{C}$-Lagrangian manifold at (x_0, ξ_0), one says that $\Lambda_1 \leq \Lambda_2$ relative to F at (x_0, ξ_0) if $\kappa(\Lambda_j) = \Lambda_{\varphi_j}$ with $\varphi_1 \leq \varphi_2$ both pl.s.h. and with $\varphi_1(0) = \varphi_2(0)$. Here κ is a $\mathbb{C}$-canonical transformation with $\kappa((x_0, \xi_0)) = (0, 0)$ and $\kappa(F) = \{x = 0\}$. We first note that this definition does not depend on the choice of κ. First, we well have the desired invariance if we replace κ by $\kappa_1 \circ \kappa$, where κ_1 is linear with $\kappa_1(\{x = 0\}) = \{x = 0\}$. It then suffices to consider the case where we replace κ by $\kappa_2 \circ \kappa$, where κ_2 satisfies $\kappa_2((0, 0)) = (0, 0)$ and $d\kappa_2(0, 0) = I$. Hence, if $\varphi(x, \eta)$ is the (holomorphic) generating function of κ_2 with $\varphi(0, 0) = 0$, then we have $\varphi(x, \eta) = x\eta + \mathcal{O}(|x|^3 + |\eta|^3)$. With $\Phi(x, \eta) = -\mathrm{Im}\, \varphi(x, \eta)$ and $\kappa_2(\Lambda_{\varphi_j}) = \Lambda_{\tilde{\varphi}_j}$, we have

$$\tilde{\varphi}_j(x) = \mathrm{v.c.}_{(y,\eta)}(\Phi(x, \eta) + \mathrm{Im}\,(y\eta) + \varphi_j(y))$$

and the Fundamental Lemma shows that $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are pl.s.h. and that $\tilde{\varphi}_1 \leq \tilde{\varphi}_2$ near 0.

The same type of reasoning also gives the independence of F.

Proposition 11.16. *Let $\Lambda_1 \leq \Lambda_2$ relative to F at (x_0, ξ_0), and let $\tilde{F}$ be another $\mathbb{C}$-Lagrangian manifold passing through (x_0, ξ_0) such that Λ_2 is $\tilde{F}$-pseudoconvex. Then $\Lambda_1 \leq \Lambda_2$ relative to $\tilde{F}$.*

We have thus defined "$\Lambda_1 \leq \Lambda_2$ at (x_0, ξ_0)".

Definition 11.17. *A real (C^∞) function $\varphi(x, \theta)$ defined near $(x_0, \theta_0) \in \mathbb{C}^{n+N}$ is an admissible phase function if φ is pl.s.h. and if there exists a pl.s.h. function $\psi(x, w)$, $w \in \mathbb{C}^M$, such that $\psi(x, w) + \varphi(x, \theta)$ has a saddle at (x_0, θ_0, w_0).*

As in the linear situation, we note that the admissible phases are non-degenerate phases in the sense of Hörmander, and that in Definition 11.17 it suffices to consider pl.h. functions $\psi = \psi(x)$. If $\psi(x, w)$ is a non-degenerate pl.h. phase, and $\nabla(\psi + \varphi) = 0$ at (x_0, θ_0, w_0), then Λ_φ is $\Lambda_{-\psi}$–pseudoconvex at $(x_0, \xi_0) = (x_0, \frac{2}{i}\frac{\partial \varphi}{\partial x}(x_0))$ if and only if $\varphi + \psi$ has a saddle at (x_0, θ_0, w_0). (Here the I-Lagrangian manifold is defined as before.) If $\varphi_1 \leq \varphi_2$ and $\varphi_1(x_0, \xi_0) = \varphi_2(x_0, \xi_0)$, with φ_1 pl.s.h. and φ_2 admissible, then $\Lambda_{\varphi_1} \leq \Lambda_{\varphi_2}$.

