IAGOLNITZER'S MICROLOCAL UPPER BOUNDS

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In this handout we summarize the work of Iagolnitzer [1], [2] and consider how it relates to our previous papers [4], [5]. Iagolnitzer proved certain microlocal exponentially decreasing upper bounds in the framework of axiomatic quantum field theory, with concrete estimates on the rate of exponential decrease. To prove such estimates, we include a general exposition of the theory.

1. The Wightman Axioms

Our quantum field theory will simply consist of a single ("neutral", bosonic, scalar) field acting on a separable Hilbert space $\mathcal{H}$ of physical states. What we call a "field" is an operator-valued tempered distribution A, and the corresponding test functions are [Schwartz] functions of d-dimensional Minkowski space-time. Working in Minkowski space-time, we use the notation:

$$p^2 = p_0^2 - \sum_{k=1}^{d-1} p_k^2 \equiv p_0^2 - \mathbf{p}^2, \qquad \text{and}$$

$$|p|^2 = p_0^2 + \mathbf{p}^2.$$

With these as our basic objects, we assume the Wightman axioms:

1: Assumptions of Relativistic Quantum Theory. The relativistic transformation law of the states is given by a continuous unitary representation U in $\mathcal{H}$ of the Poincaré group (the inhomogeneous $SL(2,\mathbb{C})$), which associates to each element $g = (a, \Lambda)$ of this group a unitary operator $U(g)$ in $\mathcal{H}$.

Since $U(a,1)$ is unitary it can be written as $\exp(iP^\mu a_\mu)$, where P^μ is an unbounded hermitian operator, interpreted as the energy-momentum operator of the theory. The operator $P^\mu P_\mu = m^2$ is interpreted as the square of the mass. The eigenvalues of P^μ (comprising the so-called "physical spectrum") lie in or on the forward light cone

$$\overline{V}_+ := \{p;\, p^2 = p_0^2 - \mathbf{p}^2 \geq 0,\, p_0 \geq 0\}.$$

There is an invariant state, called the vacuum, Ω:

$$U(g)\Omega = \Omega$$

unique up to a constant phase factor.

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2: Assumptions about the Domain and Continuity of the Field. For each test function $f \in \mathcal{S}$, defined on space-time, the operator $A(f)$ and its adjoint $A(f)^*$ are defined on a domain D of vectors, dense in $\mathcal{H}$. Furthermore, D is a linear set containing Ω, and the operators $U(g)$, $A(f)$, and $A(f)^*$ carry vectors in D into vectors in D.

If $\Phi, \Psi \in D$, then $\langle \Phi | A(f) | \Psi \rangle$ is a tempered distribution, regarded as a functional of f.

3: Transformation Law of the Field. The equation

$$U(g)A(f)U(g)^{-1} = A(f_g)$$

is valid when each side is applied to any vector in D. Here

$$f_g(x) = f(g^{-1}(x)), \qquad g(x) = \Lambda x + a.$$

This transformation law is especially simple because we are only considering A to be a *scalar* field.

4: Local Commutativity, sometimes called Microscopic Causality. If the support of f and the support of g are space-like separated–that is, if $f(x)g(y) = 0$ for all pairs of points such that $(x-y)^2 \geq 0$–then

$$[A(f), A(g)] = 0$$

whenever the left-hand side is applied to any vector in D. We are taking the commutator instead of the anticommutator because we are only considering a single *bosonic* field.

And as a substitute for the assumption of canonical commutation relations, we assume:

5: Cyclicity. The subspace of states generated by vectors $A(f_1) \cdots A(f_n)\Omega$ is dense in $\mathcal{H}$.

These axioms alone do not give life to "particles". For this, we will assume:

6: The One-Mass ($\mu > 0$) Spectral Condition. The mass spectrum is contained in the union of the origin, the hyperboloid $H_+(\mu) := \{p;\, p^2 = \mu^2,\, p_0 > 0\}$, and $\overline{V}_+(2\mu) := \{p;\, p^2 \geq (2\mu)^2,\, p_0 > 0\}$.

We need the mass spectrum to contain the whole continuum $[2\mu, \infty)$ (corresponding to masses of multiparticle states) in order to have the physical spectrum be closed under addition, as the following lemma shows.

