COMMENTARY ON *MICROLOCAL ANALYTIC SINGULARITIES*

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1. H_φ, etc.

The "ϵ" in the definition of $H_\varphi^{\mathrm{loc}}(\Omega)$ allows us to neglect powers of h^{-1}. Here is a basic example:

Example 1.1. *We take* $\varphi(z) := \frac{1}{2}[(Im\, z)^2 - (Re\, z)^2]$. *Then for any* $k \in \mathbb{R}$ *we have*

$$u_k(z;h) := h^{-k} e^{-z^2/(2h)} \in H_\varphi^{loc}(\mathbb{C}^n).$$

We may also define equivalence in $H_\varphi^{\mathrm{loc}}(\Omega)$ as follows:

Definition 1.2. *For* $u, v \in H_\varphi^{loc}(\Omega)$, *we say* $u \approx v$ *if for all compact* $K \subset \Omega$ *there exist* $C_K > 0$ *and* $\delta_K > 0$ *such that*

$$|u(z;h) - v(z;h)| \leq C_K\, e^{(\varphi(z)-\delta_K)/h} \qquad \forall z \in K, \quad \forall h \in (0,1].$$

This definition is equivalent to the one given by Sjöstrand. That is, with the notation of [2], Section 1:

Exercise 1. $u \approx v$ *if and only if* $u \sim v$.

Proof. $\approx \Rightarrow \sim$: Let $K_j \subset \Omega$, $j = 1, 2, \ldots$, be a sequence of compact sets such that

$$K_j \subset K_{j+1}^\circ.$$

We may take each K_j to be, say, a union of cubes. Then, by hypothesis, for each K_j there exist $C_j > 0$ and $\delta_j > 0$ such that

$$|u(z;h) - v(z;h)| \leq C_j\, e^{(\varphi(z)-\delta_j)/h} \qquad \forall z \in K_j, \quad \forall h \in (0,1],$$

and we may of course assume that

$$\delta_1 \geq \delta_2 \geq \ldots > 0.$$

We first define φ_0 to be the function

$$\varphi_0(z) := \varphi(z) - \delta_j \qquad \text{for } z \in K_j \backslash K_{j-1}.$$

We then modify φ_0 in a neighborhood of ∂K_j intersected with K_j to obtain a continuous function φ_1 on Ω such that

$$\varphi_1 - \varphi_0 \geq 0 \qquad \text{and} \qquad \varphi_1 < \varphi \quad \text{on } \Omega.$$

Then

$$\begin{aligned} |u(z;h) - v(z;h)| &\leq C_j\, e^{\varphi_0(z)/h} \\ &\leq C_j\, e^{\varphi_1(z)/h} \qquad \forall z \in K_j \backslash K_{j-1}, \quad \forall h \in (0,1]. \end{aligned}$$

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Hence

$$|u(z;h) - v(z;h)| \leq \tilde{C}_j \, e^{\varphi_1(z)/h} \qquad \forall z \in K_j, \quad \forall h \in (0,1].$$

Thus $u - v \in H^{\mathrm{loc}}_{\varphi_1}(\Omega)$. As previously mentioned, we also have that φ_1 is continuous and $\varphi_1 < \varphi$, so φ_1 is thus a suitable function.

$\approx \Leftarrow \sim$: This is the easy direction. $\square$

Remark 1.3. Note that a classical formal analytic symbol has no "$e^{+\epsilon/h}$" in the estimate:

$$\begin{aligned} |a_{\tilde{\Omega}}(z;h)| &\leq C_{\tilde{\Omega}} \left(\frac{1 - (small)^{1/h}}{1 - e^{-1}} \right) \\ &\leq const. \,\text{(independent of h)} < \infty. \end{aligned}$$

This is important in the proof of Proposition 6.2.

Re: the proof of Proposition 1.2. We use twice that $\sum_{j=1}^{N} \chi_j = 1$ on $\overline{\tilde{\Omega}}$. For Hörmander's existence theorem, see Theorem 4.4.2 of [1]. The last line of the proof uses Cauchy's integral formula; see Theorem 2.2.3 of [1].

Example 1.4. *Here is a* non-*example of* Ω_t*'s: For* $0 \leq t \leq t_0 < \frac{2}{3}$*,*

$$\tilde{\Omega}^{n.e.}_t := \{(x,\xi) \in \mathbb{C}^{2n};\ |x|^2 + |\xi|^2 < t^4\}.$$

Example 1.5.

$$\Omega^1_t := \{(x,\xi) \in \mathbb{C}^{2n};\ |x| < t\}.$$

Example 1.6. *For any* $r_j > 0$*,* $j = 1, \ldots, n$*,*

$$\Omega^2_t := \{(x,\xi) \in \mathbb{C}^{2n};\ |x_j| < r_j + t\}.$$

We recall that a set $D \subset \mathbb{C}^n$ is called a polydisc if there are discs $D_1, \ldots, D_n$ in $\mathbb{C}$ such that

$$D = \prod_1^n D_j = \{z;\ z_j \in D_j,\ j = 1, \ldots, n\}.$$

A natural norm for points on a polydisc is given by

$$|x|_{p.d.} := \max_j |x_j|,$$

and of course we have

$$|x|_{p.d.} \leq |x| \leq \sqrt{n}|x|_{p.d.}.$$

The set $\prod_1^n \partial D_j$ is called the distinguished boundary of D and we denote it by $\partial_0 D$.

Exercise 2. *With the notation of* [2], *Section 1,*

$$||D_x^\alpha||_{t,s} \le \frac{C_0^{|\alpha|}|\alpha|^{|\alpha|}}{(t-s)^{|\alpha|}}.$$

Proof. Let $(x,\xi) \in \Omega_s$, $u \in B(\Omega_t)$, and let D be a polydisc in $\mathbb{C}^n$, centered at x and of radius $\frac{t-s}{\sqrt{n}}$. Hence $D \times \{\xi\} \subset \Omega_t$. The factor $\sqrt{n}$ is not necessary if Ω_t is a polydisc of radius *constant* $+\, t$—that is, as in Example 1.6 above, but where all the r_j's are equal.

By Cauchy's integral formula, for all multi-indices α we have

$$\frac{\partial^\alpha u}{\partial x^\alpha}(x,\xi) = \frac{\alpha!}{(2\pi i)^n}\int_{\partial_0 D}\frac{u(\zeta_1,\dots,\zeta_n,\xi)}{(\zeta_1-x_1)^{\alpha_1+1}\cdots(\zeta_n-x_n)^{\alpha_n+1}}\,d\zeta_1\dots d\zeta_n.$$

By the choice of D, we have

$$|\zeta_j - x_j| = \frac{t-s}{\sqrt{n}} \qquad \forall j,$$

so that

$$|D_x^\alpha u(x,\xi)| \le \frac{n^{|\alpha|/2}\alpha!}{(t-s)^{|\alpha|}}||u||_{B(\Omega_t)}.$$

This proves the result with $C_0 = \sqrt{n}$.

