10.7: Planar Graphs!

A 1. Which of K_3, K_4, K_5 do you think are planar?
(no need to justify)

2. Is C_n, W_n planar?

3. Is $K_{3,3}$ planar? Try to justify. (this is hard).

B Try to prove the following corollaries:

Cor 2: If G is a connected, planar simple graph then G has a vertex of degree not exceeding 5

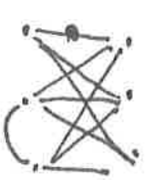
Cor 3: If a conn plan. simple graph has e edges and v vertices and no circuits of length three, then $e \leq 2v-4$.

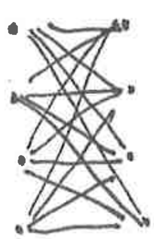
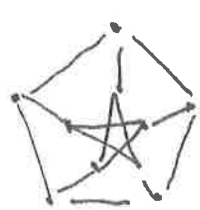
C Let G be a connected planar simple graph with 20 verts, each of deg 3. How many regions are there?

D Use cor 1,2,3 to show

1. $K_{3,3}$ is not planar
2. K_5 is not planar.

E Use Kuratowski's thm to show that the following graphs are nonplanar



F: Challenge: Draw $K_{3,3}$ on a torus (surface of a donut) so that no edges cross.

Solutions

A

1. K_3, K_4 are planar, K_5 is not.

2. Yes, C_n, W_n are planar.

3. $K_{3,3}$ is not planar. (We proved this in class, let me know if you have any questions)

B

Proof of corollary 2.: This is on page 757 on the book.

Proof of corolary 3.: Let G be a connected, planar simple graph with e edges, v vertices, $v \geq 3$. and no circuits of length 3.

Since every circuit is of length 4 or more, the degree of every region (degree of region was defined in class) on the interior is ≥ 4. Since there are at least 3 vertices, the outer region also has degree ≥ 4.

Thus $\sum_{\text{all regions } R} \deg(R) \geq 4r$.

In class, we showed $2e = \sum_{\text{all regions } R} \deg R$.

Putting this together, we get

$$2e \geq 4r \rightarrow \tfrac{1}{2}e \geq r.$$

Using $r = e - v + 2$, we get

$$e - v + 2 \leq \tfrac{1}{2}e$$ which rearranges to

$$e \leq 2v - 4.$$

C We compute the number of edges by using the handshaking lemma: $2e = \sum_{v \in V} \deg(v) = 20 \cdot 3 = 60$.

So there are 30 edges.

By Euler's formula, $r = e - v + 2 = 30 - 20 + 2 = 12$.

D 1. We use corollary 3. Note that $K_{3,3}$ has no 3-cycles, so it is valid to use corolary 3. It has 6 vertices, and 9 edges.

Since $e = 9 \not\leq 2 \cdot 6 - 4 = 8$, this cannot be planar.

2. We use corollary 1. By the handshake lemma, K_5 has $\frac{5 \cdot 4}{2} = 10$ edges. Since $e = 10 \not\leq 3 \cdot 5 - 6 = 9$, this cannot be planar.

E. This subgraph:

is homeomorphic to $K_{3,3}$ since you can remove this vertex.

For the second graph, the subgraph

is $K_{3,3}$.

For the last one, see page 760 for a picture.