If $\varphi(x, \theta)$ is an admissible phase defined near (x_0, θ_0), on denotes by I_φ the space of formal objects

$$u(x; h) = \int a(x, \theta; h)\, d\theta \quad , \quad a(x, \theta; h) \in H_{\varphi,(x_0,\theta_0)}.$$

Of course, if $\theta \mapsto \varphi(x_0, \theta)$ has a saddle at θ_0, we have a choice of a good contour and a $u \in H_{\Phi, x_0}$, where $\Phi(x) = \mathrm{v.c.}_\theta(\varphi(x, \theta))$. In general, one does not seek to define $u(x; h)$ directly, but one observes that if $\psi(x, w)$ is another admissible phase such that $\varphi(x, \theta) + \psi(x, w)$ has a saddle at (x_0, θ_0, w_0) with critical value 0, then, if $v(x; h) \in \int b(x, w; h) \in I_\psi$, one defines the scalar product

$$\langle u, v\rangle = \iiint_\Gamma a(x, \theta; h)\, b(x, w; h)\, dx\, d\theta\, dw$$

modulo the sign (which depends on the choice of orientation of Λ) and modulo an exponentially decreasing term. Here Γ is a good contour for $\varphi+\psi$.

If $\tilde{\varphi}(x,\tilde{\theta})$ is another admissible phase with

$$\tilde{\varphi}(x_0,\tilde{\theta}_0)=\varphi(x_0,\theta_0),$$

$$\frac{2}{i}\frac{\partial\tilde{\varphi}}{\partial x}(x_0,\tilde{\theta}_0)=\frac{2}{i}\frac{\partial\varphi}{\partial x}(x_0,\theta_0)=:\xi_0,$$

and

$$\Lambda_\varphi=\Lambda_{\tilde{\varphi}},$$

then $\psi+\varphi$ has a saddle at (x_0,θ_0,w_0) if and only if $\psi+\tilde{\varphi}$ has a saddle at $(x_0,\tilde{\theta}_0,w_0)$; in fact, Proposition 11.13 is still valid in the nonlinear case. We will identify $u\in I_\varphi$ and $\tilde{u}\in I_{\tilde{\varphi}}$ if $\langle u,v\rangle\equiv\langle\tilde{u},v\rangle$, modulo an exponentially small term, for all $v\in I_\psi$ and all ψ as above. One writes then $u=\tilde{u}$.

Theorem 11.18. *(Equivalence of Phases.): Let $\varphi(x,\theta)$ and $\tilde{\varphi}(x,\tilde{\theta})$ be two admissible phases defined near (x_0,θ_0) and $(x_0,\tilde{\theta}_0)$, respectively, with $\tilde{\varphi}(x_0,\tilde{\theta}_0)=\varphi(x_0,\theta_0)$, $\frac{2}{i}\frac{\partial\tilde{\varphi}}{\partial x}(x_0,\tilde{\theta}_0)=\frac{2}{i}\frac{\partial\varphi}{\partial x}(x_0,\theta_0)=:\xi_0$, and $\Lambda_\varphi=\Lambda_{\tilde{\varphi}}$. Then $I_\varphi\equiv I_{\tilde{\varphi}}$.*

Proof. After multiplication of I_φ and $I_{\tilde{\varphi}}$ by $e^{\frac{g(x)}{h}}$, where $g(x)$ is holomorphic, we can reduce to the case where $\xi_0=0$ and $\Lambda_\varphi=\Lambda_{\tilde{\varphi}}$ is pseudoconvex with respect to $\xi_0=0$. We can also easily reduce to the case where $x_0=0$, $\theta_0=0$, and $\tilde{\theta}_0=0$. Hence $\varphi(x,\theta)$ and $\tilde{\varphi}(x,\tilde{\theta})$ have saddles at (x_0,θ_0) and $(x_0,\tilde{\theta}_0)$, respectively, and, as before, we introduce the pl.s.h. function:

$$H(\xi)=\text{v.c.}(\varphi(x,\theta)+\operatorname{Im}(x\xi))=\text{v.c.}(\tilde{\varphi}(x,\tilde{\theta})+\operatorname{Im}(x\xi)).$$

