PARENTAL ADVISORY: THE FOLLOWING DESCRIBES MY WORK TOWARD A PH.D. IN MATH

MICHAEL VANVALKENBURGH

1. Introduction

I am now partway through my sixth and final year in graduate school. The climax was the past couple months: finishing papers, submitting them to journals, and applying for jobs. In just these first three weeks of December, I turned in about thirty job applications, one paper was published [3], and two more papers were accepted for publication [1],[4]—and with luck it was barely in time. Most of the job applications are online, allowing amendments after the first submission, but one can only hope that the hiring committees look at the most current version.

My dissertation will consist of the three soon-to-be-published papers along with one or two more, still in progress, and it is my goal in the following pages to describe the results in plain English. The past couple months were some preparation for this, as I was forced to look back and take stock of my work, trying to justify it to journals and committees and, as always, to myself. This is an attempt to describe my work one more time, for my most forgiving audience, my parents, not as a justification, but because they have always been curious and interested in my work.

2. Quantum Mechanics

"Mechanics" is the study of physical systems in terms of energy, force, mass, speed, and the like. Quantum mechanics is then the study of physical systems in which certain quantities, like energy, can only take on a discrete set of values. It is surprising but apparently true that such systems exist. For example, if the hydrogen atom were a car, it would only drive at 0 miles per hour, 1 mile per hour, 2 miles per hour, and so on, but nothing in between—a bumpy ride to say the least. This realization, that quantum mechanical systems exist, caused a scientific revolution in the early twentieth century, and scientists of all stripes, especially chemists, physicists, and mathematicians, but also increasing numbers of engineers, are still sorting out the consequences.

Though the name "quantum mechanics" comes from the study of discrete physical quantities, like quantum energy levels, in our discussion we will focus on another aspect of the theory—the probabilistic interpretation of energy states and the concept of "uncertainty", described in the next section. To a sensible human being, the tenets should be strange and perhaps unbelievable, as they were to Albert Einstein, but they have a redeeming quality: they are very useful in describing experimental results to an astounding degree of accuracy. The fundamental experiments were in atomic physics—the world of the very small, so small that it is beyond our direct experience and hence naturally counterintuitive—thus we tend to think of quantum mechanics as only holding on the very small scale. But actually it is a current field of research to make quantum phenomena appear in macroscopic systems. It is believed that something called "quantum decoherence" (see wikipedia) causes quantum phenomena to dissipate in the transition to the visible world, leaving us with the familiar everyday world in which our human senses evolved; but one day we may learn to mitigate quantum decoherence and see quantum mechanics on a larger scale.

3. Uncertainty

Experimenting with very small particles is a tricky business. For example, to see an electron one would have to shine light on it and then have his eye process the reflected light. But an electron is so small that light will affect it, so the observer and the observed are interdependent. Moreover, the frequency of light (or color, for visible light) is proportional to its energy (or ability to cause a sunburn):

$$E = h\nu$$

where E stands for the energy, ν stands for the frequency, and h is Planck's constant, a very small but fixed number ($h = 6.626196 \times 10^{-34}$ joule-second). Hence a blue light will affect a particle more than a red light. Of course, you already knew this, that ultraviolet light affects your skin more than visible light.

This unavoidable interdependence between observer and observed has strange consequences that were (and still are!) shocking to the world of classical physics, often defying intuition. For example, in the world of the very small, the world of quantum mechanics, we can never perfectly know both the position and momentum of a particle. In fact, the uncertainty of a particle's position times the uncertainty of the particle's momentum can never be smaller than Planck's constant. This statement is known as Heisenberg's Uncertainty Principle. The better we know the position, the worse we know the momentum, and conversely. This is a new way of viewing the world, and it required a new mathematical language to describe it.

In classical physics, observable physical quantities are real numbers; I am 150 pounds and am 6 feet tall. But in quantum mechanics, observable quantities are in general not mere real numbers. An observable behaves like a real number only when it has a definite value, and the product[1] AB of two observables A and B is only an observable if both quantities simultaneously have definite values, that is, if they can be precisely known at the same time. Moreover, this happens precisely when $AB = BA$.