Lemma 1. *$H_+(\mu) \cup \overline{V}_+(2\mu)$ is closed under addition.*

Proof. To show that $H_+(\mu) + H_+(\mu) \subset \overline{V}_+(2\mu)$, it clearly suffices to show that

$$p^2 = q^2 = 1 \Rightarrow (p+q)^2 \geq 4.$$

But this is easy to check:

$$\{(p+q)^2 \geq 4\} \Leftrightarrow \{p \text{ and } q \text{ are space-like separated}\}.$$

And it is easy to see geometrically that

$$\{p, q \in H_+(\mu)\} \Leftarrow \{p - q \text{ is space-like}\}.$$

It is also easy to see geometrically that

$$H_+(\mu) + \overline{V}_+(2\mu) \subset \overline{V}_+(2\mu) \qquad \text{and} \qquad \overline{V}_+(2\mu) + \overline{V}_+(2\mu) \subset \overline{V}_+(2\mu).$$

□

We now give a lemma which will play the essential role of creating support properties from the spectral condition.

Lemma 2. *For any states* Φ *and* Ψ*, we have*

$$\int e^{-ip\cdot a} da \langle \Phi | U(a,1) \Psi \rangle = 0$$

unless p *belongs to the physical spectrum.*

Proof. The usual (non-rigorous) way is to write an expansion over the [intermediate] states of physical energy-momentum Q and some other quantum numbers α. To be precise, the eigenstates of the energy-momentum operator form a complete orthonormal system in $\mathcal{H}$, and the eigenvalues are all contained in $\{0\} \cup H_+(\mu) \cup \overline{V}_+(2\mu)$; we label these eigenstates as $|Q\alpha\rangle$, where Q is the eigenvalue and the α are some other quantum numbers that distinguish between different eigenstates with the same eigenvalue Q. We may then give an expansion in terms of these eigenstates:

$$\begin{aligned} \int da\, e^{-ip\cdot a} \langle \Phi | U(a,1) \Psi \rangle &= \sum_\alpha \int dQ \int da\, e^{-i(p-Q)\cdot a} \langle \Phi | Q\alpha \rangle \langle Q\alpha | \Psi \rangle \\ &= (2\pi)^d \sum_\alpha \int dQ\, \delta(Q-p) \langle \Phi | Q\alpha \rangle \langle Q\alpha | \Psi \rangle, \end{aligned}$$

since

$$\langle Q\alpha | U(a,1) \Psi \rangle = e^{iQ\cdot a} \langle Q\alpha | \Psi \rangle.$$

A rigorous proof is sketched in [3], using the SNAG (Stone, Naimark, Ambrose, Godement) Theorem. □

That our assumptions give rise to "one-particle states" is made precise by the following lemma:

Lemma 3. *Let f be a test function in x-space whose Fourier transform $\tilde{f}$ is a C^∞ function of $p = (p_0, \mathbf{p})$ with support in a neighborhood of a point P of $H_+(\mu)$ that does not intersect the region $\overline{V}_+(2\mu)$ (and stays within the cone $\overline{V}_+$). Then $A(f)\Omega$ depends only on the restriction $\hat{f}$ of $\tilde{f}$ to the mass-shell hyperboloid $H_+(\mu)$.*

If $A(f)\Omega$ is non-zero, it is naturally interpreted as the free one-particle state $|\hat{f}\rangle$ with wave function $\hat{f}$. More precisely,

$$A(f)\Omega = const|\hat{f}\rangle.$$

Proof. We will only give a non-rigorous argument; however, the reader may take comfort in the fact that we will not use this lemma for anything.

Let f be a test function as in the statement, but with the additional condition that $\tilde{f}(p) = 0$ for all $p \in H_+(\mu)$. Then we are to show that

$$\langle \Phi | A(f) | \Omega \rangle = 0$$

for all physical states Φ.

We compute

$$\begin{aligned} \int e^{-ip\cdot a} \langle \Phi | A(f(\cdot - a)) | \Omega \rangle da &= \int e^{-ip\cdot a} \langle \Phi | U(a,1) A(f) U(a,1)^{-1} | \Omega \rangle da \\ &= \int e^{-ip\cdot a} \langle \Phi | U(a,1) A(f) | \Omega \rangle da \\ &= 0 \qquad \text{if } p \notin \{0\} \cup H_+(\mu) \cup \overline{V}_+(2\mu), \end{aligned}$$

where the last line follows from Lemma 2.