Again, if Ω_t is a polydisc of radius *constant* $+\, t$, the factor $n^{|\alpha|/2}$ may be omitted. □

Remark 1.7. Sjöstrand replaced $\alpha!$ by the often much bigger number $|\alpha|^{|\alpha|}$, and I believe there are two reasons for this. One is to emphasize that the bound k^k fits into the framework of classical formal analytic symbols, as defined earlier in the section. (And presumably k^k is the biggest number so that the general theory of classical formal analytic symbols works.) Another reason is so that we can have an estimate that only depends on the order of α, so that all derivatives of equal order can be treated equally. (However, even then $|\alpha|!$ would be a much smaller upper bound.)

Exercise 3. *"We leave as an exercise to show that r is in fact a classical analytic symbol and that if $\tilde{p}, \tilde{q}$ are local representatives of p, q and if $\tilde{r} = \tilde{p} \circ \tilde{q}$ is defined by (1.9), then $\tilde{r}$ is in fact a representative of r."*

Proof. Let $\tilde{\Omega} \subset\subset \Omega$ be a neighborhood of (x_0,ξ_0). With local representatives in this neighborhood, we have

$$\begin{aligned} r(x,\xi;h) &= \sum_\alpha \sum_{j,k} \frac{h^{|\alpha|+j+k}}{\alpha!} i^{-|\alpha|}\frac{\partial^\alpha p_j}{\partial \xi^\alpha}\frac{\partial^\alpha q_k}{\partial x^\alpha} \\ &= \sum_{m=0}^\infty h^m r_m(x,\xi), \end{aligned}$$

where

$$r_m(x,\xi) = \sum_{|\alpha|+j+k=m}\frac{1}{\alpha!} i^{-|\alpha|}\frac{\partial^\alpha p_j}{\partial \xi^\alpha}\frac{\partial^\alpha q_k}{\partial x^\alpha}. \tag{1.1}$$

By the given conditions on the p_j, q_k, and by the Cauchy integral formula, we have $C = C_{\tilde{\Omega}} > 0$ such that

$$\begin{aligned}\frac{1}{\alpha!}\left|\frac{\partial^\alpha p_j}{\partial \xi^\alpha}\right|\left|\frac{\partial^\alpha q_k}{\partial x^\alpha}\right| &\leq \alpha!\, C^{|\alpha|+1}\left(C^j j^j\right)\left(C^k k^k\right)\\ &\leq |\alpha|^{|\alpha|} C^{|\alpha|+j+k+1}(j+k)^{j+k}\\ &\leq C^{|\alpha|+j+k+1}(|\alpha|+j+k)^{|\alpha|+j+k}.\end{aligned}$$

Since there are $\leq (m+1)^{n+1}$ terms in (1.1), we then have

$$\begin{aligned}|r_m(x,\xi)| &\leq (m+1)^{n+1} C^{m+1} m^m\\ &\leq \tilde{C}^{m+1} m^m\end{aligned}$$

for some new constant $\tilde{C} = \tilde{C}(\tilde{\Omega}) > 0$. So indeed r is a classical formal analytic symbol. □

Exercise 4. *The operator associated to* $r = p \circ q$ *is*

$$C = A \circ B = \sum_{M=0}^{\infty} h^M C_M, \qquad \textit{where} \qquad C_M = \sum_{\nu+\mu=M} A_\nu \circ B_\mu.$$

Proof. By definition, the operator associated to r is

$$\begin{aligned}r(x,\xi + hD_x; h) &:= \sum_\beta \frac{1}{\beta!} r^{(\beta)}(x,\xi;h)(hD_x)^\beta\\ &= \sum_{M=0}^{\infty} h^M R_M,\end{aligned}$$

where

$$R_M = \sum_{j+k+|\alpha|+|\gamma|+|\delta|=M} \frac{1}{\alpha!\,\gamma!\,\delta!} i^{-|\alpha|} p_j^{(\alpha+\gamma)}(x,\xi)\frac{\partial^\alpha q_k^{(\delta)}}{\partial x^\alpha}(x,\xi) D_x^{\gamma+\delta}.$$

On the other hand, one may check that

$$C_M = \sum_{\nu+\mu=M} A_\nu \circ B_\mu$$

has the same expression, proving the result. □

In Theorem 1.5: Note that q is elliptic (?). (See the proof of Theorem 4.5.)

2. The Method of Stationary Phase in the Analytic Case

The error bound (2.14) is tricky at first glance, since we are using (2.13) with $2n$ replacing n. See later, when $L(dx)$ is Lebesgue measure on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$.

Exercise 5. *Deduce the following version of the Cauchy inequalities from the proof of Theorem 2.1:*

$$\frac{1}{k!}\left|\left(\frac{\Delta}{2}\right)^k u(0)\right| \leq C(n)(k+1)^{n/2}(k-1)!\,2^k \sup_{\tilde{B}} |u| \qquad \textit{for} \quad k \geq 1.$$

Proof. Later. □

Check the remark before Theorem 2.8, and check Remark 2.10.

Lemma 2.1. *Re* $\varphi(z) \geq C_\delta |z|^2$ *on* Γ_δ.

Proof. We have

$$|z|^2 \approx |x|^2 + \delta^2 |\varphi'(x)|^2,$$

and

$$\operatorname{Re}\varphi(z) \approx \operatorname{Re}\varphi(x) + \delta|\varphi'(x)|^2$$

when $0 < \delta \ll 1$, so we are to show that

$$\operatorname{Re}\varphi(x) + \delta|\varphi'(x)|^2 \gtrsim_\delta |x|^2.$$

By hypothesis, $\det \varphi''(0) \neq 0$, so we have

$$\begin{aligned}|\varphi'(x)| &= |\varphi''(0)x^2 + \mathcal{O}(x^3)| \\ &\gtrsim |x|^2.\end{aligned}$$

