NOTES ON THE CALDERÓN PROBLEM WITH PARTIAL DATA

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Abstract. These notes are my commentary on the paper "The Calderón Problem with Partial Data" by Kenig, Sjöstrand, and Uhlmann.

1. One of Calderón's Main Characteristics

In the following pages, I give a nearly line-by-line discussion of the paper "The Calderón Problem with Partial Data" by Kenig, Sjöstrand, and Uhlmann [14]. But before I begin, it is best to reflect on Alberto P. Calderón himself, and the way he worked. We recall that "one of Calderón's main characteristics [is that] he always sought his own proofs, developed his own methods. From the start, Calderón worked in mathematics that way: he rarely read the work of others farther than the statements of theorems, and after grasping the general nature of the problem, went ahead by himself. In this process, Calderón not only rediscovered results, but added new insights to the subject" [4]. For those who wish to consider the problem themselves before reading about someone else's methods, in this brief section I only state the main problem.

Let $\Omega \subset\subset \mathbb{R}^n$ be a bounded open connected set with, say, C^∞ boundary. For $q \in L^\infty(\Omega)$ we consider the operator $-\Delta + q : L^2(\Omega) \to L^2(\Omega)$ with domain $H^2(\Omega) \cap H_0^1(\Omega)$, and we assume that

$$0 \text{ is not an eigenvalue of } -\Delta + q : H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega).$$

Under this assumption, we have a well-defined Dirichlet-to-Neumann (DN) map

$$\mathcal{N}_q : H^{1/2}(\partial\Omega) \ni v \mapsto \partial_\nu u|_{\partial\Omega} \in H^{-1/2}(\partial\Omega),$$

where ν denotes the exterior unit normal and u is the unique solution in

$$H_\Delta(\Omega) := \{u \in H^1(\Omega);\ \Delta u \in L^2(\Omega)\}$$

of the problem

$$(-\Delta + q)u = 0 \ \text{ in } \Omega,\ u|_{\partial\Omega} = v.$$

Question: Let q_1, q_2 be two functions as above. Given two subsets $\Gamma_1, \Gamma_2 \subset \partial\Omega$, we would like to say that if

$$\mathcal{N}_{q_1} u = \mathcal{N}_{q_2} u \text{ in } \Gamma_1, \qquad \text{for all } u \in H^{1/2}(\partial\Omega) \cap \mathcal{E}'(\Gamma_2),$$

then $q_1 = q_2$. What are conditions on Γ_1 and Γ_2 such that this is true? ■

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2. Introduction

In medical imaging and geophysics, one wishes to determine the electrical conductivity of a body be making voltage and current measurements at the boundary of the body. This is known as Calderón's problem, after Calderón's seminal paper [2]. The problem may be reduced to studying the Dirichlet-to-Neumann map associated to the Schrödinger equation (see, for example, [14] or [16]), so in this paper we restrict to that formalism.

In these notes we study the Calderón problem with partial data, following Kenig, Sjöstrand, and Uhlmann [14], the main result being Theorem 2.1; however, one would like even weaker conditions on the data, in which case the problem is not fully resolved. The problem with full data was first proven by Sylvester and Uhlmann [19]. We also note that the problem with full data is equivalent to the inverse scattering problem at fixed energy (see Section 29 of Eskin's lecture notes [9]).

We let $\Omega \subset\subset \mathbb{R}^n$ be an open connected set with C^∞ boundary and assume that $n \geq 3$. For $q \in L^\infty(\Omega)$ we consider the operator $-\Delta + q : L^2(\Omega) \to L^2(\Omega)$ with domain $H^2(\Omega) \cap H^1_0(\Omega)$ as a bounded perturbation of minus the usual Dirichlet Laplacian. Then $-\Delta + q$ has discrete spectrum, and we assume that

$$0 \text{ is not an eigenvalue of } -\Delta + q : H^2(\Omega) \cap H^1_0(\Omega) \to L^2(\Omega). \tag{1}$$

Under this assumption, we have a well-defined Dirichlet-to-Neumann (DN) map

$$\mathcal{N}_q : H^{1/2}(\partial\Omega) \ni v \mapsto \partial_\nu u|_{\partial\Omega} \in H^{-1/2}(\partial\Omega),$$

where ν denotes the exterior unit normal and u is the unique solution in

$$H_\Delta(\Omega) := \{u \in H^1(\Omega);\ \Delta u \in L^2(\Omega)\}$$

of the problem

$$(-\Delta + q)u = 0 \ \text{ in } \Omega,\ u|_{\partial\Omega} = v.$$

Bukhgeim and Uhlmann [1] used a slightly different definition of H_Δ (with $H^1(\Omega)$ replaced by $L^2(\Omega)$), but it seems to be unimportant. We used the domain of the Dirichlet Laplacian only in imposing the condition (1), so that the DN map is well-defined, but that is the only time we used or will use any spectral theory. Also, there may be another way to do it; as Sjöstrand says, we assume the condition (1) "for simplicity" [18]. Hence for the remainder of the paper we may take a very large domain for the operator. Once we say $u \in H^1(\Omega)$, then $v \in H^{1/2}(\Omega)$ and and $\partial_\nu u \in H^{-1/2}(\Omega)$, since "restriction to the boundary loses half a derivative" (see Eskin's book [8] or lecture notes [9], or Section 1 of Bukhgeim and Uhlmann's paper [1]).

Let $x_0 \in \mathbb{R}^n \backslash \overline{\mathrm{ch}(\Omega)}$, where $\mathrm{ch}(\Omega)$ denotes the convex hull of Ω. Define the front and back faces of $\partial\Omega$ by

$$F(x_0) = \{x \in \partial\Omega;\ (x - x_0) \cdot \nu(x) \leq 0\}$$

and

$$B(x_0) = \{x \in \partial\Omega;\ (x - x_0) \cdot \nu(x) > 0\},$$

respectively. (It would be nice to have a more symmetric definition, but it's not a big deal–see the statement of Theorem 2.1.)

The main result of [14] is the following:

Theorem 2.1. *With* Ω, x_0, $F(x_0)$, *and* $B(x_0)$ *as above, let* $q_1, q_2 \in L^\infty(\Omega)$ *be two potentials satisfying (1), and assume that there exist open neighborhoods* $\tilde{F}, \tilde{B} \subset \partial\Omega$ *of* $F(x_0)$ *and* $B(x_0) \cup \{x \in \partial\Omega;\ (x - x_0) \cdot \nu(x) = 0\}$, *respectively, such that*

$$\mathcal{N}_{q_1} u = \mathcal{N}_{q_2} u \text{ in } \tilde{F}, \text{ for all } u \in H^{1/2}(\partial\Omega) \cap \mathcal{E}'(\tilde{B}). \tag{2}$$

Then $q_1 = q_2$.

Notice that by Green's formula $\mathcal{N}_q^* = \mathcal{N}_{\overline{q}}$. It follows that $\tilde{F}$ and $\tilde{B}$ can be permuted in (3) and we get the same conclusion.

If $\tilde{B} = \partial\Omega$ then we obtain the following result.

