A FUN LINEAR ALGEBRA PROBLEM

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Question: Let $\Psi = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in \mathrm{GL}(2n, \mathbb{R})$. What is the relationship between $\det \Psi \in \mathbb{R}$ and $\det(X + iY) \in \mathbb{C}$?

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By considering multiplication by J_0 (which does not affect the determinant) and scaling, it is easy enough to guess the formula

$$\det \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} = |\det(X+iY)|^2.$$

The proof I found is completely elementary, but it may be easier to understand with some knowledge of basic symplectic linear algebra. In fact, the question is taken verbatim from the book *Introduction to Symplectic Topology* (2nd edition) by McDuff and Salamon, where it appears as Exercise 2.27. It is related to the Maslov index, of importance, for example, in the global theory of Fourier integral operators.

I would be interested if anyone can find an alternative proof. I tried a proof using the fact that the determinant is the product of the eigenvalues, but I could not manage to make the argument rigorous.

Proof. We write

$$\Psi = \begin{pmatrix} \uparrow & \dots & \uparrow & \uparrow & \dots & \uparrow \\ v_1 & \dots & v_n & J_0 v_1 & \dots & J_0 v_n \\ \downarrow & \dots & \downarrow & \downarrow & \dots & \downarrow \end{pmatrix},$$

with $J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, so that the standard symplectic form ω_0 ("$= \sum dx_j \wedge d\xi_j$") in matrix form is $-J_0$. The column vectors of Ψ form a basis for $\mathbb{R}^{2n}$, but they are not orthogonal. Our approach in solving the problem is to change bases, resulting in an orthogonal basis, in such a way that the determinant is not changed. Moreover, we write $v_j = \begin{pmatrix} x_j \\ y_j \end{pmatrix}$ and $z_j = x_j + iy_j$ so that

$$\omega_0(v_j, v_k) = \operatorname{Im}(\overline{z_j} \cdot z_k)$$

and

$$\langle v_j, v_k \rangle = \operatorname{Re}(\overline{z_j} \cdot z_k).$$

We first use the Gram-Schmidt process (with respect to the standard inner product):

$$u_k := v_k - \sum_{j=1}^{k-1} \frac{\langle v_k, u_j \rangle}{|u_j|^2} u_j.$$

Since J_0 is a unitary matrix, the same coefficients work for orthogonalizing the $J_0 v_j$:

$$\begin{aligned} J_0 u_k &= J_0 v_k - \sum_{j=1}^{k-1} \frac{\langle J_0 v_k, J_0 u_j \rangle}{|J_0 u_j|^2} J_0 u_j \\ &= J_0 v_k - \sum_{j=1}^{k-1} \frac{\langle v_k, u_j \rangle}{|u_j|^2} J_0 u_j. \end{aligned}$$

The upper-triangular matrix used here has 1's on the diagonal, so we have

$$\det \Psi = \det \begin{pmatrix} \uparrow & \dots & \uparrow & \uparrow & \dots & \uparrow \\ u_1 & \dots & u_n & J_0 u_1 & \dots & J_0 u_n \\ \downarrow & \dots & \downarrow & \downarrow & \dots & \downarrow \end{pmatrix}$$
$$= (-1)^{n(n-1)/2} \det \begin{pmatrix} \uparrow & \uparrow & \uparrow & \dots & \uparrow & \uparrow \\ u_1 & J_0 u_1 & u_2 & \dots & u_n & J_0 u_n \\ \downarrow & \downarrow & \downarrow & \dots & \downarrow & \downarrow \end{pmatrix}.$$

The columns are still not pairwise orthogonal, so we use the *symplectic* Gram-Schmidt process. Let

$$\alpha_k = u_k + \sum_{j=1}^{k-1} \frac{\omega_0(u_k, \alpha_j)}{|\alpha_j|^2} J_0 \alpha_j - \sum_{j=1}^{k-1} \frac{\omega_0(u_k, J_0 \alpha_j)}{|\alpha_j|^2} \alpha_j$$

and

$$\beta_k = J_0 \alpha_k.$$

Again, the upper-triangular matrix has 1's on the diagonal, so, after reordering the columns (to undo the earlier permutation), we have

$$\det \Psi = \det \begin{pmatrix} \uparrow & \dots & \uparrow & \uparrow & \dots & \uparrow \\ \alpha_1 & \dots & \alpha_n & J_0 \alpha_1 & \dots & J_0 \alpha_n \\ \downarrow & \dots & \downarrow & \downarrow & \dots & \downarrow \end{pmatrix}.$$