On the other hand, we have the formal argument

$$\int e^{-ip\cdot a} A_x(f(x-a)) da = \tilde{f}(p) A_x(e^{-ip\cdot x})$$

so that

$$\int e^{-ip\cdot a} \langle \Phi | A_x(f(x-a)) | \Omega \rangle da = \tilde{f}(p) \langle \Phi | A_x(e^{-ip\cdot x}) | \Omega \rangle.$$

It follows that this quantity is zero in all cases. □

2. Connected Chronological Functions

Given a set of functions $f_1(x)$, $f_2(x_1, x_2)$, …, $f_N(x_1, \dots, x_N)$, …, *connected functions* $(f_N)_c$ are defined inductively by the formulae:

$$\begin{aligned} f_1(x) &= (f_1)_c(x) \\ f_2(x_1, x_2) &= (f_2)_c(x_1, x_2) + (f_1)_c(x_1)(f_1)_c(x_2) \\ f_3(x_1, x_2, x_3) &= (f_3)_c(x_1, x_2, x_3) + (f_2)_c(x_1, x_2)(f_1)_c(x_3) + (f_2)_c(x_1, x_3)(f_1)_c(x_2) \\ &\qquad (f_2)_c(x_2, x_3)(f_1)_c(x_1) + (f_1)_c(x_1)(f_1)_c(x_2)(f_1)_c(x_3), \end{aligned}$$

and, more generally,

$$f_N(x_1, \dots, x_N) = \sum_\pi \prod_{j=1}^k f_c(x_{\pi_j}),$$

where the sum is over all partitions π of $\{1, \dots, N\}$ into subsets $\pi_1, \dots, \pi_k$, $k = 1, \dots, N$.

Now let $\mathcal{T}(x_1, \dots, x_N)$ be the product of the time-ordered field operators $A(x_1)$, …, $A(x_N)$. That is, let

$$\mathcal{T}(x_1, \dots, x_N) = A(x_{\pi(1)}) \dots A(x_{\pi(N)}),$$

where π is a permutation of $\{1, \dots, N\}$ such that $(x_{\pi(j)})_0 \geq (x_{\pi(j+1)})_0$.

For a subset I of $\{1, \dots, N\}$, and $x(I) = \{x_i\}_{i \in I}$, we define $\mathcal{T}(x(I))$ to be the time-ordered product of the field operators $\{A(x_i)\}_{i \in I}$. Then we may define a *connected chronological function* to be of the form

$$T(x_1, \dots, x_N) = \langle \Omega | \mathcal{T}(x_1, \dots, x_N) | \Omega \rangle_c.$$

Moreover, for I again a subset of $\{1, \dots, N\}$, and $J = \{1, \dots, N\} \backslash I$, we define

$$T_I(x_1, \dots, x_N) = \langle \Omega | \mathcal{T}(x(I)) \mathcal{T}(x(J)) | \Omega \rangle_c.$$

We clearly have that

$$\mathcal{T}(x_1, \dots, x_N) = \mathcal{T}(x(I)) \mathcal{T}(x(J)) \quad \text{if } x(I) \gtrsim x(J),$$

where $x(I) \gtrsim x(J)$ means that $x(I)$ has no point of $x(J)$ in its closed causal future. Hence

$$(T - T_I)(x_1, \dots, x_N) = 0 \quad \text{if } x(I) \gtrsim x(J). \tag{1}$$

We define the Fourier transform by

$$\tilde{F}(p_1, \dots, p_N) = \int \exp\{i \sum_{j=1}^{N} x_j p_j\} F(x_1, \dots, x_N) dx_1 \cdots dx_N.$$

Then the (one-mass) spectral condition gives the following support property:

Lemma 4. *For $N \geq 2$,*

$$\textit{supp } \tilde{T}_I \subset \{(p_1, \dots, p_N); \sum_{k=1}^{N} p_k = 0 \textit{ and } p_I \in H_+(\mu) \cup \overline{V}_+(2\mu)\},$$

where $p_I := \sum_{i \in I} p_i$.