Theorem 2.2. *With* Ω, x_0, *and* $F(x_0)$ *as above, let* $q_1, q_2 \in L^\infty(\Omega)$ *be two potentials satisfying (1), and assume that there exists an open neighborhood* $\tilde{F} \subset \partial\Omega$ *of* $F(x_0)$ *such that*

$$\mathcal{N}_{q_1} u = \mathcal{N}_{q_2} u \text{ in } \tilde{F}, \text{ for all } u \in H^{1/2}(\partial\Omega). \tag{3}$$

Then $q_1 = q_2$.

That is, the potential is uniquely determined by knowledge of the DN map on possibly very small subsets of the boundary; for example, if Ω is a ball, we may choose x_0 as near as we like to $\partial\Omega$.

As Sjöstrand points out, the more general Theorem 2.1 was "discovered at a later stage" [18], so we may consider Theorem 2.2 as being the heart of Kenig, Sjöstrand, and Uhlmann's paper [14]. Hence we will focus on the proof of Theorem 2.2. The additional argument for the more general theorem is given in Section 7, the final section, of [14].

The arguments are generally the same as in Bukhgeim and Uhlmann's work [1], but with some extensions. For example, we use Carleman estimates for general limiting Carleman weights whereas Bukhgeim and Uhlmann only needed linear weights; we construct a richer family of complex geometric optics solutions than Bukhgeim and Uhlmann; and the final uniqueness argument uses ideas from microlocal analysis in the complex domain, as in [17], and hence is more complicated than the argument given in Bukhgeim and Uhlmann's work. However, we emphasize that the overarching method is precisely the same as Bukhgeim and Uhlmann's.

Notation. In this paper we take the semiclassical viewpoint, with semiclassical parameter $0 < h \ll 1$, and we thus let $H^s(\mathbb{R}^n)$ denote the semiclassical Sobolev space of order s with the norm $||u||_{H^s} = ||\langle hD\rangle^s u||_{L^2}$.

3. Carleman Estimates with Limiting Carleman Weights

Let $\tilde{\Omega} \subset \mathbb{R}^n$ be an open set, and let $\varphi \in C^\infty(\tilde{\Omega}; \mathbb{R})$ with $\varphi' \neq 0$ everywhere. With $0 < h \ll 1$ denoting the semiclassical parameter, we write

$$P_0 := -h^2\Delta, \qquad P := -h^2(\Delta - q),$$

$$P_{0,\varphi} := e^{\varphi/h} \circ P_0 \circ e^{-\varphi/h}, \quad \text{and } P_\varphi := e^{\varphi/h} \circ P \circ e^{-\varphi/h}.$$

We decompose $P_{0,\varphi}$ into its symmetric and antisymmetric parts:

$$P_{0,\varphi} = \sum_{j=1}^{n} (hD_{x_j} + i\partial_{x_j}\varphi)^2 = A + iB,$$

where

$$A = (hD)^2 - (\varphi')^2, \quad \text{and} \quad B = \sum_{j=1}^{n} (\partial_{x_j}\varphi \circ hD_{x_j} + hD_{x_j} \circ \partial_{x_j}\varphi).$$

The symmetric operators A and B have the semiclassical Weyl symbols

$$a = \xi^2 - (\varphi')^2, \quad \text{and} \quad b = 2\varphi' \cdot \xi.$$

We assume that φ is a limiting Carleman weight (LCW) in the sense that

$$\{a, b\}(x, \xi) = 0 \quad \text{when} \quad a(x, \xi) = b(x, \xi) = 0$$

where $\{a, b\}$ denotes the Poisson bracket of a and b. We note that if φ is a LCW, then so is $-\varphi$ (this will be important later). Also, LCW's seem to be precisely those weights that allow the solution of the complex eikonal equation in Section 4 (see (4.6) of [14]).

The LCW condition is very restrictive, but there are two basic examples.

Example 1. $\varphi(x) = x \cdot \xi_0$ with $\xi_0 \in S^{n-1}$. Bukhgeim and Uhlmann used LCW's of this type in their paper [1], the precursor to Kenig, Sjöstrand, and Uhlmann's paper [14].

Example 2. $\varphi(x) = C \log |x - x_0|$ for $x \neq x_0$. Kenig, Sjöstrand, and Uhlmann used LCW's of this type in [14], where they also introduced the general definition. This LCW is useful in that it is radial with respect to x_0, which manifests itself in the distinction between the "front" and "back" faces of $\partial\Omega$ as in the theorems.

The key tools in [14] are the following Carleman estimates.

Proposition 3.1. *Let $\tilde{\Omega}$ be as above, and let $\varphi \in C^\infty(\tilde{\Omega}; \mathbb{R})$ with $\varphi' \neq 0$ everywhere be a LCW. Let $\Omega \subset\subset \tilde{\Omega}$ and let $q \in L^\infty(\Omega)$. Then for $u \in C_0^\infty(\Omega)$ we have*

$$h(||e^{\varphi/h}u|| + ||hDe^{\varphi/h}u||) \leq C||e^{\varphi/h}(-h^2\Delta + h^2 q)u||,$$

where C depends on Ω and $h > 0$ is small enough so that $Ch||q||_{L^\infty(\Omega)} \leq 1/2$.

This is an *interior* Carleman estimate, dealing only with functions that vanish away from the boundary. The next estimate is a *boundary* Carleman estimate. We define

$$\partial\Omega_\pm := \{x \in \partial\Omega;\ \pm\varphi'(x)\cdot\nu(x) \geq 0\},$$

where again ν is the exterior unit normal.

Proposition 3.2. *Let $\tilde{\Omega}$ and φ be as in Proposition 3.1. Let $\Omega \subset\subset \tilde{\Omega}$ be an open set with C^∞ boundary and let $q \in L^\infty(\Omega)$. Then there exists a constant $C_0 > 0$ such that for every $u \in C^\infty(\overline{\Omega})$ with $u|_{\partial\Omega} = 0$ we have, for $0 < h \ll 1$,*

$$\begin{aligned} -\frac{h^3}{C_0}((\varphi'\cdot\nu)e^{\varphi/h}\partial_\nu u|e^{\varphi/h}\partial_\nu u)_{\partial\Omega_-} + \frac{h^2}{C_0}(||e^{\varphi/h}u||^2 + ||e^{\varphi/h}h\nabla u||^2) \\ \leq ||e^{\varphi/h}(-h^2\Delta + h^2 q)u||^2 + C_0 h^3((\varphi'\cdot\nu)e^{\varphi/h}\partial_\nu u|e^{\varphi/h}\partial_\nu u)_{\partial\Omega_+}. \end{aligned}$$

where C depends on Ω and $h > 0$ is small enough so that $Ch||q||_{L^\infty(\Omega)} \leq 1/2$.

As we shall see, these two propositions are used in quite different ways. The interior Carleman estimate, Proposition 3.1, is used in conjunction with the Hahn-Banach theorem to construct complex geometrical optics solutions, and the boundary Carleman estimate, Proposition 3.2, is used in a more typical way, to prove that a certain quantity goes to zero as $h \to 0$. It is important that if φ is a LCW then so is $-\varphi$, since we will need to use the Carleman estimates for both φ and $-\varphi$.