Hence

$$\operatorname{Re}\varphi(x) + \delta|\varphi'(x)|^2 \gtrsim \delta|x|^2, \tag{2.1}$$

proving the result. □

The same argument, leading to (2.1), shows that for $x \in \partial V_{\mathbb{R}}$,

$$\begin{aligned}\operatorname{Re}\varphi(z) &\geq \epsilon + C_\delta |z|^2 \\ &\geq \epsilon\end{aligned}$$

for some $\epsilon > 0$ independent of δ. Hence

$$\begin{aligned}I(h) &= \int_{V_{\mathbb{R}}} e^{-\varphi(x)/h} u(x)\,dx \\ &= \int_{\Gamma} e^{-\varphi(z)/h} u(z)\,dz + R(h),\end{aligned}$$

with

$$R(h) = \int_\Sigma e^{-\varphi(\eta)/h} u(\eta)\, d\eta.$$

Here Σ is parameterized by

$$\eta = x + \lambda\, \delta\, \overline{\varphi'(x)}$$

for $\lambda \in [0,1]$ and with $x \in \partial V_{\mathbb{R}}$. Hence

$$\begin{aligned} |R(h)| &\leq \int_\Sigma e^{-\operatorname{Re}\varphi(\eta)/h} |u(\eta)|\, d|\eta| \\ &\leq \frac{1}{\epsilon} e^{-\epsilon/h} \sup_U |u|. \end{aligned}$$

By the proof of Lemma 2.7 (Morse's Lemma), we see that $|z| \approx |\tilde{z}|$ in a neighborhood of 0, so we do indeed have (2.19).

Let $\tilde{z}(s)$ be a curve on $\tilde{\Gamma}$ such that $\tilde{z}(0) = 0$ and such that $\dot{\tilde{z}}(0) = t_x + it_y$. Then

$$\begin{aligned} (t_x + it_y)^2 &= \lim_{s\to 0} \frac{\tilde{z}(s)^2}{s^2} \\ &= \lim_{s\to 0} \frac{x(s)^2 - y(s)^2 + 2ix(s)y(s)}{s^2}, \end{aligned}$$

and so

$$t_x^2 - t_y^2 = \lim_{s\to 0} \frac{x(s)^2 - y(s)^2}{s^2}.$$

By (2.19) we have

$$x(s)^2 - y(s)^2 \geq C(x(s)^2 + y(s)^2) \geq 0 \qquad \forall s,$$

so we have shown that $t_x^2 - t_y^2 \geq 0$.

Note that the the phase is

$$-\operatorname{Re} \frac{\tilde{z}^2}{2} = -\frac{1}{2}(x^2 - y^2), \tag{2.2}$$

which clearly has a saddle point at 0, and the work above shows that $\tilde{\Gamma}$ passes through 0 and lives in the region where (2.2) is non-positive. (Even strictly negative? See below.) Draw the picture. This is where the name "saddle point method" comes from.

$W \subset \mathbb{R}^n$ is a neighborhood of 0. It's easy to see that $H(x)$ is real-analytic, from considering the construction of Γ and $\tilde{\Gamma}$.

Why do we have $\theta < 1$ and not the weaker condition $\theta \leq 1$? Does it follow from real-analyticity? He must be correct, because he says it again after the definition of good contour in Section 3.

3. "The Fundamental Lemma" and the Fourier Transform in the Complex Domain

Proposition 3.1 is easy to prove, but at least here is a slightly different way of presenting it:

Proposition 3.1. *Let $q(z)$ be a pl.s.h. quadratic form on $\mathbb{C}^n$...*

Proof. We write the positive and negative subspaces for a decomposition of q as L_+ and L_-, respectively. Since q is pl.s.h.,

$$0 \leq \ell(x) = \frac{1}{2}(q(x) + \Im q(x)) \qquad \forall x \in \mathbb{C}^n.$$

That is,

$$q(x) \geq -q(ix) \qquad \forall x \in \mathbb{C}^n. \tag{3.1}$$

Let $0 \neq x \in L_-$. Then (3.1) shows that $q(ix) > 0$. This proves (i).

To prove (ii), let $\tilde{L}_+$, $\tilde{L}_-$ be the corresponding subspaces for $\tilde{q}$. Since $\tilde{q} \leq q$,

$$\tilde{L}_+ \subset L_+ \qquad \text{and} \qquad L_- \subset \tilde{L}_-,$$

and so

$$\dim \tilde{L}_+ \leq \dim \tilde{L}_-.$$

This combined with part (i) gives equality, proving the result. □

Lemma 3.2. *If f is holomorphic, then $\frac{\partial Im\, f}{\partial x} = \frac{1}{2i}\frac{\partial f}{\partial x}$.*

Proof. Here is the shortest proof: Since f is holomorphic, we have

$$0 = \frac{\partial \overline{f}}{\partial x} = \frac{\partial \mathrm{Re}\, f}{\partial x} - i\frac{\partial \mathrm{Im}\, f}{\partial x}.$$

Hence

$$\frac{\partial f}{\partial x} = \frac{\partial \mathrm{Re}\, f}{\partial x} + i\frac{\partial \mathrm{Im}\, f}{\partial x} = 2i\frac{\partial \mathrm{Im}\, f}{\partial x}.$$

□

Lemma 3.3. *The fundamental lemma. He does mean for φ to be pl.s.h. in (x, y) and not just in y, though that is the part I understand the least. Also, the hypotheses seem to tacitly imply that $\nabla_y^2\varphi(0,0)$ is real-valued as a quadratic form.*

I'll fill in the details of the proof in the following claims.

Claim 3.4. *If $\psi(t,s)$ is a real C^∞ function defined near $(0,0) \in \mathbb{R}^{n'+n''}$, with a critical point at $(0,0)$ and such that $\nabla_s^2\psi(0,0)$ is non-degenerate, then, if we introduce the critical point $s(t)$ of*

$$s \mapsto \psi(t,s),$$

the function $f(t) = \psi(t, s(t))$ *has a critical point at* 0*. Moreover,* $\nabla_t^2 f(0)$ *is non-degenerate if and only if* $\nabla_{(t,s)}^2 \psi(0,0)$ *is non-degenerate, and*

$$sign \nabla_t^2 f(0) + sign \nabla_s^2 \psi(0,0) = sign \nabla_{(t,s)}^2 \psi(0,0).$$

Proof. First of all, we have the existence and smoothness of $s(t)$ by using the implicit function theorem. Indeed, since $\nabla_s^2 \psi(0,0)$ is non-degenerate, then for (t,s) near $(0,0)$ we have that $\nabla_s^2 \psi(t,s)$ is non-degenerate. Hence, by the implicit function theorem, the set

$$\frac{\partial \psi}{\partial s}(t,s) = 0$$

may locally be written as

$$\frac{\partial \psi}{\partial s}(t, s(t)) = 0.$$

Also, it is clear (at least, computationally) that $f(t) = \psi(t, s(t))$ has a critical point at 0.