Heisenberg's Uncertainty Principle then tells us that, if A represents the position observable and B represents the momentum observable, AB is not equal to BA. However, the difference depends on Planck's constant, so when objects are large compared to Planck's constant, that difference is negligible and we get the usual laws of classical physics. Mathematically, one may think of classical physics as the theory that results when you take the limit as h goes to zero. But we must remember that this is just a mathematical trick that helps us make approximations. After all, h is always 6.626196×10^{-34} joule-second—we can't make it go to zero any more than we can make the number 17 go to zero. But when we use the mathematical technique of taking the parameter h to be smaller and smaller,

[1]The product AB represents the process of first measuring the quantity B and then measuring the quantity A.

some quantities get small, and others stay the same, which tends to sort out what is really important and what is relatively small.

Since we cannot simultaneously know the exact position and momentum of a particle, as we can in classical mechanics, we instead think in terms of probability distributions of the position and momentum. For example, suppose we have a hillside with roundish stones. You are more likely to find one of the stones at the bottom of a hill, or better yet in a valley, and are unlikely to find a stone on the steep slope of a hill. You are also unlikely to find a stone at the top of a hill, though it is possible to find one balanced there. Suppose now that the stones are valuable, and that you want to buy land, using only a topographical map to guide your decision on where to buy. A real estate agent might price the land according to the chance of finding stones, and his map would essentially be a probability distribution.

So it is with the position of a particle. In general you only know the most likely and least likely places to find it. And Heisenberg's Uncertainty Principle tells you that the probability distributions of the momentum and position depend on one another. If you know the precise location of a particle, you have no idea of its momentum—all momenta are equally likely in that case. And, strangely enough, if you know exactly what its momentum is, it could be anywhere at all. By what seems to be a miracle of mathematics, the "Fourier transform" is the exact right tool for this. The Fourier transform of a position distribution is exactly the momentum distribution, and the Fourier transform is invertible, so you can go back. Hence the position distribution and the momentum distribution are just two ways of looking at the same object[2].

Moreover, in practice one is interested in probability distributions that correspond to physical states with definite energies. It ends up that the observed energy is described by a partial differential equation, Schrödinger's Equation, and the states with definite energy are solutions of Schrödinger's Equation; when Schrödinger's operator (the observed energy operator) is applied to these states, it has a definite value and behaves like a real number. This is important because for a given environment, like a hillside, it might be easy to find the corresponding Schrödinger's Equation, and then one would go from there, studying Schrödinger's Equation to find the properties of the states.

With just this much quantum mechanics, I can start to describe my first two papers.

4. Local Exponential Lower Bounds

My first paper [1] is called "Exponential Lower Bounds for Quasimodes of Semiclassical Schrödinger Operators", and in it I obtain lower bounds for solutions of Schrödinger's Equation on bounded surfaces that have boundaries[3]. For example, we may think of waves in a swimming pool. I prove that the water can never be completely still in any part of the pool—the energy on any part of the pool is bounded from below, but is bounded below by a quantity that gets very small as Planck's constant gets very small.

[2]To be completely honest, the main object of study associated with a particle's physical state is its so-called *wave function*: a function f such that $\int |f(x)|^2\,dx = 1$. Then the probability distribution for the position of the particle is $|f|^2$ and the probability distribution for the momentum of the particle is $|\hat{f}|^2$, where $\hat{f}$ is the Fourier transform of f.

[3]A sphere is an example of a bounded surface *without* boundary. The novelty of my paper is that I consider surfaces that *do* have boundaries.

Although the lower bound is very small, I show that my result is the best possible. There are cases where the wave is in fact very small—but is of course not smaller than my lower bound.

All of this is related to "quantum tunnelling". In classical mechanics, if a ball with low energy is near the bottom of a valley, the ball cannot get past a certain height—it cannot roll up and over a hill unless it has a substantial amount of energy. On the other hand, in quantum mechanics, a particle may "tunnel" from one valley to another, even if it has low energy. And there is even a nonzero probability of finding it on top of the hill, although this probability is very small.

5. Microlocal Lower Bounds

My second paper (in progress) is called "Exponential lower bounds for the Bargmann transform" [2] and discusses lower bounds for a particular joint representation of a particle as a function of both position and momentum. Originally I used the word "microlocal" in the paper's title—which is sometimes used to refer to the joint local study of position and momentum—but I dropped it in favor of the more precise current title. The particular microlocal representation in the paper is called the "Bargmann transform" (a slight variation is called the "Fourier-Bros-Iagolnitzer (FBI) transform"). Just as the Fourier transform represents a function in terms of plane waves, the Bargmann transform represents a function in terms of Gaussian wave packets, that is, wave packets with minimum uncertainty.