Here we give the full details for the Hahn-Banach argument. First we note that from Proposition 3.1 we have, for any $s \in [0, 1]$,

$$h||u||_{H^{1-s}} \leq C_{s,\Omega}||P_\varphi u||_{H^{-s}} \quad \forall u \in C_0^\infty(\Omega). \tag{4}$$

The extension to $s \in [0, 1]$ follows from an ellipticity argument, since the semiclassical Weyl symbol of $P_{0,\varphi} = e^{\varphi/h} \circ (-h^2\Delta) \circ e^{-\varphi/h}$, being $a + ib = \xi^2 - (\varphi_x')^2 + 2i\varphi_x'\cdot\xi$, is elliptic in the region $|\xi| \geq 2||\varphi_x'||_{L^\infty}$.

Remark 3.3. Since φ is a LCW, the same is true after replacing φ by $-\varphi$, and since the L^∞ potential plays a minor role in Carleman estimates, the same estimate holds when replacing q by $\overline{q}$. In particular, the estimate also holds when replacing P_φ by its adjoint P_φ^*.

Proposition 3.4. *Let $s \in [0, 1]$. Then for $h > 0$ small enough, for every $v \in H^{s-1}(\Omega)$ there exists $u \in H^s(\Omega)$ such that*

$$P_\varphi^* u = v, \quad h||u||_{H^s} \leq C||v||_{H^{s-1}}.$$

Proof. We consider $M := P_\varphi C_0^\infty(\Omega)$ as a subspace of $H^{-s}(\Omega)$. We then define the complex linear functional

$$\begin{aligned} L : M &\to \mathbb{C} \\ P_\varphi u &\mapsto \langle u|v\rangle_{L^2}. \end{aligned}$$

Using the estimate (4) we see that L is well-defined. Using the estimate (4) for a second time, we obtain the upper bound

$$\begin{aligned}|L(P_\varphi u)| &\leq ||u||_{H^{1-s}}||v||_{H^{s-1}}\\ &\leq \frac{C}{h}||P_\varphi u||_{H^{-s}}||v||_{H^{s-1}}.\end{aligned}$$

By the Hahn-Banach theorem, we may extend L to all of $H^{-s}(\Omega)$ without increasing the operator norm. Hence we have

$$\hat{L} : H^{-s}(\Omega) \to \mathbb{C}$$

such that $\hat{L}|_M = L$ and such that

$$||\hat{L}||_{H^{-s}(\Omega)\to\mathbb{C}} \leq \frac{C}{h}||v||_{H^{s-1}}.$$

Then by the Riesz representation theorem, we know that there exists some $u \in H^s(\Omega)$ with $||u||_{H^s} \leq \frac{C}{h}||v||_{H^{s-1}}$ such that $\hat{L}(w) = \langle w|u\rangle_{L^2}$ for all $w \in H^{-s}(\Omega)$. In particular,

$$\begin{aligned}\langle w|v\rangle_{L^2} &= L(P_\varphi w)\\ &= \hat{L}(P_\varphi w)\\ &= \langle P_\varphi w|u\rangle_{L^2}\\ &= \langle w|P_\varphi^* u\rangle_{L^2}\end{aligned}$$

for all $w \in C_0^\infty(\Omega)$. Hence $P_\varphi^* u = v$, which concludes the proof. ☐

Remark 3.5. The proposition also holds after modifying the L^∞ potential or after replacing φ by $-\varphi$. In particular, it holds with P_φ^* replaced by P_φ. (See Remark 3.3.)

Remark 3.6. Since we are in a Hilbert space, the Riesz representation theorem is simple to prove.

4. Construction of Complex Geometrical Optics Solutions

In this section, we construct complex geometrical optics (CGO) solutions for the homogeneous equation $P_\varphi u = 0$. The CGO solutions are in general highly non-unique, and, in fact, the Carleman estimate in Proposition 3.1 allows us "to construct a much wider class of complex geometric optics than previously known" ([14], p.568) since we are allowing φ to be any LCW (previous work considered only linear weights).

Proposition 4.1. *Let φ be a limiting Carleman weight. Then there exist*

(i) a smooth, locally defined ψ such that $(\psi')^2 = (\varphi')^2$ and $\psi' \cdot \varphi' = 0$,

(ii) a nonvanishing $a \in C^\infty$ independent of h, and

(iii) some $r \in H^1(\Omega)$ with $||r||_{H^1} \leq Ch$

such that

$$P(e^{(-\varphi+i\psi)/h}(a+r))=0.$$

In fact, we will explicitly control ψ's domain of definition in the construction.

Remark 4.2. The proposition also holds after modifying the L^∞ potential or after replacing φ by $-\varphi$. (See Remarks 3.3 and 4.2.)

Proof. (of the Proposition.) (Sketch.) We first consider the operators $P_0=-h^2\Delta$ and $P_{0,\varphi}=e^{\varphi/h}\circ P_0\circ e^{-\varphi/h}$. We find a function ψ that solves the eikonal equation for $P_{0,\varphi}$, and we take care to control the size of ψ's domain of definition. Then we find a non-vanishing smooth function $a\in C^\infty$, independent of h, that solves the first transport equation; we find a that is well-defined in a neighborhood of $\overline{\Omega}$. (For this, Kenig, Sjöstrand, and Uhlmann refer to Duistermaat and Hörmander's paper [7]. For details in a special case, see Section 6.)

Then, returning to $P=P_0+h^2q$, we have

$$P(e^{(-\varphi+i\psi)/h}a)=e^{-\varphi/h}h^2d, \tag{5}$$

with $d=\mathcal{O}(1)$ in L^∞ and hence in L^2. Explicitly, d is defined by

$$(-\Delta+q)a=e^{-i\psi/h}d.$$

However, with Proposition 3.4 we are able to remove the error term in (5); that is, we use Proposition 3.4 with P_φ (see Remark 4.2), with $v:=h^2d$, $s=1$, and writing $r:=-e^{-i\psi/h}u$ to complete the proof. ☐

We will only need CGO solutions for a particular choice of φ, in which case we may directly construct ψ and a. This is done in Section 6.

5. The Heart of the Proof: the Bukhgeim-Uhlmann Argument

Let $q_1,q_2\in L^\infty(\Omega)$ be two potentials as in Theorem 2.2. Let $\mathcal{N}_{q_1}$ and $\mathcal{N}_{q_2}$ denote the DN maps as before, and define

$$\partial\Omega_{-,\epsilon_0}:=\{x\in\partial\Omega;\ \nu(x)\cdot\varphi'(x)<\epsilon_0\}$$

and

$$\partial\Omega_{+,\epsilon_0}:=\{x\in\partial\Omega;\ \nu(x)\cdot\varphi'(x)\geq\epsilon_0\},$$

for some fixed $\epsilon_0>0$, so that $\partial\Omega_{+,\epsilon_0}\subset\partial\Omega_+$ and $\partial\Omega_-\subset\partial\Omega_{-,\epsilon_0}$. Here again $\nu(x)$ denotes the unit outer normal to $\partial\Omega$.