For $u=\int a(x,\theta;h)\,d\theta\in I_\varphi$ one can define the Fourier transform

$$\mathcal{F}u(\xi;h)=\iint a(x,\theta;h)e^{-\frac{ix\xi}{h}}\,dx\,d\theta\in H_{H,0}$$

and similarly one defines $\mathcal{F}\tilde{u}\in H_{H,\xi_0}$ for $\tilde{u}\in I_{\tilde{\varphi}}$. The mapping $H_{\varphi,(x_0,\theta_0)}\ni a(x,\theta;h)\mapsto\mathcal{F}u\in H_{H,0}$ is not injective, and we add then (as before) the variables θ^* dual to θ, and one defines a more complete Fourier transform

$$Ta(\xi,\theta^*;h)=\iint a(x,\theta;h)e^{-\frac{i}{h}(x\xi+\theta\theta^*)}\,dx\,d\theta\in H_{\kappa,(0,0)},$$

where $\kappa(\xi,\theta^*)=\text{v.c.}_{(x,\theta)}(\varphi(x,\theta)+\operatorname{Im}(x\xi+\theta\theta^*))$ is pl.s.h. with a saddle at $(0,0)$. By Proposition 3.3, $T:H_{\varphi,(0,0)}\to H_{\kappa,(0,0)}$ is bijective with inverse given by

$$a(x,\theta;h)=STa(x,\theta;h)=(2\pi h)^{-n-N}\iint e^{\frac{i}{h}(x\xi+\theta\theta^*)}(Ta)(\theta,\theta^*;h)\,d\xi\,d\theta^*.$$

Lemma 11.19. *$\mathcal{F}:I_\varphi\to H_{H,0}$ is surjective.*

Proof. By our discussion, it suffices to find for each element $f(\xi;h)\in H_{H,0}$ an element $g(\xi,\theta^*;h)\in H_{\kappa,(0,0)}:g(\xi,0;h)=f(\xi;h)$. We do it one variable at a time, and so it suffices to consider the case where $\theta^*=z$ is a single complex variable. We let $\chi\in C_0^\infty(\mathbb{C})$, $=1$ near 0. Then we first put $\tilde{g}(\xi,z;h)=f(\xi;h)\chi(\frac{z}{h})\in L^2_{\kappa,(0,0)}$. We then look for g of the form $\tilde{g} + z\,k(\xi, z; h)$. It then suffices to solve $\overline{\partial}k = -\frac{f(\xi;h)}{z}\overline{\partial}(\chi(\frac{z}{h}))$ in $L^2_{\kappa,(0,0)}$. Placing ourselves in a small pseudoconvex neighborhood of $(0,0)$, we then find $k \in L^2_{\kappa,(0,0)}$ by the standard results on the solvability of $\overline{\partial}$ of Hörmander [14]. ☐

We now let $\psi(x,w)$ be admissible and such that $\varphi + \psi$ has a saddle at $(0,0,w_0)$ with critical value 0. Let $v = \int b(x,w;h)\,dw \in H_\psi$, $u = \int a(x,\theta;h)\,d\theta \in H_\varphi$, and $A(\xi,\theta^*;h) = Ta$. Then, with a natural contour,

$$\langle u, v\rangle \equiv (2\pi h)^{-n-N} \iiiint \int e^{\frac{i}{h}(x\xi+\theta\theta^*)} b(x,w;h)A(\xi,\theta^*;h)\,d\xi\,d\theta^*\,d\theta\,dx\,dw.$$

The weight function $\psi(x,w) + \kappa(\xi,\theta^*) - \operatorname{Im}(x\xi + \theta\theta^*)$ has a saddle with respect to all the variables, but also with respect to the variables (θ,θ^*). After a contour deformation we can then apply the method of stationary phase in (θ,θ^*), so we obtain (since $A(\xi,0;h) = \mathcal{F}u(\xi;h)$)

$$\langle u, v\rangle \equiv (2\pi h)^{-n} \iiint e^{\frac{i}{h}x\xi} b(x,w;h)\mathcal{F}u(\xi;h)\,dx\,dw\,d\xi.$$