The Bargmann transform takes a function u of n variables (the position variables) and transforms it into a function Tu of $2n$ variables (position and momentum); the value $Tf(x,\xi)$ can be considered as the contribution to u of the Gaussian wave packet centered at the point (x,ξ) in the $2n$-dimensional position-momentum space (a.k.a. the phase space). This introduces redundancy—in using $2n$ variables to describe a function of n variables—but also produces beautiful regularity properties. Whereas the original function could have been very rough, the Bargmann transform is always smooth; in fact, it is better than smooth: it is *holomorphic*, which means that it enjoys many of the same properties as simple polynomials, written with respect to the complex variable $z = x - i\xi$. For example, unless it is the zero function, it cannot be identically zero in an interval, just as a polynomial cannot be identically zero in an interval unless it is the trivial zero polynomial, 0. My second paper then addresses the question: since it cannot be zero in a whole interval, how small can it be?

Writing the $2n$ variables $(x,\xi) \in \mathbb{R}^{2n}$ in complex form, $z = x - i\xi \in \mathbb{C}^n$, the precise definition of the Bargmann transform is

$$Tu(z;h) = 2^{-\frac{n}{2}}(\pi h)^{-\frac{3n}{4}} \int e^{-\frac{1}{2h}(z-y)^2} u(y)\, dy.$$

The term $e^{-\frac{1}{2h}(z-y)^2}$ is the Gaussian wave packet centered at $z \in \mathbb{C}^n$, considered as a function of $y \in \mathbb{R}^n$. Moreover, the Bargmann transform is invertible (you can recover a function from its Bargmann transform), so the transform gives an honest representation—it does not lose any information.

When using only the Fourier transform, we must study the position distribution separately from the momentum distribution; the Fourier transform takes us from one viewpoint to the other. But with the Bargmann or FBI transform, we can look at both at the same time. We can make the analogy with a musical score. In a musical score we have the time axis in one direction and the frequency axis in the perpendicular direction, so that we have a single joint picture of the time and frequency. Using different methods than in the first paper, I prove lower bounds for this joint representation. This lower bound again gets very small as Planck's constant gets very small.

6. Zakharov-Shabat

In my third paper (in progress), "On the Pseudospectrum of the Zakharov-Shabat System", I construct "almost solutions" (or "quasimodes") to a system of differential equations, called the Zakharov-Shabat system, that is related to the Schrödinger Equation. The way I do this is by making a now-standard assumption that my quasimode will look like a beam of light: a relatively fixed amplitude multiplied by a rapidly oscillating factor. This assumption is called the "Geometric Optics Ansatz". The main result is not surprising to physicists working in fiber optics, but I was able to prove it rigorously using a familiar method from microlocal analysis.

7. Twisted Beams of Light

My fourth paper [3] is called "Laguerre-Gaussian Modes and the Wigner Transform", and in it I give a new analytical description of Laguerre-Gaussian (LG) beams, examples of so-called "twisted beams of light" of interest in laser physics. Whereas the most common type of laser beam has a bell-curve-type distribution of (cross-sectional) intensity, LG beams have annular distributions of intensity, bright in a ring but dark inside and outside the ring. Hence the laser beam looks like a bright hollow tube as it propagates. To be more precise, the light has helical phase fronts—as it travels, the surfaces of constant phase corkscrew along the bright tube. It is hoped that these twisting beams may be used as "optical wrenches", to rotate tiny particles merely by shining light on them. As far as I know, optical wrenches are in an early stage of development, but "optical tweezers" already have successful applications in biology—using laser beams to trap tiny particles. In fact, President-elect Obama's nominee for Energy Secretary, Stephen Chu, won the Nobel Prize in Physics for his experimental work on using laser beams as optical tweezers!

The central observation in my paper is that LG modes can be described using the Wigner transform; to say it exactly as I do in the paper, LG modes are simply Wigner transforms of the more common Hermite-Gaussian (HG) modes. The Wigner transform is a close relative of the Bargmann and FBI transforms described above—it is another version of a function's "musical score". Using this point of view I was able to give simplified proofs of some known results.