The hypothesis of Theorem 2.2 is precisely that

$$\mathcal{N}_{q_1}(f)=\mathcal{N}_{q_2}(f)\quad\text{in }\partial\Omega_{-,\epsilon_0}\quad\forall f\in H^{1/2}(\partial\Omega).$$

To prove the theorem we must show that this implies $q_1=q_2$.

Step 1. Let

$$u_2 = e^{(\varphi+i\psi_2)/h}(a_2 + r_2(x;h))$$

be as in Section 4, with

$$(\Delta - q_2)u_2 = 0, \quad \text{and} \quad ||r_2||_{H^1} = \mathcal{O}(h).$$

Note that for this we use the (interior) Carleman estimate, Proposition 3.1, for $P_{-\varphi}$.

Step 2. Let $u_1 \in H^1(\Omega)$ solve

$$(\Delta - q_1)u_1 = 0, \quad u_1|_{\partial\Omega} = u_2|_{\partial\Omega}.$$

Intermission. Now let $u = u_2 - u_1$ and $q = q_2 - q_1$. Then

$$\operatorname{supp}(\partial_\nu u|_{\partial\Omega}) \subset \partial\Omega_{+,\epsilon_0}$$

and

$$(\Delta - q_1)u = (\Delta - q_1)u_2 = qu_2, \quad u|_{\partial\Omega} = 0.$$

For $v \in H^1(\Omega)$ with $\Delta v \in L^2(\Omega)$, we then get

$$\begin{aligned} \int_\Omega q u_2 \overline{v}\, dx &= \int_\Omega [(\Delta - q_1)u]\overline{v}\, dx \\ &= \int_\Omega u\overline{(\Delta - \overline{q_1})v}\, dx + \int_{\partial\Omega_{+,\epsilon_0}} (\partial_\nu u)\overline{v}\, S(dx). \end{aligned} \tag{6}$$

Step 3. Let

$$v = e^{-(\varphi+i\psi_1)/h}(a_1 + r_1(x;h))$$

be as in Section 4, with

$$(\Delta - \overline{q_1})v = 0, \quad \text{and} \quad ||r_1||_{H^1} = \mathcal{O}(h).$$

Note that for this we use the (interior) Carleman estimate, Proposition 3.1, for P_φ.

Then, with this v, (6) becomes

$$\int_\Omega q e^{i(\psi_1+\psi_2)/h}(a_2 + r_2)\overline{(a_1 + r_1)}\, dx = \int_{\partial\Omega_{+,\epsilon_0}} (\partial_\nu u) e^{-(\varphi - i\psi_1)/h}\overline{(a_1 + r_1)}\, S(dx).$$

After some work, including a use of the *boundary* Carleman esimate, Proposition 3.2, we get

$$\int_\Omega q(x) a_2(x)\overline{a_1(x)} e^{if(x)}\, dx = 0 \tag{7}$$

in the $h \to 0$ limit, where f is a function such that

$$\frac{1}{h}(\psi_1 + \psi_2) \to f \quad \text{as } h \to 0.$$

Since we may change the signs of ψ_1 and ψ_2 without changing the results, we seek f of the form

$$f(x) := \lim_{h\to 0} \frac{\psi_1(x) - \psi_2(x)}{h}. \tag{8}$$

In the next section we will find φ, ψ_1, and ψ_2 so that an appropriate such f exists.

6. Conclusion of the Proof: Construction of a Bargmann-FBI Transform

The following material is taken both from the paper of Kenig, Sjöstrand, and Uhlmann [14] and that of Dos Santos Ferreira, Kenig, Sjöstrand, and Uhlmann [5].

From now on we assume that $n \geq 3$, and we choose $\varphi(x) = \log|x - x_0|$ for x_0 varying in a small open set separated from $\overline{\Omega}$ by some fixed affine hyperplane H. In Section 5 the main result was the equation (7). In this section, we begin by constructing a suitable function f; we will find an analytic family $\psi(x, \alpha)$ depending on the additional parameters $\alpha = (\alpha_1, \ldots, \alpha_k)$, with $\psi(\cdot, \alpha)$ satisfying

$$(\psi'_x)^2 = (\varphi'_x)^2 \quad \text{and} \quad \psi'_x \cdot \varphi'_x = 0, \tag{9}$$

and then we will take

$$f(x) = \psi'_\alpha(x, \alpha) \cdot \nu(\alpha), \tag{10}$$

where $\nu(\alpha)$ is a tangent vector in the α-variables. That is, we will find ψ and then take

$$\psi_1(x) := \psi(x, \alpha + h\nu(\alpha)) \quad \text{and} \quad \psi_2(x) := \psi(x, \alpha)$$

so that

$$\lim_{h\to 0} \frac{\psi_1(x) - \psi_2(x)}{h} = \psi'_\alpha(x, \alpha) \cdot \nu(\alpha) =: f(x).$$

We first discuss the choice of ψ. By hypothesis, $x_0 \in \mathbb{R}^n \backslash \overline{\mathrm{ch}(\Omega)}$, so we are working on a convex cone with vertex at x_0. Thus a function ψ satisfies (9) if and only if $(\psi'_x)^2 = (\varphi'_x)^2$ on a suitable open subset $x_0 + r_0 W$ of $x_0 + r_0 S^{n-1}$ for some fixed $r_0 > 0$. Indeed, we can extend ψ to be a positively homogeneous function of degree 0 in the variables $x - x_0$, satisfying the second condition of (9); then $(\psi')^2$ is of course positively homogeneous of degree -2. Since $(\varphi'_x)^2 = |x - x_0|^{-2}$ is also positively homogeneous of degree -2, the first condition of (9) is then also satisfied. Moreover, there is an obvious choice for the open subset $W \subset \partial B(0, 1)$: We let $r_0 > 0$ be large enough so that $\overline{\Omega} \subset B(x_0, r_0)$, and then we let $x_0 + r_0 W \subset \partial B(x_0, r_0)$ be defined by

$$x_0 + r_0 W = \partial B(x_0, r_0) \cap H_+,$$

where H_+ is the open half-space delimited by the affine hyperplane H for which $x_0 \notin H_+$ (so that $\overline{\Omega} \subset H_+$).

With the above reduction, we may simply take

$$\psi(x, y) = \mathrm{dist}_{S^{n-1}}(x, y),$$

and from there, as noted above, we may extend ψ by homogeneity to obtain the desired function,

$$\psi(x,y) = \frac{\pi}{2} - \arctan \frac{y\cdot(x-x_0)}{\sqrt{(x-x_0)^2 - (y\cdot(x-x_0))^2}} \quad \text{for}\, y \in S^{n-1}.$$

If we let $y_0 \in \partial B(0,1)\backslash\overline{W}$ be such that the antipodal point $-y_0$ is also outside of $\overline{W}$, then the (extended) function ψ is in $C^\infty(\overline{\Omega} \times \text{ neigh}(y_0))$.