We have the same formula with u replaced by $\tilde{u} \in I_{\tilde{\varphi}}$. If $\tilde{u} \in I_{\tilde{\varphi}}$ is given, we can, by Lemma 11.19, find $u \in I_\varphi$ with $\mathcal{F}u = \mathcal{F}\tilde{u}$ in $H_{H,0}$, and hence $\langle u, v\rangle \equiv \langle \tilde{u}, v\rangle$. ☐

Remark 11.20. If $a(x,\theta;h)\,d\theta = \frac{h}{i}d_\theta b(x,\theta;h)$ in $H_{\varphi,(x_0,\theta_0)}$ with $b \in H_{\varphi,(x_0,\theta_0)}$, then $\int a(x,\theta;h)\,d\theta \equiv 0$ in I_φ. Conversely, if $\int a(x,\theta;h)\,d\theta \equiv 0$ in I_φ and $\mathcal{F}u \equiv 0$, then $Ta(\xi,\theta^*;h) \equiv \sum_1^N A_j(\xi,\theta^*;h)\theta_j^*$ in $H_{\kappa,(\xi_0,0)}$ with $A_j \in H_{\kappa,(\xi_0,0)}$. (This decomposition which does not result directly from the Taylor formula is left as an exercise.) By the Fourier inversion formula, we obtain

$$a(x,\theta;h) = \sum \frac{h}{i}\frac{\partial}{\partial\theta_j} b_j(x,\theta;h)$$

where

$$b_j = (2\pi h)^{-n-N} \iint e^{\frac{i}{h}(x\xi+\theta\theta^*)} A_j\,d\xi\,d\theta^* \in H_{\varphi,(x_0,\theta_0)}.$$

Hence $a(x,\theta;h)\,d\theta = \frac{h}{i}\,d_\theta b(x,\theta;h)$ with

$$b = \sum (-1)^{j+1} a_j\,d\theta_1 \wedge \ldots \wedge \widehat{d\theta_j} \wedge \ldots \wedge d\theta_N.$$

Example 11.21. We consider the phase

$$\varphi = \operatorname{Im} x \operatorname{Re}\theta + C_1(\operatorname{Im} x)^2 + C_2(\operatorname{Im}\theta)^2$$

with C_1, $C_2 > 0$. This is clearly a non-degenerate phase with $C_\varphi : \operatorname{Im} x = 0, \operatorname{Im}\theta = 0$. The Levi matrix is

$$\begin{pmatrix} \frac{\partial^2\varphi}{\partial\bar{x}\partial x} & \frac{\partial^2\varphi}{\partial\bar{x}\partial\theta} \\ \frac{\partial^2\varphi}{\partial\bar{\theta}\partial x} & \frac{\partial^2\varphi}{\partial\bar{\theta}\partial\theta} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}C_1 I & -\frac{1}{4i}I \\ -\frac{1}{4i}I & \frac{1}{2}C_2 I \end{pmatrix}$$

So φ is pl.s.h. if $C_1C_2 \geq \frac{1}{4}$. With this hypothesis we verify that φ is admissible: Let $\psi(x)$ be a pl.h. quadratic form that is < 0 on $\dot{\mathbb{R}}^n$. Then $\psi(x) \leq -C_3(\operatorname{Re} x)^2 + C_4(\operatorname{Im} x)^2$

for $C_3 > 0$ and $C_4 > 0$. Let $\Gamma \subset \mathbb{C}^{2n}$ be the subspace of real dimension $2n$ given by $\operatorname{Im} x = -\epsilon \operatorname{Re} \theta$ and $\operatorname{Im} \theta = 0$, with $\epsilon > 0$. Then

$$\begin{aligned}\varphi + \psi|_\Gamma &\leq -\epsilon(\operatorname{Re}\theta)^2 + (C_1 + C_4)(\epsilon \operatorname{Re}\theta)^2 - C_3(\operatorname{Re} x)^2 \\ &\leq -\frac{\epsilon}{2}(\operatorname{Re}\theta)^2 - C_3(\operatorname{Re} x)^2,\end{aligned}$$

if $\epsilon > 0$ is sufficiently small. Then Γ is a good contour for $\varphi + \psi$, and consequently $\varphi + \psi$ is non-degenerate of signature 0. Finally we find

$$\Lambda_\varphi = \{(x, -\theta);\, \operatorname{Im} x = \operatorname{Im}\theta = 0\} = T^*\mathbb{R}^n.$$