The columns now form a basis that is not normalized, but is otherwise a standard symplectic basis:

$$\omega_0(\alpha_j, \alpha_k) = 0, \qquad \omega_0(\beta_j, \beta_k) = 0, \qquad \omega_0(\alpha_j, \beta_k) = \delta_{jk} |\alpha_j|^2.$$

Moreover, it is an orthogonal basis for $\mathbb{R}^{2n}$:

$$\langle \alpha_j, \alpha_k \rangle = \delta_{jk} |\alpha_j|^2, \qquad \langle \beta_j, \beta_k \rangle = \delta_{jk} |\alpha_j|^2, \qquad \langle \alpha_j, \beta_k \rangle = 0.$$

Hence

$$\det \Psi = |\alpha_1|^2 \cdots |\alpha_n|^2.$$

On the other hand,

$$\det(X + iY) = \det \begin{pmatrix} \uparrow & \dots & \uparrow \\ z_1 & \dots & z_n \\ \downarrow & \dots & \downarrow \end{pmatrix}.$$

We note that the changes of bases above may be written in terms of the complex symplectic form and complex inner product. Letting ζ_j be the complexification of u_j, and η_j the complexification of α_j, we have

$$v_k = u_k + \sum_{j=1}^{k-1} \frac{\operatorname{Re}(\overline{z_k} \cdot \zeta_j)}{|\zeta_j|^2} u_j$$

and

$$\alpha_k = u_k + \sum_{j=1}^{k-1} \frac{\mathrm{Im}\,(\overline{\zeta_k} \cdot \eta_j)}{|\eta_j|^2} J_0 \alpha_j - \sum_{j=1}^{k-1} \frac{\mathrm{Im}\,(\overline{\zeta_k} \cdot i\eta_j)}{|\eta_j|^2} \alpha_j.$$

This is equivalent to

$$z_k = \zeta_k + \sum_{j=1}^{k-1} \frac{\mathrm{Re}\,(\overline{\zeta_j} \cdot z_k)}{|\zeta_j|^2} \zeta_j$$

and

$$\zeta_k = \eta_k + \sum_{j=1}^{k-1} \frac{(\overline{\eta_j} \cdot \zeta_k)}{|\eta_j|^2} \eta_j.$$

Hence

$$\det(X + iY) = \det \begin{pmatrix} \uparrow & \dots & \uparrow \\ \eta_1 & \dots & \eta_n \\ \downarrow & \dots & \downarrow \end{pmatrix}.$$

Now the columns are orthogonal: We of course have

$$\overline{\eta_j} \cdot \eta_k = \langle \alpha_j, \alpha_k \rangle + i\omega_0(\alpha_j, \alpha_k),$$

but by our construction of the α's,

$$\omega_0(\alpha_j, \alpha_k) = 0 \ \forall j, k, \qquad \text{and} \qquad \langle \alpha_j, \alpha_k \rangle = \delta_{jk} |\alpha_j|^2.$$

Hence, with $*$ denoting the conjugate transpose,

$$\begin{aligned} |\det(X + iY)|^2 &= \det \left(\begin{pmatrix} \uparrow & \dots & \uparrow \\ \eta_1 & \dots & \eta_n \\ \downarrow & \dots & \downarrow \end{pmatrix}^* \begin{pmatrix} \uparrow & \dots & \uparrow \\ \eta_1 & \dots & \eta_n \\ \downarrow & \dots & \downarrow \end{pmatrix} \right) \\ &= \det \begin{pmatrix} |\alpha_1|^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & |\alpha_n|^2 \end{pmatrix} \\ &= |\alpha_1|^2 \cdots |\alpha_n|^2. \end{aligned}$$

So we finally have the formula

$$\det \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} = |\det(X + iY)|^2.$$

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