Back on $W \subset S^{n-1}$: Since the domain of definition W does not contain antipodal points, we have that

$$\psi''_{x,y} \text{ is of rank } n-2, \quad \mathcal{R}(\psi''_{x,y}) = (\psi'_x)^\perp, \quad \text{and}\, \mathcal{N}(\psi''_{x,y}) = (\psi'_y). \tag{11}$$

(For an interesting discussion of related topics, see Chapter 1 of [10].) For completeness, we provide the details: Suppose $\psi''_{x,y}(x,y)u = 0$. We then take a curve $y(t)$, on a geodesic through y, such that $y(0) = y$ and $\dot{y}(0) = u$. Then

$$\frac{d}{dt}\Big|_{t=0} \psi'_x(x, y(t)) = \psi''_{x,y}(x,y)u = 0.$$

That is, $\psi'_x(x,y(t))$ (the arrival direction) is $\psi'_x(x,y) + \mathcal{O}(t^2)$, and hence the points $y(t)$ lie on the geodesic from x to y. (This is especially intuitive in our case, working on S^{n-1}. Also, note that $|\psi'_x(x,y)|$ is independent of y.) Here we are using the fact that x and y are not conjugate points. Thus $u \in (\psi'_y)$, and we have shown that $\mathcal{N}(\psi''_{x,y}) = (\psi'_y)$. Now for any v we have

$$(\psi'_x)^T \psi''_{x,y}(v) = (v^T \psi''_{y,x}(\psi'_x))^T = 0.$$

Thus $\mathcal{R}(\psi''_{x,y}) \subset (\psi'_x)^\perp$. Since both sides have dimension $n-2$, we are done: $\mathcal{R}(\psi''_{x,y}) = (\psi'_x)^\perp$.

Remark 6.1. By a translation and rotation we may assume that $x_0 = 0$ and $y = (1,0,\ldots,0)$, and we write

$$z := x_1 + i|x'| \in \mathbb{C}.$$

Then

$$\varphi = \operatorname{Re}\log z \quad \text{and} \quad \psi = \operatorname{Im}\log z \quad \text{whenever } \operatorname{Im} z > 0.$$

It is then easy to check that ψ satisfies (9).

Now we find the amplitude a. With the above phase ψ we have

$$e^{-i\psi/h} \circ P_\varphi \circ e^{i\psi/h} = (hD)^2 + 2\psi'\cdot hD + 2i\varphi'\cdot hD + h\Delta\varphi + h^2 q - ih\Delta\psi.$$

We want a to solve this $\text{mod}\mathcal{O}(h^2)$; that is, we want a such that

$$(\psi' + i\varphi')\cdot Da + \frac{1}{2}(\Delta\varphi - i\Delta\psi)a = 0.$$

Moreover, we seek a of the form

$$a = e^\Phi.$$

Again with $z = x_1 + i|x'|$ and $\log z = \varphi + i\psi$, we define

$$w := \log \overline{z} = \varphi - i\psi.$$

Then we wish to find Φ such that

$$\nabla w \cdot \nabla \Phi + \frac{1}{2}\Delta w = 0.$$

To simplify this we use cylindrical coordinates so that $z = x_1 + ir$. Then

$$\nabla w \cdot \nabla = \frac{2}{\overline{z}}\frac{\partial}{\partial z}$$

and

$$\Delta w = \frac{-i(n-2)}{r\overline{z}}.$$

Hence we take Φ to solve the inhomogeneous Cauchy-Riemann equations

$$\frac{\partial \Phi}{\partial z} = \frac{-(n-2)}{2(z-\overline{z})}.$$

We may then use Proposition 3.4 to complete the CGO construction as in Section 4.

We are not quite done constructing f—we need even more parameters than just y. (I think the basic idea is to keep introducing parameters until $f''_{x,\theta}$ has maximal rank n. In the end the parameter is some $\theta = (y, \tilde{x}, \nu)$.) In what follows we give a rather technical-looking construction of f, but the main point is to create a function so that (7) can be made into a Bargmann-FBI transform. For this we need some conditions on the phase, but once we set up the Bargmann-FBI transform the final result $q = 0$ will follow from the analytic wavefront version of Holmgren's uniqueness theorem, a basic result in microlocal analysis in the complex domain.

As a lower-dimensional version of f, for $x \in W \subset S^{n-1}$, $(y, \nu) \in TS^{n-1}$, and $y \in \operatorname{neigh}(y_0)$, we take

$$\tilde{f}(x; y, \nu) = \psi'_y(x, y) \cdot \nu.$$

Then

$$\tilde{f}'_x(x; y, \nu) = \psi''_{x,y}(x, y)\nu.$$

In view of (9) we see that this vanishes precisely when $\nu \parallel \psi'_y(x, y)$, i.e. when ν is parallel to the (arrival) direction of the minimal geodesic from x to y. By restricting the set of ν to nonvanishing directions which are close to parallel to the plane H, we can arrange for

$$\tilde{f}'_x(x; y, \nu) \neq 0. \qquad (12)$$

(This is where we use that $n \geq 3$.) Also, we have the following non-degeneracy result:

Lemma 6.2. *$\tilde{f}''_{x,(y,\nu)}$ has maximal rank $n-1$.*

Proof. First of all we already know that $\tilde{f}''_{x,\nu} = \psi''_{x,y}$ has rank $n-2$ and that the image of this matrix is $(\psi'_x)^\perp$. (See (11).) Hence it suffices to find a vector in the image of $\tilde{f}''_{x,y}$ that is not orthogonal to ψ'_x. For this we introduce

$$\begin{aligned} g(y) &:= \psi'_x(x,y_0)\cdot\psi''_{x,y}(x,y)\nu \\ &= \psi'_x(x,y_0)\cdot\tilde{f}'_x(x;y,\nu), \end{aligned}$$

which vanishes at $y=y_0$. Then

$$g'(y_0) = \psi'_x(x,y_0)\cdot\tilde{f}''_{x,y}(x;y_0,\nu).$$

We wish to show that this is not the zero vector. We rewrite

$$\begin{aligned} g(y) &= \langle\psi''_{x,y}(x,y)|(\psi'_x(x,y_0)-\psi'_x(x,y))\otimes\nu\rangle \\ &= \langle\psi''_{x,y}(x,y)(\nu)|\,\psi''_{x,y}(x,y)(y_0-y)\rangle + \mathcal{O}((y_0-y)^2). \end{aligned}$$

Thus

$$g'(y_0)=0 \quad \text{if and only if} \quad (\psi''_{x,y}(x,y_0))^T\psi''_{x,y}(x,y_0)\nu = 0.$$

Now of course

$$\psi''_{x,y}(x,y_0)^T = \psi''_{y,x}(x,y_0),$$

so by (9) we have

$$\psi''_{x,y}(x,y_0)^T u = 0 \quad \text{if and only if} \quad u\in\operatorname{span}(\psi'_x).$$

Also by (9), we have

$$\psi''_{x,y}(x,y_0)\nu\in(\psi'_x)^\perp,$$

so that

$$g'(y_0)=0 \quad \text{if and only if} \quad \psi''_{x,y}(x,y_0)\nu = 0.$$

Since we arranged for

$$\tilde{f}'_x(x;y_0,\nu) = \psi''_{x,y_0}(x,y_0)\nu \neq 0,$$

we see that $g'(y_0)\neq 0$, concluding the proof of the lemma. □

Now we consider

$$\Psi(x;y,\tilde{x}) = \psi(\frac{x-\tilde{x}}{|x-\tilde{x}|},y) \in C^\infty(\overline{\Omega}\times\operatorname{neigh}(y_0,S^{n-1})\times\operatorname{neigh}(x_0,\mathbb{R}^n)).$$