Proof. For convenience we let

$$y_k = x_{i_k} \text{ for } k = 1, \dots, |I|,$$

$$y_{k+|I|} = x_{j_k} \text{ for } k = 1, \dots, |J|,$$

and we let

$$\mathcal{W}(y_1, \dots, y_N) = \langle \Omega | A(y_1) \cdots A(y_N) | \Omega \rangle.$$

Then we are to show that

$$\text{supp } \tilde{\mathcal{W}}_c(p_1, \dots, p_N) \subset \{(p_1, \dots, p_N); \sum_{k=1}^{N} p_k = 0 \text{ and } \sum_{k=1}^{|I|} p_k \in H_+(\mu) \cup \overline{V}_+(2\mu)\}.$$

Say that

$$\text{supp } f \subset \{p(J);\, p_J = 0 \text{ and } p_{I \cap J} \in H_+(\mu) \cup \overline{V}_+(2\mu)\}$$

and that

$$\text{supp } g \subset \{p(K);\, p_K = 0 \text{ and } p_{I \cap K} \in H_+(\mu) \cup \overline{V}_+(2\mu)\}$$

for $J \cap K = \emptyset$. Then, since $H_+(\mu) \cup \overline{V}_+(2\mu)$ is closed under addition,

$$\text{supp } (f \otimes g) \subset \{(p(J), p(K));\, p_{J\cup K} = 0 \text{ and } p_{I\cap(J\cup K)} \in H_+(\mu) \cup \overline{V}_+(2\mu)\}.$$

So by induction it suffices to show that

(i) supp $\tilde{\mathcal{W}}(p_1) \subset \{p_1 = 0\}$ (that is, the case $N = 1$), and
(ii) supp $\tilde{\mathcal{W}}(p_1, \ldots, p_N) \subset \{(p_1, \ldots, p_N);\, \sum_{k=1}^{N} p_k = 0$ and $\sum_{k=1}^{|I|} p_k \in H_+(\mu) \cup \overline{V}_+(2\mu)\}$.

By Lemma 2, we have that

$$\begin{aligned}
\int e^{ip_1\cdot a} da\, \langle\Omega|U(a,1)A(y)|\Omega\rangle &= \int e^{ip_1\cdot a} da\, \langle\Omega|U(a,1)A(y)U(a,1)^{-1}U(a,1)\Omega\rangle \\
&= \int e^{ip_1\cdot a} da\, \langle\Omega|A(y+a)|\Omega\rangle \\
&= \tilde{\mathcal{W}}(p_1) \\
&= 0 \quad \text{unless } p_1 = 0,
\end{aligned}$$

which proves (i).

Now, to prove (ii), let

$$\xi_j = y_j - y_{j+1} \quad \text{for } j = 1, \ldots, N-1.$$

Since the vacuum is translation-invariant, there exists a tempered distribution W such that

$$W(\xi_1, \ldots, \xi_{N-1}) = \mathcal{W}(y_1, \ldots, y_N).$$

Moreover,

$$\begin{aligned}
&\tilde{\mathcal{W}}(p_1, \ldots, p_N) \\
&\quad = \int \exp\{i\sum_{k=1}^{N} y_k p_k\}\mathcal{W}(y_1, \ldots, y_N) dy_1 \cdots dy_N \\
&\quad = \int \exp\{i(y_1 - y_2)p_1 + i(y_2 - y_3)(p_1 + p_2) + \cdots + i(y_{N-1} - y_N)(p_1 + \cdots + p_N)\} \\
&\qquad\qquad \exp\{iy_N(p_1 + \cdots + p_N)\}\mathcal{W}(y_1, \ldots, y_N) dy_1 \cdots dy_N \\
&\quad = \int \exp\{i\xi_1 p_1 + i\xi_2(p_1 + p_2) + \cdots + i\xi_{N-1}(p_1 + \cdots + p_N)\} \\
&\qquad\qquad \exp\{iy_N(p_1 + \cdots + p_N)\}W(\xi_1, \ldots, \xi_{N-1}) dy_1 \cdots dy_N \\
&\quad = (2\pi)^d \delta(p_1 + \cdots + p_N)\tilde{W}(p_1, p_1 + p_2, \ldots, p_1 + \cdots + p_{N-1}).
\end{aligned}$$

So it suffices to show that

$$\tilde{W}(q_1, \ldots, q_{N-1}) = 0 \quad \text{if } q_{|I|} \notin H_+(\mu) \cup \overline{V}_+(2\mu).$$