To prove the "moreover" part, we make a change of variables in s. Let

$$\tilde{\psi}(t,s) := \psi(t, s + s(t)).$$

Then $s \mapsto \tilde{\psi}(t,s)$ has a critical point at $s = 0$ for all t. Also,

$$\frac{\partial^2 \tilde{\psi}}{\partial t \partial s}(0,0) = \frac{\partial^2 \psi}{\partial t \partial s}(0,0) + \frac{\partial^2 \psi}{\partial s^2}(0,0)\dot{s}(0).$$

But since $\frac{\partial \psi}{\partial s}(t, s(t)) = 0$ for all t, we have

$$\frac{\partial^2 \psi}{\partial t \partial s}(t, s(t)) + \frac{\partial^2 \psi}{\partial s^2}(t, s(t))\dot{s}(t) = 0 \qquad \forall t.$$

In particular, it is true for $t = 0$, which shows that

$$\frac{\partial^2 \tilde{\psi}}{\partial t \partial s}(0,0) = 0.$$

We also have

$$\tilde{f}(t) := \tilde{\psi}(t,0) = \psi(t, s(t)) = f(t),$$

so

$$\frac{\partial^2 f}{\partial t^2}(0) = \frac{\partial^2 \tilde{\psi}}{\partial t^2}(0,0).$$

Since we are given that

$$\frac{\partial^2 \tilde{\psi}}{\partial s^2}(0,0) = \frac{\partial^2 \psi}{\partial s^2}(0,0)$$

is non-degenerate, and since we just showed that $\frac{\partial^2 \tilde{\psi}}{\partial t \partial s}(0,0) = 0$, we thus have the result for $\tilde{\psi}$.

Now clearly $\frac{\partial^2 \tilde{\psi}}{\partial s^2}(0,0) = \frac{\partial^2 \psi}{\partial s^2}(0,0)$, and the other equivalences between the Hessians of ψ and $\tilde{\psi}$ follow from the coordinate invariance of the Hessian (at a critical point). Right? □

Proof. So far we have not used the pl.s.h. assumption. The rest of the proof doubtless follows from basic analysis of several complex variables. We reduce to the case of quadratic forms, and then we decrease to a pl.h. form by simply deleting the component of Levi type. □

Remark 3.5. The moral of the fundamental lemma seems to be: "We can decrease $f''(y_0)$ until it is a pl.h. form." (See the discussion of φ^* on page 15 and also the discussion after (4.13′).) The lemma then guarantees that all the necessary properties still hold for the resulting pl.h. form.

Since $\frac{\partial}{\partial x}(\varphi + \mathrm{Im}\, x \cdot \xi)(x(\xi), \xi) = 0$, we have

$$\frac{\partial \varphi^*}{\partial \xi}(\xi) = \frac{1}{2i} x(\xi).$$

To prove that

$$\varphi(x) = \text{v.c.}_{\xi}(\varphi^*(\xi) - \mathrm{Im}\, x \cdot \xi),$$

it's probably best to think of Λ_φ as facilitating this relationship.

In the proof of Theorem 2.8, Γ_δ is a good contour with respect to $-\mathrm{Re}\,\varphi$. That's what Morse's Lemma did for us! I think we only need *local* injectivity.

In showing that $U(x; h) \in H_{\Phi,0}$, we seem to be using the saddle point method.

In (3.1), the composed contour is indeed a good contour:

$$\varphi(y) - \mathrm{Im}\,(x - y) \cdot \xi - \varphi(x) \leq -C|y|^2 - C|\xi|^2.$$

To show that the point is a saddle point, it might be best to use the "$m_\pm = \max \dim_{\mathbb{R}} L$" formulation.

Lemma 3.6. *$\tilde{\Gamma}(x)$ is a good contour.*

Proof. First, recall that

$$\begin{aligned}\varphi(y) &= \varphi(x) + \frac{\partial \varphi}{\partial x}(x) \cdot (y - x) + \frac{\partial \varphi}{\partial \overline{x}}(x) \cdot (\overline{y - x}) + \mathcal{O}(|y - x|^2) \\ &= \varphi(x) + 2\mathrm{Re}\, \frac{\partial \varphi}{\partial x}(x) \cdot (y - x) + \mathcal{O}(|y - x|^2).\end{aligned}$$

Secondly, we must realize that the *general* good contour condition is

$$\text{“}\varphi(y) - \varphi(y_0) \leq -C|y - y_0|^2.\text{”}$$

Now

$$(y, \xi) \mapsto \varphi(y) - \mathrm{Im}\,(x - y) \cdot \xi$$

has a saddle at $y = x$, $\xi = \frac{2}{i}\frac{\partial\varphi}{\partial x}(x)$, so of course the critical value is $\varphi(x)$. Then on $\tilde{\Gamma}(x)$,

$$\begin{aligned}\varphi(y) - \mathrm{Im}\,(x-y)\cdot\xi - \varphi(x) &= \varphi(y) - \varphi(x) - 2\mathrm{Re}\,\frac{\partial\varphi}{\partial x}(x)\cdot(y-x) - C|x-y|^2 \\ &\leq M|x-y|^2 - C|x-y|^2 \\ &\leq -\frac{C}{2}|x-y|^2.\end{aligned}$$

This shows that

$$\varphi(y) - \mathrm{Im}\,(x-y)\cdot\xi - \varphi(x) \leq -C_0|y-x|^2 - C_0\left|\xi - \frac{2}{i}\frac{\partial\varphi}{\partial x}(x)\right|^2,$$

so we are done. □

4. ΨDO's and Fourier Integrals in the Complex Domain

After (4.2), the "$+\epsilon/h$" from estimating a gets absorbed into the "$-(R-C)/h$" term.

"We then have an obvious continuity result on the spaces L^2_φ." It follows from Folland's Theorem 6.18, which is easy to prove once you know the "Schur norm" trick. We then get:

$$||Au||_{L^2_\varphi} \leq C_0 h^{-n/2}||u||_{L^2_\varphi}.$$

Recall that $L^2_\varphi(\Omega) = L^2(\Omega;\, e^{-2\varphi/h}L(dx))$ (as defined in the proof of Proposition 1.2).

In the proof of Lemma 4.1, recall that a is an analytic symbol defined in a neighborhood of $(x_0, x_0, \xi_0) \in \mathbb{C}^n$; that is, $a \in H^{\mathrm{loc}}_0(\Omega)$. So u is of class H^{loc}_φ near $(0, \theta_0)$.

I still need to check (4.6).

I still need to check that $|x|^2 + |y - y(x)|^2 \sim |x|^2 + |y|^2$ on p.19.