Example 11.22. Here we have a second example of an admissible phase which generates $T^*\mathbb{R}^n$: We consider $\varphi(\theta, y)$, $\varphi_1(\theta, y) = -\operatorname{Im}\varphi(\theta, y)$, and $\Phi(\theta) = \sup_{y\in\mathbb{R}^n} \varphi_1(\theta, y)$, as in the beginning of Section 7, verifying (7.1)-(7.3). We consider then the pl.s.h. phase:

$$\tilde{\varphi}(y, \theta) = -\varphi_1(\theta, y) + \Phi(\theta).$$

Let $\psi(y)$ be a pl.h. function with $\frac{2}{i}\frac{\partial\psi}{\partial y} = -\eta_0$ and $\nabla^2\psi|_{\mathbb{R}^n} < 0$. We then verify that, for $\epsilon > 0$ sufficiently small, the contour

$$\{(y, \theta) \in \mathbb{R}^n \times \mathbb{C}^n;\, \theta \in \Gamma(y)\} \ni (y, \theta) \mapsto (y - \epsilon\nabla_y(\tilde{\varphi}(y, \theta) + \psi(y)), \theta)$$

is a good contour for $\tilde{\varphi} + \psi$. Here ∇_y denotes the gradient in the real direction on $\mathbb{C}^n$; this is then a vector in $\mathbb{C}^n$. Having shown that $\tilde{\varphi}$ is admissible, we then find that C_φ is given by $\theta \in \Gamma(y)$ and $y \in \mathbb{R}^n$, and that $\Lambda_\varphi = T^*\mathbb{R}^n$.

To finish this section, we finally discuss Fourier integral operators. If $\varphi(x, \theta)$ and $\psi(x, w)$ are admissible phases such that $\varphi(x, \theta) + \psi(x, w)$ has a saddle at (x_0, θ_0, w_0), then we have seen that this condition is equivalent to the $N^*(\{x = y\})$–pseudoconvexity of $\Lambda_\varphi \times \Lambda_\psi$ at $(x_0, \xi_0, x_0, -\xi_0)$, with $\xi_0 = \frac{2}{i}\frac{\partial\varphi}{\partial x}(x_0, \theta_0)$. We write then $\Lambda_\varphi \$ \Lambda_\psi$. When ψ is pl.h., then $\Lambda_{-\psi}$ is $\mathbb{C}$-Lagrangian, and a third equivalent condition is that Λ_φ is $\Lambda_{-\psi}$–pseudoconvex.

We now let $\varphi_1(y, \omega)$ be an admissible phase, defined near $(y_0, \omega_0) \in \mathbb{C}^{n_y + N_\omega}$, and we let $\varphi(x, y, \theta)$ be an admissible phase defined near $(x_0, y_0, \theta_0) \in \mathbb{C}^{n_x + n_y + N_\theta}$. We suppose that

$$-\frac{2}{i}\frac{\partial\varphi}{\partial y}(x_0, y_0, \theta_0) = \frac{2}{i}\frac{\partial\varphi_1}{\partial y}(y_0, \omega_0) =: \eta_0,$$

and we put

$$\xi_0 = \frac{2}{i}\frac{\partial\varphi}{\partial x}(x_0, y_0, \theta_0).$$

We introduce the following geometric hypothesis:

Hypothesis (H). There exists an admissible phase $\psi(x, w)$ defined near $(x_0, w_0) \in \mathbb{C}^{n_x + N_w}$ with $\xi_0 = -\frac{2}{i}\frac{\partial\psi}{\partial x}(x_0, w_0)$, such that $\Lambda_\varphi \ \$ \ \Lambda_\psi \times \Lambda_{\varphi_1}$.