Now we compute

$$\begin{aligned}
&\int e^{ip\cdot a}da\,\langle\Omega|A(y_1)\cdots A(y_{|I|})U(-a,1)A(y_{|I|+1})\cdots A(y_N)|\Omega\rangle \\
&\qquad = \int e^{ip\cdot a}da\,\langle\Omega|A(y_1+a)\cdots A(y_{|I|}+a)A(y_{|I|+1})\cdots A(y_N)|\Omega\rangle \\
&\qquad = \int e^{ip\cdot a}da\,\mathcal{W}(y_1+a,y_2+a,\ldots,y_{|I|}+a,y_{|I|+1},\ldots,y_N) \\
&\qquad = \int e^{ip\cdot a}da\,W(y_1-y_2,y_2-y_3,\ldots,y_{|I|}-y_{|I|+1}+a, \\
&\qquad\qquad\qquad\qquad y_{|I|+1}-y_{|I|+2},\ldots,y_{N-1}-y_N) \\
&\qquad = \int e^{ip\cdot a}da\,W(\xi_1,\ldots,\xi_{|I|}+a,\xi_{|I|+1},\ldots,\xi_{N-1})
\end{aligned}$$

which, by Lemma 2, is equal to 0 unless $p\in H_+(\mu)\cup\overline{V}_+(2\mu)$. This proves (ii). □

Macrocausality properties are described by exponential decay of T when applied to test functions of the form

$$\varphi_{i,\tau}(x_i)=\mathrm{const}(\gamma\tau)^{-\frac{d}{2}}\exp\{ip_i\cdot(x_i-\tau u_i)\}\exp\{-\frac{|x_i-\tau u_i|^2}{4\gamma\tau}\} \tag{2}$$

where the constant is chosen such that

$$\tilde{\varphi}_{i,\tau}(p_i')=\exp\{ip_i'\cdot\tau u_i)\}\exp\{-\gamma\tau|p_i'-p_i|^2\}.$$

When the associated field operators are applied to the vacuum state, the resulting state,

$$A(\varphi_{i,\tau})|\Omega\rangle,$$

can be interpreted as asymptotic to a one-particle state, in the $\tau\to\infty$ limit.

To precisely study such macrocausality properties, we present some general mathematical results in the next section.

3. Mathematical Results

For a tempered distribution f, we let

$$\tilde{f}(p)=\int e^{ix\cdot p}f(x)dx,$$

and define the *generalized Fourier transform* of f to be

$$F(x,p;\gamma)=\int\tilde{f}(p')e^{-ip'\cdot x}e^{-\gamma|x||p'-p|^2}dp',$$

defined for all $\gamma>0$.

For the following theorem, let C be a cone with apex at the origin, let

$$d(x):=\mathrm{dist}(x,\complement C),$$

and let

$$C_a:=\{x;\,d(x)\geq a\}.$$

Moreover, let

$$d_a(x) := \mathrm{dist}(x, \complement C_a),$$

and let $\hat{x} := \frac{x}{|x|}$.

Theorem 1. *If $f(x) = 0$ in the cone C, then, given any $\epsilon > 0$, $F(x,p;\gamma)$ satisfies, for $x \in C_\epsilon$,*

$$|F(x,p;\gamma)| < [c_\epsilon(\gamma|x|)^{-\frac{\nu}{2}}\mathcal{P}(|x|,|p|,\sqrt{\gamma|x|})]\exp\{-\frac{d_\epsilon(x)^2}{4\gamma|x|}\}$$

where $\mathcal{P}$ is a polynomial and ν is an integer. Here $d_\epsilon(x) := (d(\hat{x}) - \frac{\epsilon}{|x|})|x|$.

Proof. First of all, we have the convolution formula

$$(\gamma|x|)^{\frac{n}{2}}F(x,p;\gamma) = \int f(x')e^{ip\cdot(x-x')}e^{-\frac{|x-x'|^2}{4\gamma|x|}}dx'. \tag{3}$$

Since f is in general a tempered distribution (supported in $\complement C$), the equation (3) is formally equal to

$$f_{x'}\left(e^{ip\cdot(x-x')-\frac{|x-x'|^2}{4\gamma|x|}}\right).$$

Then, by the continuity property of tempered distributions, there are C, r, and s such that

$$\left|f_{x'}\left(e^{ip\cdot(x-x')-\frac{|x-x'|^2}{4\gamma|x|}}\right)\right| \leq C||e^{ip\cdot(x-\cdot)-\frac{|x-\cdot|^2}{4\gamma|x|}}||_{r,s},$$