In (4.13′):

$$\left(x, \frac{2}{i}\frac{\partial\psi}{\partial x}, y, -\frac{2}{i}\frac{\partial\psi}{\partial y}\right) = \left(x, \frac{\partial\varphi}{\partial x}, y, -\frac{\partial\varphi}{\partial y}\right),$$

and I think he means that

$$\left(x, \frac{\partial\varphi}{\partial x}\right) \mapsto \left(y, -\frac{\partial\varphi}{\partial y}\right)$$

is a symplectomorphism in the sense that, writing $\xi = \frac{\partial\varphi}{\partial x}$, $\eta = -\frac{\partial\varphi}{\partial y}$, we have

$$dx \wedge d\xi = \frac{\partial^2\varphi}{\partial x\partial y}dx \wedge dy = dy \wedge d\eta.$$

After (4.13′): At that one point, $\left(y, -\frac{2}{i}\frac{\partial\psi}{\partial y}\right) = \left(y, \frac{2}{i}\frac{\partial f}{\partial y}\right)$. That is, $\frac{\partial}{\partial y}(\psi + f) = 0$ at that one point. Since $\nabla_y^2(\psi + f)$ is non-degenerate, we can use the implicit function theorem:

$$\frac{\partial\psi}{\partial y}(x, y) + \frac{\partial f}{\partial y}(y) = 0$$

is given by

$$\frac{\partial\psi}{\partial y}(x, y(x)) + \frac{\partial f}{\partial y}(y(x)) = 0.$$

So $y(x)$ is the critical point of

$$y \mapsto \psi(x, y) + f(y).$$

Hence

$$g(x) = \psi(x, y(x)) + f(y(x))$$

by definition. Thus

$$\kappa : \quad \left(y, \frac{2}{i}\frac{\partial f}{\partial y}\right) = \left(y, -\frac{2}{i}\frac{\partial\psi}{\partial y}\right) \mapsto \left(x, \frac{2}{i}\frac{\partial\psi}{\partial x}\right)$$

and

$$\begin{aligned} \frac{\partial g}{\partial x}(x) &= \frac{\partial\psi}{\partial x}(x, y(x)) + \frac{\partial\psi}{\partial y}(x, y(x))\frac{\partial y}{\partial x} + \frac{\partial f}{\partial y}(x, y(x))\frac{\partial y}{\partial x} \\ &= \frac{\partial\psi}{\partial x}(x, y(x)). \end{aligned}$$

So indeed

$$\kappa : \quad \left(y, \frac{2}{i}\frac{\partial f}{\partial y}(y)\right) \mapsto \left(x, \frac{2}{i}\frac{\partial g}{\partial x}(x)\right)$$

where $y = y(x)$ is the critical point. (Amazing?) That is, $\kappa(\Lambda_f) = \Lambda_g$.

When he says “which is equivalent to the fact that the map $\kappa(\Lambda_f) \ni (x, \xi) \mapsto x \in \mathbb{C}^n$ is a local diffeomorphism,” the point is that we can go backwards... (Right?)

φ is acting like a mirror: from x to y and back.

In the proof of Theorem 4.5, I guess we’re looking for an *elliptic* classical analytic symbol of order zero b (and $\tilde{b}$). Otherwise we wouldn’t “recognize here an elliptic classical pseudodifferential operator.” Maybe note this is Theorem 1.5: the symbol q is elliptic (?).

“as we saw earlier in the section...” I think he’s referring to Remark 4.3 which I admit I didn’t read. It seems trivial.

5. ΨDO’s in the Real Domain and Resolutions of the Identity

In the second sentence, he seems to be referring to Hörmander’s approach of using carefully chosen cut-off functions to define WF_a (see p.283 of Hörmander Vol.I).

The operators defined in this section are used in the proof of Proposition 6.2. In particular, that proof illustrates the role played by the "extra" variables y in (5.1) and (5.2).

The point of Section 5 seems to be: How to take an operator from Section 4 and turn it into an operator

$$\mathcal{D}'(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n).$$

He calls this process the "realization" of the operator, in the sense of "real-ize." We don't get to this until (5.9). The rest seems to be set-up.

The point of Section 5 seems to be: To define our "FIO with complex phases" for distributions on $\mathbb{R}^n$. We do this by putting further conditions on the phase and by introducing cutoffs in a simple way. The point of the cutoff is so that we can define it on distributions in $\mathcal{D}'(Y)$. If we didn't have it, we might only be able to define it on $\mathcal{S}'(Y)$.

The main point about A and A^V: they act equivalently on complex geometric optics functions. Think of the "black box" point of view as stated by Guillemin (Intro to *FIO: Past and Present* book) and by Meyer in his Wavelets book. For Au, we are integrating over $\Gamma(x)$, which is of real dimension $3n$. For $A^V u$ we are also integrating over a "contour" of real dimension $3n$. So apparently given such a phase φ, satisfying (5.1) and (5.2), deforming the contour to the real domain gives an operator, A^V, which is equivalent to A when acting on complex geometric optics functions.

At first the factor $h^{-3n/2}$ seems irrelevant in the definition of A in (5.8). After all, $h^{-3n/2}$ can just be incorporated into the analytic symbol a. However, in Section 6 we will restrict to classical analytic symbols a, and for such symbols there is no "$e^{+\epsilon/h}$" in the estimate. He is writing $h^{-3n/2}$ to set up for Section 6, and at any rate it emphasizes the fact that we are integrating over $3n$ real variables, so that the "inverse" of A would likely have the same factor, as is the case with the Fourier transform.

After (5.10): By definition, $\sigma_A = 1$ means

$$A\left(e^{i(\cdot)\xi/h}\right) = e^{ix\xi/h}.$$

So A is apparently the identity on Fourier transforms, hence on, e.g., $\mathcal{S}'$ (right?). We saw that A only depends on σ_A (up to equivalence), and the same is apparently true for A^V. Since morally (and maybe actually?) the identity operator I has $\sigma_I = 1$, we should have that if $\sigma_A = 1$ then $A^V \equiv I$. For a use of these "partitions of unity," see the proof of Proposition 7.4.