More explicitly, $\psi(x, w) + \varphi(x, y, \theta) + \varphi_1(y, \omega)$ must have a saddle at $(x_0, y_0, w_0, \theta_0, \omega_0)$, and Hypothesis (H) is equivalent to saying that

$$\varphi_2 := \varphi(x, y, \theta) + \varphi_1(y, \omega)$$

is an admissible phase with y, θ, ω as fiber variables. If one introduces the I-canonical relation $\Lambda'_\varphi = \{(x, \xi, y, -\eta);\ (x, \xi, y, \eta) \in \Lambda_\varphi\}$, then Hypothesis (H) gives that

$$T_{(x_0,\xi_0,y_0,\eta_0)}(\Lambda'_\varphi) \pitchfork \left(T_{(x_0,\xi_0)}(\mathbb{C}^{2n_x}) \times T(\Lambda_{\varphi_1})\right),$$

and, as observed by Hörmander [15], this transversality alone shows that $C_\varphi(\Lambda_{\varphi_1})$ is an (I-)Lagrangian manifold. Of course,

$$\Lambda_{\varphi_2} = \Lambda'_\varphi(\Lambda_{\varphi_1}).$$

We now let $u(y; h) = \int f(y, \omega; h)\, d\omega \in I_{\varphi_1}$ and $A(x, y; h) = \int a(x, y, \theta; h)\, d\theta \in I_\varphi$. Then, with Hypothesis (H),

$$Au(x; h) = \int A(x, y)u(y; h)\, dy = \iiint a(x, y, \theta; h)u(y, \omega; h)\, dy\, d\theta\, d\omega$$

is an element of I_{φ_2}. With ψ as in (H), that is to say $\Lambda_\psi \$ \Lambda_{\varphi_2}$, and normalized with a constant such that $\psi(x_0, w_0) + \varphi(x_0, y_0, \theta_0) + \varphi_1(y_0, \omega_0) = 0$, let

$$v(x; h) = \int g(x, w; h)\, dw \in I_\psi.$$

Then one can define

$$\begin{aligned}\langle v, Au\rangle &= \iiiint \int a(x, y, \theta; h)g(x, w; h)f(y, \omega; h)\, dx\, dy\, d\theta\, dw\, d\omega \\ &= \langle A, v \otimes u\rangle.\end{aligned}$$

It is clear that if $\tilde{\varphi}(x, y, \tilde{\theta})$ satisfies $\Lambda_\varphi = \Lambda_{\tilde{\varphi}}$, $\nabla\tilde{\varphi}(x_0, y_0, \tilde{\theta}_0) = \nabla\varphi(x_0, y_0, \theta_0)$, and $\tilde{\varphi}(x_0, y_0, \tilde{\theta}_0) = \varphi(x_0, y_0, \theta_0)$, and if $\tilde{A} \in I_{\tilde{\varphi}}$ satisfies $\tilde{A} \equiv A$, then $\tilde{A}u \equiv Au$. Similarly, if $\tilde{u} \in I_{\tilde{\varphi}}$ and $\tilde{u} \equiv u$, then $Au \equiv A\tilde{u}$. In fact,

$$\langle v, Au\rangle = \langle A, v \otimes u\rangle = \langle {}^tAv, u\rangle \equiv \langle {}^tAv, \tilde{u}\rangle = \langle v, A\tilde{u}\rangle.$$