Of course, this is the same sort of function as before, but with x_0 replaced by $\tilde{x}$; we allow $\tilde{x}$ to vary and consider it as a parameter. We can take $\alpha = y$ and then we finally define f by

$$f(x) = f(x;\theta) = \Psi'_y\cdot\nu, \qquad \theta = (y,\tilde{x},\nu),$$

with $(y,\nu)\in TS^{n-1}$. (See (10).) The following is taken verbatim from [14]:

Lemma 6.2 shows that $f''_{x,(y,\nu)}$ has rank $n-1$ and indeed the image of this matrix is the tangent space of $\partial B(\tilde{x}, |x-\tilde{x}|)$ at x. Since f'_x is a non-vanishing element of $T_x(\partial B(\tilde{x}, |x-\tilde{x}|))$, we can vary $\tilde{x}$ infinitesimally to see that $f''_{x,\tilde{x}}(\tilde{\mu}) \notin T_x(\partial B(\tilde{x}, |x-\tilde{x}|))$ for a suitable $\tilde{\mu} \in \mathbb{R}^n$. It is then clear that

$$f''_{x,\theta} = f''_{x,(y,\tilde{x},\nu)} \quad \text{has maximal rank } n,$$

and hence the map

$$\operatorname{neigh}(\overline{\Omega}) \ni x \mapsto f'_\theta(x,\theta) \in \mathbb{R}^{3n-2} \tag{13}$$

has injective differential.

For this, it is enough to consider

$$F(t) := f'_x(x; y, (1-t)\tilde{x} + \tilde{\tilde{x}}, \nu)$$

for any $\tilde{\tilde{x}}$ off the line between x and $\tilde{x}$. Then of course

$$F'(0) = f''_{x,\tilde{x}}(x; y, \tilde{x}, \nu)(\tilde{\tilde{x}} - \tilde{x}),$$

and one may draw a figure to see that this is not in $T_x(\partial B(\tilde{x}, |x-\tilde{x}|))$. (Right?)

Again, I think the point is to keep adding parameters until $f''_{x,\theta}$ has maximal rank n.

Lemma 6.3. *The map (13) is injective.*

This is a global statement—an improvement on the fact that it has injective differential.

Proof. Let $x_1, x_2 \in \operatorname{neigh}(\overline{\Omega})$ be two points such that

$$f'_\theta(x_1,\theta) = f'_\theta(x_2,\theta) \qquad \text{with} \quad \theta = (y_0, \tilde{x}, \nu). \tag{14}$$

Taking the ν component, we have

$$f'_\nu(x_1,\theta) = f'_\nu(x_2,\theta).$$

That is,

$$\Psi'_y(x_1; y_0, \tilde{x}) = \Psi'_y(x_2; y_0, \tilde{x}).$$

Defining

$$\tilde{x}_j := \frac{x_j - \tilde{x}}{|x_j - \tilde{x}|},$$

we may rewrite this as

$$\psi'_y(\tilde{x}_1, y_0) = \psi'_y(\tilde{x}_2, y_0), \tag{15}$$

and we recall that, by definition,

$$\psi(\tilde{x}_j, y_0) = \operatorname{dist}_{S^{n-1}}(\tilde{x}_j, y_0) =: d(\tilde{x}_j, y_0).$$

Now (15) says that $\tilde{x}_1, \tilde{x}_2, y_0$ belong to the same geodesic γ, and this geodesic is minimal (i.e. distance minimizing) on some segment that contains these three points on its interior. (Recall that we are working on S^{n-1}, so the geometric considerations are fairly simple.)

If $\tilde{x}_1 \neq \tilde{x}_2$, then we may assume $d(\tilde{x}_2, y_0) < d(\tilde{x}_1, y_0)$. For $y \in \operatorname{neigh}(y_0, S^{n-1})$, we define

$$g(y) := d(\tilde{x}_1, \tilde{x}_2) + d(\tilde{x}_2, y) - d(\tilde{x}_1, y).$$

That is,

$$g(y) = (d(\tilde{x}_1, y_0) - d(\tilde{x}_1, y)) + (d(\tilde{x}_2, y) - d(\tilde{x}_2, y_0)).$$

Hence, by (15),

$$g(y) = \mathcal{O}((y_0 - y)^2) \sim d(y_2, \gamma)^2.$$

Thus $g'(y_0) = 0$, and $g''(y_0)\nu \neq 0$ since ν is not parallel to $\dot{\gamma}$ at y_0.

On the other hand,

$$g'(y) = \psi'_y(\tilde{x}_2, y) - \psi'_y(\tilde{x}_1, y),$$

and hence

$$\begin{aligned} g'(y_0) \cdot \nu &= \Psi'_y(x_2; y_0, \tilde{x}) \cdot \nu - \Psi'_y(x_1; y_0, \tilde{x}) \cdot \nu \\ &= f(x_2, \theta) - f(x_1, \theta). \end{aligned}$$

Thus by (14) we have

$$g''(y_0) \cdot \nu = f'_y(x_2, \theta) - f'_y(x_1, \theta) = 0.$$

This gives a contradiction; hence $\tilde{x}_1 = \tilde{x}_2$. That is, we have shown that x_1 and x_2 lie on the same half-ray through $\tilde{x}$.

Now taking the $\tilde{x}$ component of (14), we get precisely that

$$\nabla_{\tilde{x}} \langle \psi'_y(\frac{x - \tilde{x}}{|x - \tilde{x}|}), \nu \rangle |_{x = x_1} = \nabla_{\tilde{x}} \langle \psi'_y(\frac{x - \tilde{x}}{|x - \tilde{x}|}), \nu \rangle |_{x = x_2}, \tag{16}$$

and these are non-vanishing.

Since x_1 and x_2 lie on the same half-ray through $\tilde{x}$, we have

$$x_1 - \tilde{x} = \lambda(x_2 - \tilde{x}),$$

and thus

$$\nabla_{\tilde{x}}(\frac{x - \tilde{x}}{|x - \tilde{x}|})|_{x = x_1} = \lambda^{-1} \nabla_{\tilde{x}}(\frac{x - \tilde{x}}{|x - \tilde{x}|})|_{x = x_2},$$

by homogeneity. Then (16) requires that $\lambda = 1$; that is, $x_1 = x_2$. This completes the proof of the lemma. □

Since $f(x, \theta) = f(x; y, \tilde{x}, \nu)$ depends linearly on ν, we may then rewrite (7) as

$$\int e^{i\lambda f(x,\theta)} a_2 \overline{a_1} q(x)\, dx = 0, \qquad \lambda \geq 1. \tag{17}$$