To prove the second inequality in (5.12), we first note the elementary inequality

$$|\vec{a} + \beta\vec{b}|^2 + |\vec{b}|^2 \geq (1 - \beta^2(\epsilon^{-2} - 1))|\vec{b}|^2 + (1 - \epsilon^2)|\vec{a}|^2$$

for any $\vec{a}, \vec{b} \in \mathbb{R}^n$, any $\beta \in \mathbb{R}$, and any $\epsilon \in \mathbb{R}$. In particular, for any $\beta \in \mathbb{R}$ we have

$$|\vec{a} + \beta\vec{b}|^2 + |\vec{b}|^2 \approx_\beta |\vec{a}|^2 + |\vec{b}|^2.$$

Of course, it suffices to prove these estimates for $n = 1$. Then for (5.12) we have

$$\begin{aligned} ... &\geq |y - \alpha_x|^2 + |x - \alpha_x|^2 + \tilde{\chi}|\partial_y \varphi + \alpha_\xi + \partial_y \psi(y) - \partial_y \psi(x) + \partial_x \psi(x) - \alpha_\xi|^2 \\ &= |y - \alpha_x|^2 + |x - \alpha_x|^2 + \tilde{\chi}|\mathcal{O}(1)(x - \alpha_x, y - \alpha_x) + \partial_x \psi(x) - \alpha_\xi|^2 \\ &\geq C'(|y - \alpha_x|^2 + |x - \alpha_x|^2 + |\alpha_\xi - \partial_x \psi(x)|^2). \end{aligned}$$

The point is that ψ' is now evaluated at x. We only care about points near $\nabla \overline{V}$ since otherwise $\operatorname{Im}(\varphi + \psi) > 0$ uniformly. Hence we may assume $\tilde{\chi} \geq c_0 > 0$. This finishes the proof of (5.12).

Now we use (5.12) to show that $\Gamma_\delta(x_0)$ is a good contour. When $x = x_0$, (5.12) says

$$\begin{aligned} \operatorname{Im}(\varphi(x_0, \tilde{y}, \alpha) + \psi(\tilde{y})) &\geq C'(|y - \alpha_x|^2 + |\alpha_x - x_0|^2 + |\alpha_\xi - \xi_0|^2) \\ &= C'(|y - x_0 + x_0 - \alpha_x|^2 + |\alpha_x - x_0|^2 + |\alpha_\xi - \xi_0|^2) \\ &\geq C''(|y - x_0|^2 + |\alpha_x - x_0|^2 + |\alpha_\xi - \xi_0|^2). \end{aligned}$$

Hence $\Gamma_\delta(x_0)$ is a good contour.

I still need to check the last paragraph of the proof of Lemma 5.1.

See how he uses resolutions of the identity in the proof of Proposition 7.4. The main property of ROI's seems to be: Say $\sigma_A = 1$. Then $\sigma_{B \circ A} = \sigma_B$ and $\sigma_{A \circ B} = \sigma_B$ by the usual formulas. Hence $A \circ B \equiv B$ and $B \circ A \equiv B$ (right?). Then, morally, when B acts on $I \equiv \int \Pi_{\alpha,h}\, d\alpha$, it replaces $e^{i\varphi/h} a$ by its own phase and amplitude, like a bird stealing another's nest

6. The Analytic Wavefront Set

The point of this section is to give an FBI-like characterization of WF_a (for elements of $\mathcal{D}'(\mathbb{R}^n)$) in a general context. In Section 7 we impose stronger (?) conditions on the phase to get good geometric properties.

In the proof of Proposition 6.2, Au is an integral over a good contour $\Gamma(x)$, which is "found without difficulty" as in the previous section.

In the proof of Proposition 6.2, we use that A admits a parametrix to get $\tilde{b}$ such that

$$A(\tilde{b}(\cdot, \alpha;\, h) e^{i\tilde{\varphi}/h}) \equiv \tilde{a}(x, \alpha;\, h) e^{i\tilde{\varphi}(x,\alpha)/h}.$$

Once we have that expression, we can use Lemma 5.1. Very cool!

In the proof of Proposition 6.2, the expression for Au should have the term $\overline{\varphi(\overline{y}, \overline{\alpha})}$ in the phase. This is also what appears in the expression for f, where Sjöstrand correctly writes it as $\overline{\varphi}(x, \beta)$; I am just adding all the bars for emphasis. After all, among other things, the phase is supposed to be analytic. We can easily check that (5.1) is satisfied, and (5.2) is satisfied because that condition is on the real domain.

In the proof of Proposition 6.2, in the definition of f, every parameter is real, so

$$\begin{aligned} |f(\alpha,\beta;h)| &\leq h^{-3n/2}\int e^{-C(|x-\beta_x|^2+|x-\alpha_x|^2)/h}|\tilde{b}(x,\alpha;h)|\,dx \\ &\leq C_0 h^{-3n/2}\int e^{-C(|x-\beta_x|^2+|x-\alpha_x|^2)/h}\,dx \\ &\leq C_0 h^{-3n/2}\int e^{-C|x|^2/h}\,dx \\ &= C_{00}h^{-n}. \end{aligned}$$

We have used the fact that for a classical analytic symbol, there is no "$e^{+\epsilon/h}$" in the estimate. (See Remark 1.3.) So indeed f is of temperate growth.

Now that we have proven Proposition 6.2, we can show that WF_a is conic. The proposition shows that we may take $a \equiv 1$ and the phase

$$\begin{aligned} \varphi(y,\alpha) &= \varphi(y,x,\xi) \\ &= -(x-y)\xi + i(x-y)^2/2, \end{aligned}$$

where we are writing $\alpha = (x,\xi)$. Suppose that $(x_0,\xi_0) \in \complement WF_a(u)$. Then

$$\int e^{-i(x-y)\xi/h-(x-y)^2/2h}\chi(y)\overline{u(y)}\,dy$$

is exponentially decreasing when $h \to 0$, uniformly for α in a real neighborhood of (x_0,ξ_0). Of course this is equivalent to

$$\int e^{i(x-y)\xi/h-(x-y)^2/2h}\chi(y)u(y)\,dy$$

being exponentially decreasing when $h \to 0$, uniformly for α in a real neighborhood of (x_0,ξ_0). Now let $\lambda > 0$. We define

$$\psi(y,\alpha) := -(x-y)\xi + i(x-y)^2/2\lambda.$$

This is an equivalent phase, in the sense of Proposition 6.2. That is, we have that

$$\int e^{-i(x-y)\xi/h-(x-y)^2/(2\lambda h)}\chi(y)\overline{u(y)}\,dy$$

is exponentially decreasing when $h \to 0$, uniformly for $\alpha = (x,\xi)$ in a real neighborhood of (x_0,ξ_0). Let $\tilde{h} = \lambda h$ and $\tilde{\xi} = \lambda\xi$. Then

$$\int e^{-i(x-y)\tilde{\xi}/\tilde{h}-(x-y)^2/2\tilde{h}}\chi(y)\overline{u(y)}\,dy$$

is exponentially decreasing when $\tilde{h} \to 0$, uniformly for $\alpha = (x,\tilde{\xi}/\lambda)$ in a real neighborhood of (x_0,ξ_0), that is, for $\tilde{\alpha} := (x,\tilde{\xi})$ in a real neighborhood of $(x_0,\lambda\xi_0)$. Hence we are back in the case of the standard phase, $\varphi(y,\tilde{\alpha})$, which shows that $(x_0,\lambda\xi_0) \in \complement WF_a(u)$. Hence WF_a is conic.

