CLASSIFICATION OF INTEGRABLE SYSTEMS ON TWO-DIMENSIONAL MANIFOLDS AND INVARIANTS IN FOMENKO'S MODELS

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Abstract. If M is an oriented compact surface, ω a volume form, and $\mathcal{F}$ a compact foliation with Morse singularities, we associate to the triple $(M, \omega, \mathcal{F})$ some invariants which allow to classify it. Among these invariants, there are Taylor coefficients corresponding to bifurcation points. One can find such invariants in Fomenko's models for Liouville's torus bifurcations.

If $\mathcal{F}$ is the Lagrangian foliation with compact leaves defined on the symplectic manifold (M_{2n}, ω) by a fibration π of M on Q, one knows that Q is provided with an entire affine structure ∇, and J. J. Duistermaat [1] has shown how one can reconstruct $(M, \omega, \mathcal{F})$ from an invariant of the quotient object (Q, ∇) (the "Chern class of the fibration"). Here we are interested in the case where $\mathcal{F}$ is a foliation with singularities of Morse type. In the simplest case ($n = 1$), we show (cf. I) that the quotient object is provided with natural invariants which allow for the reconstruction of $(M, \omega, \mathcal{F})$ (cf. IV). One deduces as in the models of Fomenko in dimension 4 [2] the "branching invariants" for Liouville's tori [3] (cf. V). To simplify the presentation, we only consider the case where the branching points are simple (vertices of index 3 in the graph). It is not too difficult to extend the definition to the general case.

I. Triplets $(M, \omega, \mathcal{F})$ and the Associated Reeb Graphs.

All the structures considered are of class C^∞.

I.1. **A Preliminary Study.** In all that follows, M denotes a compact and connected manifold of dimension 2, and ω is a symplectic (volume!) form on M; $\mathcal{F}$ denotes a Morse foliation in the following sense: it is a foliation with singularities such that for each leaf F of $\mathcal{F}$, there exists a neighborhood $\mathcal{U}$ of F, saturated for the foliation, and a proper Morse function $f_{\mathcal{U}}$ (to simplify, one supposes that the critical points have pairwise distinct values) such that $\mathcal{F}$ restricted to $\mathcal{U}$ is defined by the level curves of $f_{\mathcal{U}}$. One calls these level curves "leaves". Under these hypotheses, all the regular leaves are diffeomorphic to the circle, and a singular leaf is either a pin-point (an elliptic point, or center), or a figure eight (a separatrix, for which the corresponding critical point is a hyperbolic point, or saddle). Two triples $(M_1, \omega_1, \mathcal{F}_1)$

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and $(M_2, \omega_2, \mathcal{F}_2)$ are said to be equivalent if there exists a symplectomorphism from M_1 to M_2 which preserves the foliation.

I.1.1. *A Study in a Neighborhood of the Centers.* We recall the following well known result [4]: let m_0 be a center in $(M, \omega, \mathcal{F})$; then there exists a local system of coordinates (x, y) in a neighborhood of m_0 such that

$$\begin{cases} \omega = dx \wedge dy \\ \mathcal{F} : x^2 + y^2 = Const. \end{cases}$$

I.1.2. *A Study in a Neighborhood of the Saddles.* Let m be a saddle in M and let $\mathcal{U}$ be a neighborhood of m such that its saturation is provided by a Morse function f defining the foliation. The saturation of $\mathcal{U}$ by the foliation has the appearance of an enlarged figure eight with three connected parts. One calls the leaves "interior leaves" that are situated in one of the parts (1) or (2) inside the figure eight, and "exterior leaves" those situated in the exterior of the figure eight [part (3)]. Shrinking $\mathcal{U}$ if necessary, one has the following result (see [4] for part (a)):

Lemma 1. *(a) There exist local coordinates (x, y) on $\mathcal{U}$ such that $xy > 0$ on the interior leaves (resp. $xy < 0$ on the exterior leaves), and such that*

$$\begin{cases} \omega = dx \wedge dy \\ f = \lambda(xy). \end{cases}$$

(b) Every other system (X, Y) satisfying (a) is such that $XY = xy$ formally at 0.