Now represent θ by some analytic real coordinates $\theta_1, \theta_2, \ldots, \theta_N$ near some fixed given point $\theta_0 = (y_0, x_0, \nu_0)$. If $x, z \in \overline{\Omega}$, $w \in \operatorname{neigh}(\theta_0)$, we consider the (analytic) function

$$\theta \mapsto -f(z, \theta) + f(x, \theta) + \frac{i}{2}(\theta - w)^2.$$

For $x = z$ this has a unique non-degenerate critical point $\theta = w$. For $x \neq z$ there is no real critical point θ, by Lemma 6.3. However, for $x \approx z$ we have a complex critical point which is unique close to w. We then define the new phase as the critical value

$$\psi(z, x, w) = \mathrm{v.c.}_\theta \left(-f(z, \theta) + f(x, \theta) + \frac{i}{2}(\theta - w)^2 \right).$$

From (13) and standard estimates on critical values in connection with the complex stationary phase method (see [17]), we deduce that

$$\operatorname{Im} \psi(z, x, w) \sim (z - x)^2, \qquad z, x \in \overline{\Omega}, \quad z \approx x. \tag{18}$$

Moreover, when $x = z$, we have

$$\psi'_z(z, z, w) = -f'_z(z, w), \quad \psi'_x(z, z, w) = f'_z(z, w), \quad \psi(z, z, w) = 0. \tag{19}$$

We now let χ_1 be a standard cutoff to a neighborhood of 0. Then, multiplying (17) by

$$\chi_1(\theta - w) e^{-\frac{\lambda}{2}(\theta - w)^2 - i\lambda f(z,\theta)}$$

and integrating with respect to θ, we get

$$0 = \int \left[\int e^{i\lambda[-f(z,\theta)+f(x,\theta)+\frac{i}{2}(\theta-w)^2]} \chi_1(\theta - w) a_2(x, \theta) \overline{a_1(x, \theta)} \, d\theta \right] q(x) \, dx.$$

We recall that a_1 and a_2 are analytic non-vanishing functions of $x, y, \tilde{x}$ in a neighborhood of $\overline{\Omega} \times \{y_0\} \times \{x_0\}$, and we recall that $\theta = (y, \tilde{x}, \nu)$. But then the complex stationary phase method (see Theorem 2.8 and Remark 2.10 of [17]) says

$$\int e^{i\lambda[-f(z,\theta)+f(x,\theta)+\frac{i}{2}(\theta-w)^2]} \chi_1(\theta - w) a_2(x, \theta) \overline{a_1(x, \theta)} \, d\theta$$
$$= e^{i\lambda\psi(z,x,w)} a(z, x, w; \lambda) \chi_2(z - x) + \mathcal{O}(e^{-\frac{\lambda}{C}}),$$

where a is an elliptic classical analytic symbol of order 0 and χ_2 is another cutoff to a neighborhood of 0.

Remark 6.4. Another choice of cutoff would only give another $\mathcal{O}(e^{-\frac{\lambda}{C}})$ term. Also, if we wanted we could write an asymptotic expansion for a in powers of λ^{-1}.

Hence we have

$$\int e^{i\lambda\psi(z,x,w)} a(z, x, w; \lambda) \chi_2(z - x) q(x) \, dx = \mathcal{O}(e^{-\frac{\lambda}{C}}). \tag{20}$$

We now restrict w to an n-dimensional manifold Σ which passes through θ_0, and we write

$$(z, -f'_z(z, \theta)) = (\alpha_x, \alpha_\xi) = \alpha.$$

Then we rewrite (20) as

$$\int e^{i\lambda\psi(\alpha,x)} a(\alpha, x; \lambda) \chi_2(\alpha_x - x) q(x) \, dx = \mathcal{O}(e^{-\frac{\lambda}{C}}). \tag{21}$$

We are now precisely in the framework of [17]. By standard facts about Bargmann-FBI transforms (see Sections 6 and 7 of [17]), we thus have

$$(z, -f'_z(z, \theta_0)) \notin \mathrm{WF}_a(\mathbf{1}_\Omega q) \tag{22}$$

for all z in some neighborhood of $\overline{\Omega}$. (Note that we obviously need $f'_z(z, \theta_0) \neq 0$ for this to be meaningful.) Notice that (18) and (19) give

$$\psi(\alpha, x) = (\alpha_x - x) \cdot \alpha_\xi + \mathcal{O}((\alpha_x - x)^2)$$

and

$$\operatorname{Im} \psi(\alpha, x) \sim (\alpha_x - x)^2.$$

(Compare this to the standard FBI phase $\phi_{\mathrm{std}}(\alpha, x) = (x - \alpha_x) \cdot \alpha_\xi + \frac{i}{2}(x - \alpha_x)^2$.) Also, since (13) has injective differential, we can choose Σ so that the map

$$\mathrm{neigh}(z_0) \times \Sigma \ni (z, \theta) \mapsto (z, -f'_z(z, \theta))$$

is a local diffeomorphism near any given fixed point $z_0 \in \overline{\Omega}$. This justifies replacing w by $\alpha_\xi = -f'_z(z, \theta)$ in going from (20) to (21).

To conclude the proof of Theorem 2.2, let z_0 be a point in $\mathrm{supp}(q)$ where $f(\,\cdot\,; \theta_0)|_{\mathrm{supp}(q)}$ is minimal. Then $-f'_z(z_0, \theta_0)$ belongs to the exterior conormal cone of $\mathrm{supp}(q)$ at z_0, which, by the analytic wavefront version of Holmgren's uniqueness theorem (see Theorem 8.6 in [17]), contradicts (22). Hence $\mathrm{supp}(q) = \emptyset$; that is, $q = 0$. This completes the proof of Theorem 2.2.

Appendix A. A Brief Discussion of Recent Further Work

Dos Santos Ferreira, Kenig, Sjöstrand, and Uhlmann [5] extended the above results of [14] to the case of a non-zero magnetic potential. One difference is that they use a Radon transform method, replacing the Bargmann-FBI transform method of [14]. Also, their exposition differs slightly from [14].

The paper of Kenig, Sjöstrand, and Uhlmann [14] is non-constructive: They show that two equal DN maps must have equal potentials, but they don't say how to reconstruct the conductivity given a DN map. This reconstruction is the subject of a recent paper by Nachman and Street [15].

For a two-dimensional domain $\Omega \subset \mathbb{R}^2$, Imanuvilov, Uhlmann, and Yamamoto recently proved that the Cauchy data for the Schrödinger equation on an arbitrary open subset of the boundary uniquely determines the potential [12]. For a similar result, but with "front face versus back face" assumptions, see the slightly earlier paper by the same authors [11]. Their most recent paper, [13], combines the two [11], [12].

Main Open Problem. The local Calderón problem is unresolved in dimensions $n \geq 3$. Again, let $\Omega \subset\subset \mathbb{R}^n$, with $n \geq 2$, be a bounded open set with smooth boundary. For $q \in L^\infty(\Omega)$ we consider the Schrödinger equation

$$(-\Delta + q)u = 0$$

and the corresponding DN map $\mathcal{N}_q$ as described in Section 2. We formally state the open problem as a conjecture, to emphasize its importance:

Conjecture A.1. *Let* $q_1, q_2 \in L^\infty(\Omega)$ *as above, and let* Σ *be an open boundary neighborhood of some point on the boundary. If*

$$\mathcal{N}_{q_1}(f)\Big|_\Sigma = \mathcal{N}_{q_2}(f)\Big|_\Sigma \qquad \forall f \in H^{1/2}(\partial\Omega) \cap \mathcal{E}'(\Sigma),$$

then $q_1 \equiv q_2$.