The main ideas in the proof of Theorem 6.3: (1) If u is analytic in a neighborhood of x_0 then we can deform the contour in the FBI integral. It is the usual contour deformation. (2) Going in the other direction, we write u as essentially u convolved with an analytic reproducing kernel. We are integrating out the ξ, which is where we use the hypothesis.

In the proof of Theorem 6.3: the fact that a is analytic relies on the fact that we are looking at a (small) neighborhood of $\xi_0 \in \mathbb{R}^n \backslash \{0\}$.

The last step of Theorem 6.3: Note that $a(x, \xi/h) = a(x, \xi)$ for all h and that $\tilde{a}(x, y, \xi) := a(x - y, \xi)$ is in fact a classical analytic symbol defined near $(x, y, \xi) = (x_0, x_0, \xi_0)$. (Restricting the neighborhood, we have $a(x - y, \xi) \approx 1$.) Let $h := |\xi|^{-1}$ and $\xi_0 = \xi/|\xi|$. The hypothesis says that the integral decreases exponentially in a real neighborhood of (x_0, ξ_0). (Technically, we have to take the complex conjugate, but for the phase and the amplitude we can replace ξ by $-\xi$, so it's no big deal.) Moreover, Proposition 6.2 says that we can replace a by $a \equiv 1$. (And, throughout, we may assume that $u \in \mathcal{E}'(X)$, since the definitions of SS_a and WF_a involve a cutoff function anyway.) That is,

$$\int e^{i(x-y)\xi_0/h - (x-y)^2/(2h)} u(y)\, dy$$

decreases exponentially in a real neighborhood of (x_0, ξ_0). Precisely as in Martinez's Remark 3.2.4 (with no change at all!), we have that the integral decreases exponentially in a *complex* neighborhood of (x_0, ξ_0). Recall that Martinez's remark says

$$Tu(x + it, \xi + i\tau) = e^{t^2/2h + \tau^2/2h - \xi(t+i\tau)/h} Tu(x + \tau, \xi - t),$$

which shows that $|Tu|$ is exponentially small in a real neighborhood of (x_0, ξ_0) if and only if it is exponentially small in a complex neighborhood of (x_0, ξ_0). I don't see why Sjöstrand says "Proposition 6.2 (and its proof) shows that..." Why can't we just use the *statement* of the proposition?

If we were only considering the standard FBI transform (as Martinez does), the proof of Theorem 6.3 would be entirely self-contained. No need to read the rest of the book!

For the harder direction of Theorem 6.3, compare with Theorem 1.6 in Delort's book. Delort also gives credit to Lebeau for the "reproducing" result but cites Hörmander's presentation in ALPDO.

Remark 6.1. The same argument almost works for the Fourier transform

$$F(x, \xi; h) = \int e^{i(x-y)\xi/h} u(y)\, dy.$$

Well, it's almost the Fourier transform; I'm including the x so that it resembles the FBI transform. Note that with $\varphi(y, x, \xi) := (x - y)\xi$ we have $\varphi = 0$ for $y = x$, and we have $\varphi'_y = -\xi$. However, we no longer have the Gaussian property (6.2).

Suppose that u is both compactly supported and analytic (!). By that same contour deformation, $y \mapsto y - i\epsilon\chi(y)\xi$, we find that $|F(x, \xi; h)| \leq C_0 e^{-c\xi^2/h}$.

Now suppose that u is compactly supported and that $|F(x, \xi; h)| \leq C_0 e^{-c\xi^2/h}$. Then by Fourier inversion we have

$$u(x) = (2\pi h)^{-n} \int F(x, \xi; h)\, d\xi.$$

By the hypothesis on F, we can differentiate under the integral sign, even for x in a complex neighborhood of "x_0":

$$|u(x)| \leq C_{00}(2\pi h)^{-n} \int e^{-(\operatorname{Im} x)\cdot\xi/h - C\xi^2/h}\, d\xi,$$

and by the hypothesis on F this is fine for $|\xi|$ large. Hence u is analytic.

Of course, in all of this I was sloppy, assuming u to be compactly supported when I needed it and analytic when I needed it. I really need to take u to be analytic near x_0, and then use a cutoff function to localize to the set where u is analytic. So I would really want to study

$$F_2(x, \xi; h) := \int e^{i(x-y)\xi} \chi(y) u(y)\, dy.$$

But for this definition to be independent of χ (modulo exponentially decreasing terms), I need the Gaussian factor. This is the reason for the condition (6.2). That is, the main feature of the FBI transform seems to be that it allows cutoff functions when working modulo exponentially decreasing terms.

If/when teaching a class on this, maybe pose this as a thought exercise for the students: "Does the same argument hold for the Fourier transform?"

7. The FBI Transform

Example 7.1. The basic example of a Bargmann-FBI transform: Let

$$(z_0, y_0) = (y_0 - i\eta_0, y_0) \qquad \text{and} \qquad \varphi(z, y) = \frac{i}{2}(z - y)^2.$$

Writing $z = x - i\xi$, we have

$$\varphi_1(z, y) = \frac{1}{2}(\xi^2 - (x - y)^2).$$

This has a maximum at $y(z) = x$, so $\Phi(z) = \frac{1}{2}\xi^2$. The mapping in Lemma 7.1 is then the trivial map

$$z = x - i\xi \mapsto (x, \xi).$$

Say $a \equiv 1$ and say $u \in \mathcal{S}'$ (so that we don't need a cutoff function). Then the corresponding Bargmann-FBI transform is

$$Tu(z; h) = e^{\xi^2/2h} \int e^{i(x-y)\xi/h - (x-y)^2/2h} u(y)\, dy.$$

There seems to be an inconsistency between the choice of phase in Definition 6.1 and in (7.4). I feel like the signs are different in (6.1) and (7.1), since there's that blasted complex conjugation in Definition 6.1. Basic question: What is the relationship between $WF_a(u)$ and $WF_a(\overline{u})$.

After Lemma 7.1:

Claim 7.2. $T_x(\Gamma_y) = \{t_x \in \mathbb{C}^n;\ \varphi''_{yx} t_x \in \mathbb{R}^n\}$.