I.1.3. *A Geometric Study.* It follows from Lemma 1 that the foliation is defined on $\mathcal{U}$ by $xy = Const.$ If F is a leaf of the part (1) [resp. (2), (3)], defined locally by $xy = \epsilon$, $\epsilon > 0$ [resp. $\epsilon < 0$ in (3)], the expression of the area between F and the figure eight is given respectively by

$$\begin{cases} \mathcal{A}_i(\epsilon) = -\epsilon \ln(\epsilon) + h_i(\epsilon), \qquad i = 1, 2 \\ \mathcal{A}_3(\epsilon) = 2\epsilon \ln|\epsilon| + h(\epsilon) \end{cases}$$

where h, h_1, h_2 are differentiable functions. In addition, one has the relation:

$$h^{(p)}(0) = -(h_1^{(p)}(0) + h_2^{(p)}(0)) \qquad \text{for each } p.$$

Proposition 1. *The Taylor series at 0 of the functions h_1 and h_2 are invariants of $(M, \omega, \mathcal{F})$.*

I.2. **The Reeb Graph Associated to the Couple $(M, \mathcal{F})$.** This is the topological quotient space $\mathcal{G} = M/\mathcal{F}$ [5]. One denotes by p the canonical projection of M onto $\mathcal{G}$. In $\mathcal{G}$, we call a *regular point* the image by p of a regular leaf, a *bout* the image of a center, and a *bifurcation point* the image of a figure eight. The edges are the parts of $\mathcal{G}$ contained between two singular points. If s is a bifurcation point, it has three edges; one of them corresponds to the leaves that extend along the separatrix. One says that this edge is the *trunk* of s; the two other edges are the *branches* of s. Moreover, the graph is provided with the measure μ, the image by p of the natural measure defined by ω on M.

Remark 1. Let m be a saddle in M, and let (x, y) be a local system of coordinates in a neighborhood of m, satisfying Lemma 1. In such coordinates we define in $\mathcal{G}$ a function $\epsilon = xy$ in a neighborhood of the corresponding bifurcation point s, such that $xy > 0$ on the branches of s and $xy < 0$ on the trunk of s. The measure μ is expressed in this neighborhood by the formulas

$$(\star) \qquad \begin{cases} d\mu_i(\epsilon) = [\ln \epsilon + g_i(\epsilon)]d\epsilon \qquad \text{on each branch } (i = 1, 2) \\ d\mu(\epsilon) = [2 \ln |\epsilon| + g(\epsilon)]d\epsilon \qquad \text{on the trunk} \\ \text{with } g, g_1, g_2 \text{ differentiable functions satisfying, for each } p, \\ g^{(p)}(0) = (g_1^{(p)}(0) + g_2^{(p)}(0)). \end{cases}$$

Remark 2. The expression for the measure μ is deduced from that of the area. One has, as a corollary of Proposition 1, the following result: the Taylor series at 0 of the functions g_1 and g_2 are invariants of $(\mathcal{G}, \mu)$.

II. Affine Reeb Graphs.

Generalizing the notion of Reeb graph seen in 1.2, one is lead to introduce the following definition:

Definition 1. Let $\mathcal{G}$ be a topological 1-complex where the vertices are of degree 1 or 3. For each vertex s of degree 3, which one calls a bifurcation point, one distinguishes an edge and calls it the trunk of s; the two others are the branches of s.

• $\mathcal{G}$ is provided with an atlas of the following type:

1) Outside of the bifurcation points, it is a "classical" atlas of a manifold with boundary of dimension 1.

2) In a neighborhood of each bifurcation point s, there exists an open set V and a continuous map φ from V to $(-\epsilon, \epsilon)$, $\epsilon > 0$, with $\varphi(s) = 0$ and such that, if T is the trunk of s and B_1, B_2 are the branches of s,

$$\varphi|_{B_i} \quad \text{is bijective on } [0, \epsilon),\ i = 1, 2.$$

$$\varphi|_T \quad \text{is bijective on } (-\epsilon, 0].$$

One requires that the changes of charts are differentiable on each part $T \cup B_i$, $i = 1, 2$.