For recent work in this direction, see the paper of Dos Santos Ferreira, Kenig, Sjöstrand, and Uhlmann [6]. Here we just state their main result. First of all, it is easy to check that the DN map is symmetric in the following sense:

$$\int_{\partial\Omega} \mathcal{N}_q(f) g \, dS = \int_{\partial\Omega} f \, \mathcal{N}_q(g) \, dS \qquad \forall f, g \in H^{1/2}(\partial\Omega).$$

Let $q_1, q_2 \in L^\infty(\Omega)$ as above. Then one sees the equivalence of the following two properties:

$$\text{(P1)} \qquad \mathcal{N}_{q_1}(f)\Big|_\Sigma = \mathcal{N}_{q_2}(f)\Big|_\Sigma \qquad \forall f \in H^{1/2}(\partial\Omega) \cap \mathcal{E}'(\Sigma).$$

$$\text{(P2)} \qquad \int_\Omega (q_1 - q_2) u_1 u_2 \, dx = 0 \qquad \forall\, u_j \in H^1(\Omega) \text{ such that } \operatorname{supp}(u_j|_{\partial\Omega}) \subset \Sigma \text{ and such that } (-\Delta + q_j) u_j = 0.$$

Hence Conjecture A.1 is equivalent to the following: (Compare with Section 5)

Conjecture A.2. *Let* $q_1, q_2 \in L^\infty(\Omega)$. *Then the property (P1) implies that* $q_1 \equiv q_2$.

Dos Santos Ferreira, Kenig, Sjöstrand, and Uhlmann, in [6], studied a modified version of this problem (as did Calderón, in his original paper [2]). Their main result is the following theorem:

Theorem A.3. ([6]) *Let* Ω *be a connected bounded open set in* $\mathbb{R}^n$, $n \geq 2$, *with smooth boundary. The set of products of harmonic functions in* $C^\infty(\overline{\Omega})$ *which vanish on a closed proper subset* $\Gamma \subsetneq \partial\Omega$ *of the boundary is dense in* $L^1(\Omega)$.

They first prove the following local result, and then they show how it implies the global result, Theorem A.3.

Theorem A.4. ([6]) *Let* Ω *be a bounded open set in* $\mathbb{R}^n$, $n \geq 2$, *with smooth boundary, and let* $f \in L^\infty(\Omega)$. *Let* $x_0 \in \partial\Omega$ *and let* Γ *be the complement of an open boundary neighborhood* *of* x_0*. There exists* $\delta > 0$ *such that if we have the cancellation*

$$\int_\Omega f\, u\, v\, dx = 0$$

for any pair of harmonic functions u *and* v *in* $C^\infty(\overline{\Omega})$ *vanishing on* Γ*, then* f *vanishes on* $B(x_0, \delta) \cap \Omega$*.*

Appendix B. The Original Bukhgeim-Uhlmann Argument

The paper of Bukhgeim and Uhlmann [1] takes place in a simplified setting; here we sketch their proof to shed light on the more complicated argument of Kenig, Sjöstrand, and Uhlmann [14].

Let $\xi \in S^{n-1}$ and fix $k \in \mathbb{R}^n$ such that $\xi \cdot k = 0$.

Step 1. Let

$$u_2 = e^{x\cdot\rho_2/h}(1 + r_2(x;\rho_2))$$

be such that

$$(\Delta - q_2)u_2 = 0 \quad \text{and} \quad ||r_2||_{L^2} = \mathcal{O}(h).$$

Here

$$\rho_2 = \xi - \frac{ih}{2}(k + l),$$

where $l \cdot k = l \cdot \xi = 0$ and $|k + l| = 2/h$. In dimensions $n \geq 3$ we can always choose such a vector l.

Step 2. Let u_1 be such that

$$(\Delta - q_1)u_1 = 0 \quad \text{and} \quad u_1|_{\partial\Omega} = u_2|_{\partial\Omega}.$$

By hypothesis, we then have

$$\partial_\nu u_1|_{\partial\Omega_{-,\epsilon}(\xi)} = \partial_\nu u_2|_{\partial\Omega_{-,\epsilon}(\xi)}.$$

Intermission. Let $u := u_1 - u_2$ and $q := q_1 - q_2$. Then

$$(\Delta - q_1)u = qu_2 \quad \text{and} \quad u|_{\partial\Omega} = 0.$$

For any v, integration by parts gives

$$\int_\Omega q u_2 \overline{v}\, dx = \int_\Omega u\overline{(\Delta - \overline{q_1})v}\, dx + \int_{\partial\Omega} (\partial_\nu u)\overline{v}\, S(dx).$$

Step 3. Take

$$\overline{v} = e^{x\cdot\rho_1/h}(1 + r_1(x;\rho_1))$$

such that

$$(\Delta - q_1)\overline{v} = 0 \quad \text{and} \quad ||r_1||_{L^2} = \mathcal{O}(h).$$

Here

$$\rho_1 = -\xi - \frac{ih}{2}(k - l)$$

with ξ, k, l as before. We note that $\rho_1 + \rho_2 = -ihk$.

With this v, we get

$$\int_\Omega q u_2 \overline{v}\, dx = \int_{\partial\Omega_{+,\epsilon}} (\partial_\nu u)\overline{v}\, S(dx).$$

After some work, including the use of a boundary Carleman estimate, one can show that

$$\left| \int_{\partial\Omega} (\partial_\nu u)\overline{v}\, S(dx) \right| \leq Ch^{1/4} \to 0 \quad \text{as } h \to 0.$$

Hence

$$\int_\Omega q u_2 \overline{v}\, dx = \int_\Omega q e^{-ik\cdot x}(1 + r_2)(1 + r_1)\, dx \to 0 \quad \text{as } h \to 0.$$

Since r_1 and r_2 are $\mathcal{O}(h)$, we thus have

$$\int_\Omega q e^{-ik\cdot x}\, dx = 0 \qquad \forall k \in \mathbb{R}^n \text{ such that } \xi \cdot k = 0.$$

We may change $\xi \in S^{n-1}$ in a small conic neighborhood to see that

$$\int_\Omega q e^{-ik\cdot x}\, dx = 0 \qquad \forall k \in \mathbb{R}^n.$$

(So we see the "parameters" of [14] appearing here... (?))

Since $\widehat{\mathbf{1}_\Omega q}(k)$ is analytic, we have $q = 0$, concluding the proof.

Appendix C. Addition: August 24

A basic result in microlocal analysis: Using Egorov's theorem, any pseudodifferential operator with principal symbol p such that $\{\overline{p}, p\} = 0$ on $p = 0$ can be put in the normal form of the Cauchy-Riemann operator modulo a smoothing operator (see Grigis-Sjostrand).

Theorem 8.5.1 of Hörmander's old book (re: estimates for principally normal operators) is not quite a Carleman estimate for LCW's.

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