8. Manipulating Laser Beams

My fifth paper [4] is a treatment of some particularly interesting operators, designed by the physicists Gabriel F. Calvo and Antonio Picón for potential use in switching between different laser modes. The first half of the paper categorizes the "allowed" classes of laser

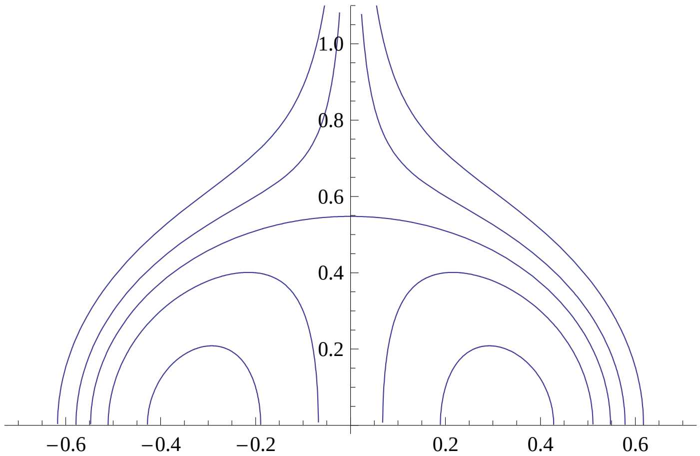

FIGURE 1. Flow lines in the upper half plane in the case $h = 1/10$, with variable C_0. The lower half plane is obtained by symmetry.

beams—it finds those classes of laser beams for which the operators of Calvo and Picón are truly self-adjoint (this is a fine point and I won't dwell on it here). And in the second half I give an approximate description of the evolving laser beams as they change in time, subject to the operators of Calvo and Picón. In an approximate sense, the laser beams evolve in phase space according to Figure 1: the dual position-momentum distributions of the laser beams, as in the musical score analogy, are distorted along the flow whose flow lines are in Figure 1 (in the most interesting two-dimensional slice of a four-dimensional space). The flow lines are actually fairly easy to describe. Just as the circle of radius r is described as the set of points (x, ξ) satisfying

$$x^2 + \xi^2 = r^2,$$

the curve in Figure 2 (which also appears in Figure 1) is described as the set of points (x, ξ) satisfying

$$C_0 = \frac{1}{2}x(x^2 + \xi^2) - \frac{3}{2}hx, \tag{1}$$

with the choice of $C_0 = 0.025$ and $h = 0.1$.

When I first computed the flow in Figure 1, I was happy to see the large circle centered at $(0, 0)$. The interior of the circle is preserved by the flow (nothing ever leaves or enters the region enclosed by the circle), which has a nice interpretation for laser beams. When one takes the two-dimensional cross-section of a laser beam, then looks at the representation of the cross-section in four-dimensional phase space, most of the beam is contained in a four-dimensional ball of radius $\sqrt{h}$ (rather, $\sqrt{3h}$ in Figure 1, or $\sqrt{5h}$ for the whole four-dimensional ball). This four-dimensional ball is preserved by the flow (an interesting two-dimensional cross-section is pictured in Figure 1), so we really are just turning one highly focused beam into another.

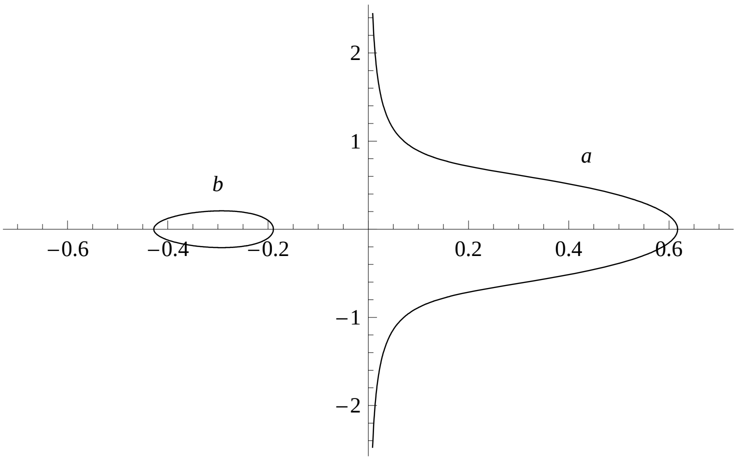

FIGURE 2. The curve described by (1), with parameters $C_0 = 0.025$ and $h = 0.1$.

9. Conclusion

So if anyone asks what I'm working on, a one-sentence answer might be: "He's using mathematics to describe the unusual behavior of physics on small scales." Or to use buzzwords: "He's learning microlocal analysis for use in optics and quantum mechanics."

References

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UCLA Department of Mathematics, Los Angeles, CA 90095-1555, USA
E-mail address: **mvanvalk@ucla.edu**