• $\mathcal{G}$ is provided with a measure given by a non-zero density, C^∞ on each edge, and such that for each vertex s of degree 3, there exists a chart φ at s in which the measure μ is written as in $(\star)$. One denotes by $(\mathcal{G}, \mathcal{D}, \mu)$ such a graph provided with its differentiable structure and with its measure, and one calls it an affine Reeb graph.

Lemma 2. *If the measure μ is written in another chart $\tilde{\varphi}$ in a neighborhood of s:*

$$\begin{cases} d\mu_i(\epsilon) = [\ln \epsilon + \tilde{g}_i(\epsilon)]d\epsilon & \text{on each branch } B_i \text{ of } s \\ d\mu(\epsilon) = [2\ln|\epsilon| + \tilde{g}(\epsilon)]d\epsilon & \text{on the trunk of } s, \end{cases}$$

then the functions g_i and $\tilde{g}_i$ have the same Taylor series at the origin. (It automatically follows that the Taylor series of g and $\tilde{g}$ at 0 are equal.)

The preceding lemma shows that the Taylor series of the functions g_i give invariants for the bifurcation points.

III. Weighted Reeb Graphs.

Definition 2. Let $\mathcal{G}$ be a combinatorial graph having vertices of degree 1 or 3. For each vertex s of degree 3, one distinguishes in the same fashion as in Definition 1 the trunk and the branches of s, and one associates to each branch a sequence of real numbers. In addition, to each edge is associated a positive real number called its length. Such a graph is called a weighted Reeb graph.

Remark 3. To each affine Reeb graph one naturally associates a weighted Reeb graph, the sequences of numbers associated to the branches corresponding to coefficients of the Taylor series of the functions g_i; the "lengths" of the edges are given by their measure.

IV. A Classification Theorem.

The preceding paragraphs show that one knows how to associate successively to a triplet $(M, \omega, \mathcal{F})$ an affine Reeb graph $(\mathcal{G}, \mathcal{D}, \mu)$, unique up to equivalence, and to such an affine Reeb graph a weighted Reeb graph, unique up to equivalence. Conversely, one has the following theorem:

Theorem 1. *(a) Every weighted Reeb graph is the graph associated to an affine Reeb graph, unique up to equivalence. (b) Every affine Reeb graph is the Reeb graph associated to a triplet $(M, \omega, \mathcal{F})$, unique up to equivalence.*

Thus the triplets $(M, \omega, \mathcal{F})$ are classified by the affine Reeb graphs or the weighted Reeb graphs.

V. Bifurcation Invariants in Fomenko's Models.

Let (M_4, ω, H) be a Hamiltonian system, c a regular value of H, Q a compact connected component of $H^{-1}(c)$, and F a first integral of Bott-Morse on Q. Fomenko has given in this situation a topological description of the singular foliation of Q by Liouville's tori. We consider a saturated neighborhood $\mathcal{U}$ of Liouville's tori of a separatrix S, containing a single orbit of X_H, critical for $F|_Q = F_Q$; transversal to this orbit, F_Q has a saddle-type singularity.

Let $\mathcal{T}$ be a connected surface in $\mathcal{U}$ which is transversal to X_H and which meets all the orbits. (Shrinking $\mathcal{U}$ if necessary, one such transversal always exists.) If ω_T is the induced volume form on $\mathcal{T}$, $\mathcal{F}_{\mathcal{T}}$ the singular foliation induced on $\mathcal{T}$ by the level surfaces of F_Q, $\sigma = S \cap \mathcal{T}$ is a saddle-type separatrix in $\mathcal{F}_{\mathcal{T}}$.

Proposition 2. *The invariants associated to σ as in Proposition 1 do not depend on the choice of transversal $\mathcal{T}$; this defines the bifurcation invariants of Liouville's tori on the separatrix S.*

This definition naturally extends to more general cases of Fomenko's models.

References

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