We now write $I_{\Lambda_{\varphi_1}}, I_{\Lambda_\varphi}, \ldots$, in place of $I_{\varphi_1}, I_\varphi, \ldots$, where the phases are now normalized: $\varphi_1(y_0, \omega_0) = \varphi(x_0, y_0, \theta_0) = \cdots = 0$. Then each $A \in I_{\Lambda_\varphi}$ gives, with Hypothesis (H), a well-defined mapping (modulo the sign); $A : I_{\Lambda_{\varphi_1}} \to I_{\Lambda_{\varphi_2}}$. We now let $B \in I_{\Lambda_{\tilde{\varphi}}}$, where $\tilde{\varphi}(z, x, \tilde{\theta})$ is an admissible phase such that $(\tilde{\varphi}, \varphi_2)$ satisfies (H). Then we have an operator $B : I_{\Lambda_{\varphi_2}} \to I_{\Lambda_{\varphi_3}}$, where $\Lambda_{\varphi_3} = \Lambda'_{\tilde{\varphi}}(\Lambda_{\varphi_2})$. We can then form the composition $C = B \circ A : I_{\Lambda_{\varphi_1}} \to I_{\Lambda_{\varphi_3}}$. Here $C \in I_\psi$, where $\psi = \tilde{\varphi}(z, x, \tilde{\theta}) + \varphi(x, y, \theta)$ is an admissible phase with $(\tilde{\theta}, y, \theta)$ as fiber variables and with (ψ, φ_1) satisfying (H). Of course, $\Lambda'_\psi = \Lambda'_{\tilde{\varphi}} \circ \Lambda'_\varphi$.

Remark 11.23. In the case $n_x = n_y$, and when φ is pl.h. and Λ'_φ is the graph of a $\mathbb{C}$-canonical transformation κ, then Hypothesis (H) is automatic for each (normalized) admissible phase φ_1. In fact, it suffices to verify this in the linear case: Let $\psi(x)$ be pl.h. and such that $\Lambda_\psi \$ \kappa(\Lambda_{\varphi_1})$, that is, $\kappa(\Lambda_{\varphi_1})$ is $\Lambda_{-\psi}$–pseudoconvex. Let $[0, 1] \ni t \mapsto \varphi_{1,t}$ be a decreasing continuous deformation of admissible phases, with $\varphi_{1,0} = \varphi_1$ and $\varphi_{1,1}$ pl.h. Then $\kappa(\Lambda_{\varphi_{1,t}}) \leq \kappa(\Lambda_{\varphi_1})$, and hence $\Lambda_\psi \$ \kappa(\Lambda_{\varphi_{1,t}})$. Now since

$$\psi(x) + \varphi(x, y, \theta) + \varphi_{1,t}(y, \omega)$$

is non-degenerate for each $t \in [0, 1]$ and is of signature 0 for $t = 1$, $\psi + \varphi + \varphi_1$ is non-degenerate of signature 0, and we then have (H).

12. A Precise Study of Pseudodifferential Operators in a Complex Domain

(I've skipped ahead: this is really Lemma 12.2.)

We will also have to recall a version of Stokes' Formula. For $t \in [-\epsilon, \epsilon]$, $\epsilon > 0$, we let Γ_t be the contour given by $W \ni x \mapsto f(t,x) \in \mathbb{R}^n$, where W is an open neighborhood of $0 \in \mathbb{R}^k$ and f is of class C^1. We put $v = \frac{\partial f}{\partial t}(0,x)$, which we can consider as a section of the normal fiber to (the image of) Γ_0. We let ω be a differential k-form of class C^1 on $\mathbb{R}^n$ such that $\Pi_x(f^{-1}(\mathrm{supp}\omega)) \subset\subset W$, where $\Pi_x : [-\epsilon, \epsilon] \times W \to W$ is the natural projection.

Lemma 12.1. *In the situation above,*

$$\left(\frac{d}{dt}\right)\Big|_{t=0} \int_{\Gamma_t} \omega = \int_{\Gamma_0} v \lrcorner\, d\omega.$$

Proof. By Stokes' formula,

$$\int_{\Gamma_s} \omega - \int_{\Gamma_0} \omega = \int_0^s \int_{x \in W} f^*(d\omega).$$

One can write $f^* d\omega = g(t,x)\, dt \wedge dx_1 \wedge \cdots \wedge dx_n$, where g is continuous. Then

$$\left(\frac{d}{ds}\right)\Big|_{s=0} \int_{\Gamma_s} \omega = \int_W g(0,x)\, dx_1 \wedge \cdots \wedge dx_n.$$

Here $g(0,x)\, dx_1 \wedge \cdots \wedge dx_n = \frac{\partial}{\partial t} \lrcorner\, f^* d\omega = f^*(v \lrcorner\, \omega)$. □

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