where

$$||g||_{r,s} := \sum_{|k|\leq r}\sum_{|\ell|\leq s}\sup_x |x^k D^\ell g(x)|.$$

Hence, by direct computation, after fixing some $\delta > 0$,

$$|(\gamma|x|)^{\frac{n}{2}}F(x,p;\gamma)| \leq C(\gamma|x|)^{-\nu}\mathcal{P}(|x|,\gamma|x|,|p|)e^{-\frac{|x-x_0|^2}{4\gamma|x|}}$$

for some x_0 such that

$$|x - x_0| \geq d(x) - \delta.$$

Hence

$$|(\gamma|x|)^{\frac{n}{2}}F(x,p;\gamma)| \leq C(\gamma|x|)^{-\nu}\mathcal{P}(|x|,\gamma|x|,|p|)e^{-\frac{|d(x)-\delta|^2}{4\gamma|x|}}.$$

Since $d(x) = |x|d(\hat{x})$, the theorem is proven by taking $\delta = \epsilon$. □

Remark 1. If f is a continuous function, we do not have to introduce cutoffs, so then we may take $\epsilon = 0$. Moreover, if $f \in L^1$, then we have

$$|F| < \mathrm{const}\exp\left\{-\frac{d(\hat{x})^2}{4\gamma}|x|\right\}.$$

Theorem 2. *If $\tilde{f}(p) = 0$ in the ball $S(P, r) := \{p;\, |p - P| < r\}$, then, for all $\epsilon > 0$, F satisfies at P the estimate*

$$|F(x, P; \gamma)| < [c_\epsilon(\gamma|x|)^{-\frac{\nu}{2}}\mathcal{P}(|x|, \sqrt{\gamma|x|})] \exp\{-(1-\epsilon)r^2\gamma|x|\} \tag{4}$$

for all $\gamma > 0$.

Proof. Again using the regularity property of tempered distributions (this time for $\tilde{f}$), for any fixed $\epsilon > 0$ there exists some p_0 such that

$$|p_0 - P| \geq (1-\epsilon)r$$

and such that

$$\begin{aligned}|F(x, P; \gamma)| &= \left|\tilde{f}_p\left(e^{-ip\cdot x - \gamma|x||p-P|^2}\right)\right| \\ &\leq [\text{whatever}]e^{-\gamma|x||p_0 - P|^2} \\ &\leq [\text{whatever}]e^{-\gamma|x|(1-\epsilon)^2 r^2}.\end{aligned}$$

□

Theorem 3. *(a) If $f = f' + f''$, $f'(x) = 0$ in a cone C and $\tilde{f}''(p) = 0$ in $S(P, r)$, then F satisfies at P bounds of the form (4) in each direction $\hat{x}$ of C if $\gamma < \frac{d(\hat{x})}{2r}$. More generally, the rate of fall-off in each direction of C is at least equal (or arbitrarily close) to $\inf\left[\frac{d(\hat{x})^2}{4\gamma}, r^2\gamma\right]$, and, in particular, to $\frac{1}{2}rd(\hat{x})$ if $\gamma = \frac{1}{2r}d(\hat{x})$.*

(b) Similarly, if $f = f_1' + f_1'' = f_2' + f_2'' = \cdots$ with $f_i'(x) = 0$ in a cone C_i and $\tilde{f}_i''(p) = 0$ in $S(P, r_i)$, $i = 1, 2, \ldots$, then $F(x, P; \gamma)$ decays exponentially in each direction $\hat{x}$ of $\cup_i C_i$. The rate of exponential fall-off is at least equal (or arbitrarily close) to the "enveloping function" of

$$\sup_i \left\{\inf\left[\frac{d_i(\hat{x})^2}{4\gamma}, r_i^2\gamma\right]\right\}.$$

Proof. This follows easily from Theorems 1 and 2. □

4. Macrocausality Properties of N-Point Functions

We define the *causal set* Σ as

$$\Sigma := \{(u, p) \equiv (u_1, \ldots, u_N, p_1, \ldots, p_N);$$
$$p_k \in H_+(\mu) \cup \overline{V}_+(2\mu)\ \forall k,\ \sum_{k=1}^N p_k = 0, \quad \text{and (C1) holds}\}$$

where the condition (C1) is:

> **(C1)** Given any proper subset I of $\{1, \ldots, N\}$ such that $u(I)$ contains no other point u_j, $j \notin I$, in its closed future, then $p_I \in H_+(\mu) \cup \overline{V}_+(2\mu)$.