Proof. For $x \in \mathbb{C}^n$ near x_0 we have $\nabla_y \varphi_1(x, y(x)) = 0$; that is,

$$\frac{\partial \varphi}{\partial \mathrm{Re}\, y}(x, y(x)) = \frac{\partial \varphi}{\partial y}(x, y(x)) \in \mathbb{R}^n.$$

Let $\gamma(s)$ be a curve in Γ_y. Then

$$\sum_{k=1}^{\infty} \frac{\partial^2 \varphi}{\partial y_j \partial x_k}(x, y)\dot{\gamma}_k(0) \in \mathbb{R} \qquad \forall j,$$

where $\gamma(0) = x$.

For the opposite inclusion, we use Lemma 7.1. □

Claim 7.3. $\Phi(x) \geq \varphi_1(x, y) + Cd(x, \Gamma_y)^2$, *for* $C > 0$. *(Recall: in this expression,* $y \in \mathbb{R}^n$*; after all,* Γ_y *is only defined for* $y \in \mathbb{R}^n$*.)*

Proof. We know that $\frac{\partial \varphi}{\partial y}(x, y(x))$, y, and $y(x)$ are all real. Also, $\mathrm{Im}\, \varphi''_{yy}(x, y(x)) > 0$, so

$$\mathrm{Im}\,(\varphi(x, y) - \varphi(x, y(x))) \geq C_0(y - y(x))^2.$$

Moreover, it is clear from the picture that

$$(y - y(x))^2 \approx d(x, \Gamma_y)^2.$$

To be precise, let $x_0 \in \Gamma_y$ be such that

$$|x - x_0| = d(x, \Gamma_y).$$

But then

$$y(x) = y(x_0) + \frac{\partial y}{\partial x}(x_0)(x - x_0) + \ldots$$

Since $\frac{\partial y}{\partial x}(x_0)$ is not orthogonal to $(x - x_0)$ (in fact, it's probably parallel–just look at the picture), we are done. □

In Proposition 7.2, at first it looks odd that we're saying "$(y(x_1), \eta(x_1)) \notin WF_a(u)$," whereas we're considering Tu at x_1 (see also Definition 6.1). But really it makes sense: we want $WF_a(u)$ to live in u's world, and we want to determine it by measuring in Tu's world. And T can be viewed as a Fourier integral operator with associated complex canonical transformation κ (defined above (7.4)). That is, x_0 and (y_0, η_0) are as in Lemma 7.1, so $x_1 \mapsto (y(x_1), \eta(x_1))$ is part of the diffeomorphism (between spaces of real dimension $2n$) in Lemma 7.1. But now we extend to a *complex* neighborhood of (y_0, η_0) and even add "directions" on the x-side. We now define it in the other direction (no big deal):

$$\kappa : \quad (y, -\frac{\partial \varphi}{\partial y}) \mapsto (x, \frac{\partial \varphi}{\partial x}).$$

It is now a local complex canonical diffeomorphism between spaces of real dimension $4n$. If we restrict to the real domain, it is a diffeomorphism onto the x-space (i.e. the first component), as we saw in Lemma 7.1. In particular, on the real domain it just says

$$(y(x_1), -\frac{\partial \varphi}{\partial y}(x_1, y(x_1))) = (y(x_1), \eta(x_1)) \mapsto (x_1, \frac{2}{i}\frac{\partial \Phi}{\partial x}(x_1)).$$

In Snowbird, someone asked me why we have the weight in the definition of WF_a. At the time, thinking on my feet, I spoke about the FBI transform being a unitary operator on L^2 and the Bargmann transform being a unitary operator on that *weighted* L^2 space. But also keep in mind that

$$(2\pi h)^{-n/2} \int e^{-(x-i\xi-y)^2/(2h)} dy = 1$$

for any $x \in \mathbb{R}^n$ and any $\xi \in \mathbb{R}^n$ (in fact, also for any $\xi \in \mathbb{C}^n$).

After Proposition 7.2, the idea seems to be: We can't expect $u \equiv STu$ (anyway, "$\equiv$" here is meaningless, since $u \in \mathcal{D}'$), but at least we can get $\tilde{T}u \equiv \tilde{T}STu$.

Re: (7.5): If the integral is over $\mathbb{R}^n$, we've got serious problems, since

$$-i(-i\varphi_1(x,y) + i\Phi(x)) = \Phi(x) - \varphi_1(x,y).$$

So we need to pick a contour that is as good as possible. However, even Γ_x might not be a good contour–it might not give exponential decrease away from the saddle point. But it *is* good if instead we act on H_Ψ, where $\Psi \le \Phi$, $\Psi(x_0) = \Phi(x_0)$, and $\nabla^2\Psi(x_0)|_{\Gamma_{y_0}} < \nabla^2\Phi(x_0)|_{\Gamma_{y_0}}$. After all, $\Psi \le \Phi$ and $\Psi(x_0) = \Phi(x_0)$ imply that $\Psi'(x_0) = \Phi'(x_0)$, by considering Taylor series, and then

$$\Psi(x) - \Phi(x) = \frac{1}{2}(\Psi''(x_0) - \Phi''(x_0))(x - x_0)^2 + \mathcal{O}((x - x_0)^3).$$

Then

$$\int_{\Gamma_{y_0}} e^{-i(\varphi(x,y)+i\Psi(x))/h} b\, e^{-\Psi(x)/h} v\, dx$$

is well-defined.

The issue now is to extend the definition of $\tilde{T}S$ (and not S—remember, that was bad on H_Φ) to $H_\Phi \to H_{\tilde{\Phi}}$.

Before Prop. 7.4 it is correct as written: $(TS)u \sim u$ in $H_\Phi^{\mathrm{loc}}(X)$. That is, he's talking about the special case when $\tilde{T} = T$.

The main point of Prop. 7.4 is that we are acting on $\mathcal{D}'(Y)$ now...

After Lemma 7.5: The critical point of $y \mapsto \varphi_1(x,y) - \Phi(x)$ is $y = y(x)$ with critical value zero. The critical point of

$$y \mapsto \varphi_1(x,y) + \psi(y) - \Phi(x)$$

is, by definition, $y = \tilde{y}(x)$, with critical value $\psi^*(x) - \Phi(x)$, again by definition.

Appendix A. Additional Comments

The real heart of the book seems to be Sections 4, 5, and 6.

The main use of plurisubharmonicity seems to be in using Lemma 3.2 (hence also (4.13′), (4.15), etc.). Moral: If we follow the critical point, it remains a saddle.

Don't forget that if φ is holomorphic then

$$\frac{\partial\varphi}{\partial z} = \frac{1}{i}\frac{\partial\varphi}{\partial \mathrm{Im}\, z}.$$

I hope I never omitted the i anywhere... I only remember writing

$$\frac{\partial\varphi}{\partial z} = \frac{\partial\varphi}{\partial \mathrm{Re}\, z},$$

which of course is ok.

References

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