The condition (C1) has a simple physical meaning in terms of collisions of particles. We take some proper subset I of $\{1,\dots,N\}$ and consider the points $u(I)$ and energy-momenta $p(I)$. We call a point (u_k, p_k) *incoming* if $(p_k)_0 < 0$ and *outgoing* if $(p_k)_0 > 0$. We then compare the incoming elements with index in I to the outgoing elements with index in I.

The condition "$u(I)$ contains no other point u_j, $j \notin I$, in its closed future" means that, in the subsystem given by I, there are no "missing" outgoing points, although there may be "missing" incoming points. Physically, we should have a net gain in energy-momentum for this subsystem. If the incoming momenta are $p_{i_1},\dots,p_{i_m}$ and the outgoing momenta are $p_{i_{m+1}},\dots,p_{i_{m+n}}$, we should have

$$-p_{i_1} - \cdots - p_{i_m} < p_{i_{m+1}} + p_{i_{m+n}}$$

in the sense that

$$p_I = \sum_{i\in I} p_i \in H_+(\mu) \cup \overline{V}_+(2\mu).$$

That is, $p_I \neq 0$ and is in the physical spectrum.

An easy special case is given in the following lemma:

Lemma 5. *Let $(u,P)\in\Sigma$. Then*

- *(i) Each outgoing point u_j is in the future cone of at least one incoming point, and*
- *(ii) Each incoming point u_i is in the past cone of at least one outgoing point.*

Proof. For (i), we may take $I = \{1,\dots,N\}\backslash\{j\}$. Then (C1) says

$$-p_j = p_1 + \cdots + p_{j-1} + p_{j+1} + \cdots + p_N \in H_+(\mu) \cup \overline{V}_+(2\mu),$$

which gives a contradiction since by hypothesis $(p_j)_0 > 0$.

For (ii), we may simply take $I = \{i\}$. □

We now state the main result of this handout.

Theorem 4. *(i) Given that test functions of the form (2), $T(\{\varphi_{i,\tau}\})$ decays exponentially, for any $\gamma > 0$, in the $\tau\to\infty$ limit, apart possibly from configurations $(u,p)\in\Sigma$.*

(ii) Given test functions of the form (2) and $(u,p)\notin\Sigma$, the rate of exponential fall-off in τ is at least equal (or arbitrarily close) to $\beta(u,p;\gamma)$ for each $\gamma>0$, where β is strictly positive and is determined as follows. For each $I(\neq\{1,\dots,N\})$, let $d_I(u)$ be the distance from u to the set of points y such that $y(I)$ contains other points in its closed future, and let $r_I(p)$ denote the distance from p to the set of points p' such that $\left\{\sum_{i=1}^N p'_i = 0\right\}$ and $\{p'_I \in H_+(\mu)\cup\overline{V}_+(2\mu)\}$. For each γ, let

$$\beta_I(u,p;\gamma) = \inf\left[\frac{d_I(u)^2}{4\gamma}, r_I^2(p)\gamma\right].$$

Then

$$\beta(u,p;\gamma) = \sup_I \beta_I(u,p;\gamma).$$

If γ is chosen equal to $\frac{d_I(u)}{2r_I(p)}$ for some I (such that $d_I(u) > 0$, $r_I(p) > 0$), then $\beta(u, p; \gamma)$ is at least equal (or arbitrarily close) to $\frac{1}{2}d_I(u)r_I(p)$.

Proof. It suffices to prove (ii). We take $I(\neq \{1, \dots, N\})$, $J = \{1, \dots, N\}\backslash I$, and consider the sum

$$T = (T - T_I) + T_I.$$

We recall (1) and Lemma 4:

$$(T - T_I)(u_1, \dots, u_N) = 0 \quad \text{if } u(I) \gtrsim u(J)$$

and

$$\tilde{T}_I(p_1, \dots, p_N) = 0 \quad \text{if } p_1 + \cdots + p_N \neq 0 \quad \text{or if } p_I \notin H_+(\mu) \cup \overline{V}_+(2\mu).$$

Then the result follows from using $f = T$, $f' = T - T_I$, and $f'' = T_I$ in Theorem 3(b). □

5. Note to Reader

A few parts of this handout (especially the statements of some of the theorems) are taken verbatim from the cited works of Iagolnitzer. Also, there are some minor inconsistencies (a few unimportant wrong signs) that result, as usual, from the multiple conflicting conventions for the Fourier transform, etc. These inconsistencies are also taken verbatim from the works cited.

References

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